

# Hey Series, Have you herd of catalanimals?

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We'll talk about

- ▶ Modules, symmetric functions, combinatorics
- ▶ Macdonald goes wild (1980's)
- ▶ Catalanimals, a series perspective

# Harmonics

Polynomials in  $n$  variables killed by symmetric diff operators

$$\mathcal{M}_n = \{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \ \forall a > 0\}$$

Some harmonics in 3 variables

$$\Delta = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix} \quad 2x_1 - x_2 - x_3 \quad 1$$

$$\mathcal{M}_3$$

$$\text{span} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_1 - x_2, 1 \}$$

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$\mathcal{M}_n$  is an  $S_n$ -module

$$\text{sp}\{\Delta, 2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2, x_3-x_1, \underbrace{x_1-x_2}_{1 \mapsto 2 \ 2 \mapsto 3 \ 3 \mapsto 1}, 1\}$$



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Irreducibles are indexed by partitions

$$\underbrace{\text{sp}\{\Delta\}}_{\boxed{\phantom{00}}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, x_1^2 - 2x_2(x_1 - x_3) - x_3^2\}}_{\boxed{\phantom{00}}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\boxed{\phantom{00}}} \oplus \underbrace{\text{sp}\{1\}}_{\boxed{\phantom{0000}}}$$

$$\lambda = \begin{array}{c} 1 \\ 1 \\ 2 \\ 4 \end{array} \boxed{\phantom{0000}} = (4, 2, 1, 1)$$

# Harmonics

Polynomials in  $n$  variables killed by symmetric diff operators

$$\mathcal{M}_n = \{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \ \forall a > 0\}$$

Some harmonics in 3 variables

$$\Delta = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix} \quad 2x_1 - x_2 - x_3 \quad 1$$

Irreducibles are indexed by partitions

$$\underbrace{\text{sp}\{\Delta\}}_{\boxed{\phantom{00}}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, x_1^2 - 2x_2(x_1 - x_3) - x_3^2\}}_{\boxed{\phantom{000}}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\boxed{\phantom{00}}} \oplus \underbrace{\text{sp}\{1\}}_{\boxed{\phantom{0000}}}$$

what is the decomposition into irreducibles?

# Symmetric functions

polynomials invariant under  $S_n$ -action:  $\sigma : z_i \mapsto z_{\sigma(i)}$

not symmetric

$$z_1 - z_2 + z_3$$

symmetric

$$z_1 + z_2 + z_3$$

$$z_1^2 z_2^5 z_3^3$$

$$z_1^2 z_2^5 z_3^3 + z_1^2 z_2^3 z_3^5 + z_1^5 z_2^2 z_3^3 + z_1^5 z_2^3 z_3^2 + z_1^3 z_2^5 z_3^2 + z_1^3 z_2^2 z_3^5$$

# Symmetric functions

polynomials invariant under  $S_n$ -action:  $\sigma : z_i \mapsto z_{\sigma(i)}$

not symmetric

symmetric

symmetrize a monomial

$$z_1^2 z_2^5 z_3^3 \longrightarrow z_1^2 z_2^5 z_3^3 + z_1^2 z_2^3 z_3^5 + z_1^5 z_2^2 z_3^3 + z_1^5 z_2^3 z_3^2 + z_1^3 z_2^5 z_3^2 + z_1^3 z_2^2 z_3^5$$

$$(5, 3, 2) = \begin{array}{c} \square \square \square \\ \square \square \square \\ \square \end{array}$$

Bases are indexed by partitions

# Symmetric functions

polynomials invariant under  $S_n$ -action:  $\sigma : z_i \mapsto z_{\sigma(i)}$

not symmetric

symmetric

symmetrize a monomial

$$z_1^2 z_2^5 z_3^3 \longrightarrow z_1^2 z_2^5 z_3^3 + z_1^2 z_2^3 z_3^5 + z_1^5 z_2^2 z_3^3 + z_1^5 z_2^3 z_3^2 + z_1^3 z_2^5 z_3^2 + z_1^3 z_2^2 z_3^5$$

$$(5, 3, 2) = \begin{array}{ccccc} \square & \square & \square & \square & \square \\ & \square & \square & \square & \end{array}$$

## Schur function basis

Weyl symmetrize a monomial

$$z_1^2 z_2 \longrightarrow 2z_1 z_2 z_3 + z_1^2 z_2 + z_1 z_2^2 + z_1^2 z_2 + z_2^2 z_3 + z_1 z_3^2 + z_2 z_3^2$$

$$s_\lambda = \sigma(z_1^{\lambda_1} \cdots z_\ell^{\lambda_\ell}) \quad \text{where} \quad \sigma(f(\mathbf{z})) = \sum_{w \in S_n} w \left( \frac{f(\mathbf{z})}{\prod_{i < j} (1 - z_j/z_i)} \right)$$

# Harmonics Module

Polynomials in  $n$  variables

$$\mathcal{M}_n = \{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \ \forall a > 0\}$$

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, x_1^2 - 2x_2(x_1 - x_3) - x_3^2\}}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

Frobenius



irreducible  $\chi^\lambda \mapsto s_\lambda$

$$????? = s_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

# Harmonics Module

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Frobenius



irreducible  $\chi^\lambda \mapsto s_\lambda$

$$????? = s_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

What can we say about this symmetric function ??? ?

# The Frobenius image

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$$\text{????} = s_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \end{array}} + s_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

Expand  $s_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \end{array}} + s_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$

$$z_1 z_2 z_3 + z_1 z_1 z_2 + z_2 z_1 z_3 + z_3 z_2 z_1 + z_2 z_1 z_1 + z_1 z_2 z_2 + z_2 z_1 z_2 + z_2 z_2 z_1 + z_3 z_1 z_2 + \dots$$

$$(z_1 + z_2 + z_3)^3$$

# The Frobenius image

$$\text{????} = s_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

Expand  $s_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$

$$\begin{aligned} z_1z_2z_3 + z_1z_1z_2 + z_2z_1z_3 + z_3z_2z_1 + z_2z_1z_1 + z_1z_2z_2 + z_2z_1z_2 + z_2z_2z_1 + z_3z_1z_2 + \dots \\ (z_1 + z_2 + z_3)^3 \end{aligned}$$

Find Schur expansion

$$\text{words } (z_1 + z_2 + z_3)^3 = \sum_{\substack{\text{words } w_1 w_2 w_3 \\ \text{in } 1, 2, 3}} z_{w_1} z_{w_2} z_{w_3}$$

# The Frobenius image

$$\text{????} = s_{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}} + s_{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}} + s_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}} + s_{\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}}$$

Expand  $s_{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}} + s_{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}} + s_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}} + s_{\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}}$

$$z_1 z_2 z_3 + z_1 z_1 z_2 + z_2 z_1 z_3 + z_3 z_2 z_1 + z_2 z_1 z_1 + z_1 z_2 z_2 + z_2 z_1 z_2 + z_2 z_2 z_1 + z_3 z_1 z_2 + \dots \\ (z_1 + z_2 + z_3)^3$$

Find Schur expansion

$$\text{words } (z_1 + z_2 + z_3)^3 = \sum_{\substack{\text{words } w_1 w_2 w_3}} z_{w_1} z_{w_2} z_{w_3}$$

$$\text{tableaux} = s_{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}} + s_{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \\ \hline\end{array}} + s_{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 3 \\ \hline\end{array}} + s_{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}}$$

# Modules, symmetric functions, combinatorics

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## ► $S_n$ -Module

$\mathcal{M}$  = Harmonic polynomials

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, x_1^2 - 2x_2(x_1 - x_3) - x_3^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{\mathbf{1}\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

## ► Identify symmetric function = Frobenius image

$$\mathcal{F}(\mathcal{M}) = (z_1 + z_2 + z_3)^3$$

## ► Explore combinatorics

words to Young tableaux

$$(z_1 + z_2 + z_3)^3 = s_{\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline 3 \\ \hline 1 & 2 \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline 2 \\ \hline 1 & 3 \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}}$$

# Quantum leap

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- ▶ Combinatorics: basic hypergeometric series,  $q$ -counting

$$\begin{array}{c} \square \\ \square \\ \square \end{array} + \begin{array}{c} \blacksquare \\ \square \end{array} + \begin{array}{c} \blacksquare \\ \blacksquare \end{array} + \begin{array}{c} \square \\ \blacksquare \\ \square \end{array} + \begin{array}{c} \square \\ \square \\ \blacksquare \end{array} \rightarrow (1 + q + q^2 + q^3 + q^4)$$

- ▶ Geometry:  $q$ -deformation of cohomology
- ▶ Physics: conformal and topological quantum field theory

Modules/symmetric functions/combinatorics?

# Harmonics Module

Polynomials in  $n$  variables

$$\mathcal{M}_n = \{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \ \forall a > 0\}$$

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} \text{ degree 2 polynomials}} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} \text{ degree 1 polynomials}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

what polynomial degrees occur for each irreducible?

# Harmonics Module

Polynomials in  $n$  variables

$$\mathcal{M}_n = \{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \forall a > 0\}$$

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} \text{ degree 2 polynomials}} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} \text{ degree 1 polynomials}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

Frobenius



degree  $d$  irreducible  $\mapsto q^d s_\lambda$

$$????? = q^3 s_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + q s_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

Identify the symmetric function  $\mathcal{F}(\mathcal{M})$

# Harmonics Module

Polynomials in  $n$  variables

$$\mathcal{M}_n = \{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \ \forall a > 0\}$$

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|}\hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, x_1^2 - 2x_2(x_1 - x_3) - x_3^2\}}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} \text{ degree 2 polynomials}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} \text{ degree 1 polynomials}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline \end{array}}$$

Frobenius



degree  $d$  irreducible  $\mapsto q^d s_\lambda$

Hall-Littlewood polynomial

$$\sum_{\text{words } w} q^{\text{inv}(w)} z_{w_1} z_{w_2} z_{w_3} = q^3 s_{\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}} + q^2 s_{\begin{smallmatrix} 3 \\ 2 \\ 1 \\ 2 \end{smallmatrix}} + q s_{\begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}} + s_{\begin{smallmatrix} 1 & 2 & 3 \end{smallmatrix}}$$

# Modules, symmetric functions, combinatorics

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## ► Graded $S_n$ -Module

$\mathcal{M}$  = Harmonic polynomials

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, x_1^2 - 2x_2(x_1 - x_3) - x_3^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{\mathbf{1}\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

## ► Identify symmetric function = $q$ -Frobenius image

$\mathcal{F}(\mathcal{M})$  = Hall-Littlewood polynomial

## ► Explore combinatorics

words, Young tableaux, inv and cocharge statistic

$$\sum_{\text{words } w} q^{\text{inv}(w)} z^w = q^3 s_{\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline 3 \\ \hline 1 & 2 \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline 2 \\ \hline 1 & 3 \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}}$$

We'll talk about

- ▶ Modules, symmetric functions, combinatorics
- ▶ Macdonald goes wild (1980's)
- ▶ Catalanimals, a series perspective

## Macdonald initiates $q, t$ -theory

Generalization of Selberg's integral led to  $q, t$  symmetric functions

$$\frac{-4q}{t-1}z_1^2 z_2 + \frac{-4q}{t-1}z_2^2 z_3 + \frac{-4q}{t-1}z_1^2 z_3 + (t^2 - 7q)z_1 z_2 z_3$$

Macdonald prove the existence of a basis

- defined by orthogonality
- specializing to Hall-Littlewoods
- conjectured to have certain positive expansion

(Garsia-modified) conjecture:  $q, t$ -sum of Schur functions

$$H_{2,1} = qt s_{\begin{array}{c} 3 \\ 2 \\ 1 \end{array}} + q s_{\begin{array}{c} 3 \\ 1 \\ 2 \end{array}} + t s_{\begin{array}{c} 2 \\ 1 \\ 3 \end{array}} + s_{\begin{array}{c} 1 \\ 2 \\ 3 \end{array}}$$

# Garsia-Haiman approach

harmonic polynomials in  $x$

$$= \{f(x) : (\partial_{x_1}^a + \dots + \partial_{x_n}^a) f(x) = 0 \forall a > 0\}$$

some module of polynomials in  $x$  and  $y$

$$\underbrace{\text{sp}\{1\}}_{\begin{smallmatrix} \square\square\square \\ (0,0) \end{smallmatrix}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\begin{smallmatrix} \boxplus \\ \text{degree 0 in } y \quad \text{degree 1 in } x \end{smallmatrix}} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\begin{smallmatrix} \boxplus \\ (1,0) \end{smallmatrix}} \oplus \underbrace{\text{sp}\{x_1 y_1 - x_2 y_2\}}_{\begin{smallmatrix} \boxplus \\ (1,1) \end{smallmatrix}}$$

$$\tilde{H}_{2,1} = t^0 q^0 s_{\square\square\square} + t^0 q^1 s_{\boxplus} + t^1 q^0 s_{\boxplus} + t^1 q^1 s_{\boxplus}$$

modified Macdonald polynomial

# Two bi-Graded trilogies

harmonic polynomials in  $x$

$$\{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \ \forall a > 0\}$$



polynomials in  $x$  and  $y$   
 $f(x, y)$  killed by  $\sum_i \partial_{x_i}^a \partial_{y_i}^b$



not a Macdonald polynomial

# TWO BI-GRADED TRILOGIES

harmonic polynomials in  $x$

$$\{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \ \forall a > 0\}$$

span of all partial derivatives of Vandermonde



Garsia-Haiman modules  
polynomials in  $x$  and  $y$   
Vandermondes in  $x$  and  $y$



Macdonald polynomials



# Two bi-Graded trilogies

harmonic polynomials in  $x$

$$\{f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a) f(x) = 0 \ \forall a > 0\}$$



polynomials in  $x$  and  $y$   
 $f(x, y)$  killed by  $\sum_i \partial_{x_i}^a \partial_{y_i}^b$



not a Macdonald polynomial

# Diagonal Harmonics Module

Polynomials in 2 sets of variables

$$\mathcal{DH}_n = \left\{ f(x, y) : \sum_i \partial_{x_i}^a \partial_{y_i}^b f(x, y) = 0 \ \forall a + b > 0 \right\}$$

$$\mathcal{DH}_2 = \text{sp}\{x_1 - x_2, y_1 - y_2, 1\}$$

$$??? = q^{\color{green}1} s_{\square} + t^{\color{blue}1} s_{\square} + s_{\square}$$

↙      ↓      ↘

What symmetric function is the Frobenius image of  $\mathcal{DH}_n$ ?

# The Frobenius image of $\mathcal{F}(\mathcal{DH}_n)$

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$$\mathcal{F}(\mathcal{DH}_2) = (q + t) s_{\square} + s_{\square\square} \quad \leftarrow \text{What is this?}$$

Macdonald expansion of  $\mathcal{F}(\mathcal{DH}_2)$

$$(q + t) s_{\square} + s_{\square\square} = \frac{t}{t - q} H_{1,1} - \frac{q}{t - q} H_2$$

Macdonald expansion of  $s_{1^3}$

$$s_{1^3} = \frac{1}{t - q} H_{1,1} - \frac{1}{t - q} H_2$$

# The Frobenius image of $\mathcal{F}(\mathcal{DH}_n)$

$$\mathcal{F}(\mathcal{DH}_2) = (q + t) s_{\square} + s_{\square\square} \quad \leftarrow \text{What is this?}$$

Macdonald expansion of  $\mathcal{F}(\mathcal{DH}_2)$

$$(q + t) s_{\square} + s_{\square\square} = \frac{t}{t - q} H_{1,1} - \frac{q}{t - q} H_2$$

Macdonald expansion of  $s_{1^3}$

↑  
define  $\nabla$  to be a  
Macdonald  
eigenoperator

$$s_{1^3} = \frac{1}{t - q} H_{1,1} - \frac{1}{t - q} H_2$$

[Haiman'02]

$$\mathcal{F}(\mathcal{DH}_n) = \nabla s_{1^n}$$

# Two scenarios

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## Bi-Graded Module

Garsia-Haiman

$\mathcal{M}_\mu = \text{span of partials of } \Delta_\mu$

Diagonal harmonics

$\mathcal{D}\mathcal{H}_n = \text{polynomials killed by differential operators}$

## Symmetric function

$\mathcal{F}(\mathcal{M}_\mu) = \text{modified Macdonald } H_\mu$

$\mathcal{F}(\mathcal{D}\mathcal{H}_n) = \nabla s_{1^n}$

QUESTION:  $\sum_{\text{words } w} q^{??} t^{??} z_{w_1} z_{w_2} z_{w_3}$

$\sum_{??} q^{??} t^{??} z^{??}$

## Schur expansion

QUESTION:  $\sum_{\text{tableaux } T} q^{??} t^{??} s_{\text{shape}(T)}$

$\sum_{??} q^{??} t^{??} s^{??}$

# The Shuffle Conjecture [HHLRU'05]

$$\nabla s_{1^n} = \sum_{\pi} t^{\text{area}(\pi)} \sum_{P \text{ labels } \pi} q^{\text{dinv}(P)} z^P$$

Dyck Paths  $\pi$ :

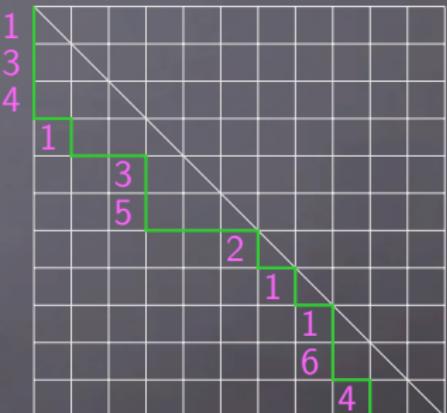
lattice path from  $(0, n)$  to  $(n, 0)$  below diagonal

LLT-polynomial

Labellings  $P$

increase down vertical runs

$\text{dinv}(P)$  counts inverted labels



# The Shuffle Conjecture [HHLRU'05]

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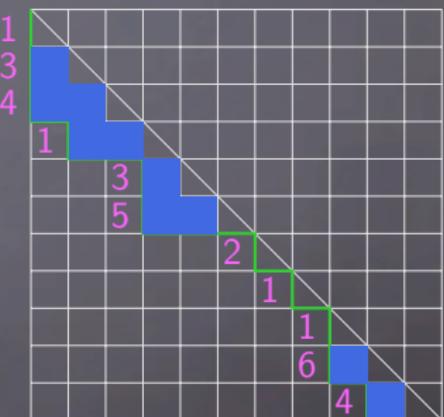
area counts squares above  $\pi$

LLT-polynomial

Labellings  $P$

increase down vertical runs

$\text{dinv}(P)$  counts inverted labels



# One Scenario

---

- Bigraded Module (Garsia/Haiman'92)

$\mathcal{DH}_n$  = harmonics in  $2n$  variables

- Symmetric functions = Frobenius image

$$\mathcal{F}(\mathcal{DH}_n) = \nabla s_{1^n} \quad (\text{Haiman'02})$$

$$= \sum_{\substack{\text{Dyck paths } \pi}} t^{\text{area}(\pi)} LLT_\pi(z; q) \quad (\text{Carlsson/Mellit'15})$$

attach an LLT to each path,  $t$ -weighted by its area



- Schur expansion

Good question!

## Second scenario

---

- Bi-Graded Module (Garsia/Haiman'92)

$\mathcal{M}_\mu$  = linear span of partial derivatives of  $\Delta_\mu$

- Symmetric function = Frobenius image

$\mathcal{F}(\mathcal{M}_\mu)$  = modified Macdonald polynomial (Haiman'02)

$$= \sum_{\text{ribbons } R} q^{\text{maj}(R)} t^{\text{leg}(R)} LLT_R(x; q) \quad (\text{HHL'05})$$

- Schur expansion

Good question!

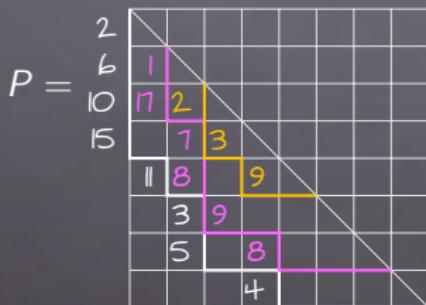
## Other $q, t$ -frameworks

Building on  $\mathcal{F}(\mathcal{DH}_n) = \nabla s_{1^n}$

Conjecture: Loehr-Warrington (2008)

$$\nabla s_\lambda = \pm \sum_{\substack{\pi = \text{nested paths} \\ \text{of lengths from } \lambda}} t^{\text{area}(\pi)} LLT_\pi(z; q)$$

attach an LLT to each nest of paths,  
 $t$ -weighted by the sum of areas of each path in a nest



We'll talk about

- ▶ Modules, symmetric functions, combinatorics
- ▶ Macdonald goes wild (1980's)
- ▶ Catalanimals, a series perspective

What we're gonna do right here is go back

---

Schur function  $s_\lambda = \sigma(z^\lambda)$  is obtained by symmetrizing a monomial with

$$\sigma(f(z)) = \sum_{w \in S_n} w \left( \frac{f(z)}{\prod_{i < j} (1 - z_j/z_i)} \right)$$

classical symmetric function theory develops from here

What about  $q$  and  $q, t$  symmetric functions?

- ▶ Hall-Littlewood polynomials (just  $q$ )
- ▶ LLT polynomials (just  $q$ )
- ▶ Macdonald polynomials
- ▶  $\nabla s_\lambda$  (includes  $\nabla s_{1^n}$ )
- ▶  $\nabla$ Hall-Littlewood

# Catalan functions

$q$ -ify Schur function definition  $s_\lambda(z) = \sigma(z^\lambda)$

Catalan function

for Dyck path  $\pi$ ,

$$H_{\pi, \lambda}(z; q) = \text{pol} \left( \sigma \left( \frac{z^\lambda}{\prod_{(i,j) \in R_\pi} (1 - q z_i / z_j)} \right) \right)$$

$$R_\pi = \begin{array}{c} \text{Diagram of a Dyck path} \\ \text{--- set of roots above Dyck path } \pi \end{array}$$

example:



$= \{(1,2), (1,3)\}$  determines denominator terms

## Example

---

$$\pi = \begin{array}{|c|c|c|} \hline & \textcolor{blue}{\square} & \\ \hline & \diagup & \\ \hline & \diagdown & \\ \hline \end{array} = \{(1,2), (1,3)\} \quad \lambda = (1, 1, 1)$$

$$H_{\pi, \lambda}(z; q) = \text{pol} \left( \sigma \left( \frac{z_1 z_2 z_3}{(1 - qz_1/z_2)(1 - qz_1/z_3)} \right) \right)$$

Series expand

$$z_1 z_2 z_3 + q z_1^2 z_3 + q^2 z_1^3 z_3/z_2 + q z_1^2 z_2 + q^2 z_1^3 + q^3 z_1^5/z_2 z_3 + \dots$$

Apply  $\sigma$

$$\begin{array}{ccccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \sigma(z^{111}) & 0 & -q^2 \sigma(z^{300}) + q \sigma(z^{210}) & + q^2 \sigma(z^{300}) & + q^3 \sigma(z^{5-1-1}) & & \end{array}$$

straightening

every  $\sigma(z^\gamma)$  is  $\pm \sigma(z^\lambda)$  for some weakly decreasing  $\lambda$ , or zero

## Example

---

$$\pi = \begin{array}{|c|c|c|} \hline & \textcolor{blue}{\square} & \\ \hline & \diagup & \\ \hline & \diagdown & \\ \hline \end{array} = \{(1,2), (1,3)\} \quad \lambda = (1, 1, 1)$$

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straightening

every  $\sigma(z^\gamma)$  is  $\pm \sigma(z^\lambda)$  for some weakly decreasing  $\lambda$ , or zero

$\text{pol}(s_\lambda) = 0$  when  $\lambda_n < 0$

$$s_{111} + q s_{21}$$

# What are Catalan functions?

---

For Dyck path  $\pi$  and partition  $\lambda$ ,

$$H_{\pi, \lambda}(z; q) = \text{pol} \left( \sigma \left( \frac{z^\lambda}{\prod_{(i,j) \in R_\pi} (1 - q z_i / z_j)} \right) \right)$$

- ▶ Hall-Littlewood polynomials
  - $q$ -Schur positive
- ▶ parabolic Hall-Littlewoods
  - Schur positivity sought since (Broer'1992)
  - crystals, rigged configurations, katabolism, higher cohomology vanishing, etc
- ▶ (Haiman-Panyshev) in general,
  - conjectured to be  $q$ -Schur positive
  - graded Euler characteristics of vector bundles on the flag variety

# Side hustle: Schubert calculus

**Theorem.** [Blasiak-M-Pun]

$k$ -Schur functions are Catalan functions  
all Catalan functions are  $q$ -Schur positive

$$s_{\lambda}^{(k)}(x; q) = \text{pol} \left( \sigma \left( \frac{z^{\lambda}}{\prod_{(i,j) \in R_{\lambda}} (1 - q z_i / z_j)} \right) \right) \quad \text{for } R_{\pi} =$$



At  $q = 1$ :

- ▶ represent Schubert classes for  $H^*(\text{Gr}_{SL_n})$
- ▶ 3-point Gromov-Witten invariants (genus 0) arise as coefficients in  $k$ -Schur expansion of Catalan functions
- ▶ An adaptation [Blasiak-M-Seelinger] supports quantum K-theory with new advances by [Ikeda-Iwao-Naito]

# Another one parameter family

Hall-Littlewood polynomial = a Catalan function

$$\sum_{\text{words } w} q^{\text{inv}(w)} z^w = H_{\text{staircase}, (1^n)}(z; q)$$

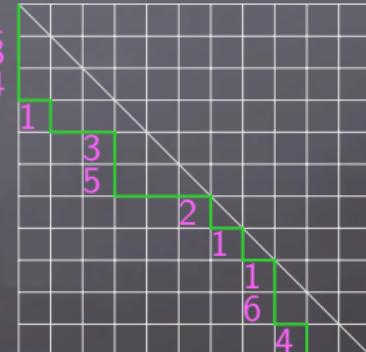
## LLT polynomials

$$LLT_\pi(z; q) = \sum_{\text{labelling } P} q^{\text{dinv}(P)} z^P$$

Labellings  $P$

increase down vertical runs

$\text{dinv}(P)$  counts inverted labels



# Another one parameter case

Hall-Littlewood polynomials

$$\mathcal{F}_q(\mathcal{M}_n) = \text{pol}\left(\sigma\left(\frac{z_1 \cdots z_n}{\prod_{i < j}(1 - qz_i/z_j)}\right)\right)$$



LLT polynomials (Haiman-Grojnowski)

$$LLT_\nu(z; q) = \text{pol}\left(\sigma\left(\frac{w_0(F_\beta(z; q)\overline{E_\alpha(z; q)})}{\prod_{i < j}(1 - qz_i/z_j)}\right)\right)$$

for non-symmetric Hall-Littlewood polynomials

$E_\alpha = (-q)^{\ell(w)} T_w z^{\alpha_+}$  and their duals

enabled their proof of Schur positivity

It's time to mind your  $t$ 's and  $q$ 's!

experiment with adding  $t$  to Hall-Littlewood formula

Example.

$$H = \text{pol} \left( \sigma \left( \frac{z_1 z_2}{(1 - qz_1/z_2)(1 - tz_1/z_2)} \right) \right)$$

sage: s(H)

sage:  $(q + t)s_{\square} + s_{\square\square}$

It's time to mind your  $t$ 's and  $q$ 's!

experiment with adding  $t$  to Hall-Littlewood formula

Example.

$$H = \text{pol} \left( \sigma \left( \frac{z_1 z_2}{(1 - qz_1/z_2)(1 - tz_1/z_2)} \right) \right)$$

sage: s(H)

sage:  $(q + t)s_{\square} + s_{\square\square}$

Example.

$$\mathcal{D}\mathcal{H}_2 = \text{sp}\{x_1 - x_2, y_1 - y_2, 1\}$$

$$\nabla s_{1^2} = q^1 s_{\square} + t^1 s_{\square} + s_{\square\square}$$

↖      ↓      ↘

# Series for diagonal harmonics

$$\nabla s_{1^n} = \sum$$



polynomial part of

$$\sigma \left( \frac{z_1 \cdots z_n \prod_{i+1 < j} (1 - q t z_i / z_j)}{\prod_{i < j} (1 - q z_i / z_j) (1 - t z_i / z_j)} \right)$$

$$= \sum_{\pi} t^{\text{area}(\pi)} LLT_{\pi}(z; q)$$

# Series for diagonal harmonics

$$\nabla s_{1^n} = \sum$$



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$$\sigma \left( \frac{z_1 \cdots z_n \prod_{i+1 < j} (1 - q t z_i / z_j)}{\prod_{i < j} (1 - q z_i / z_j) (1 - t z_i / z_j)} \right)$$

$$= \sum_{\pi} t^{\text{area}(\pi)} LLT_{\pi}(z; q)$$

polynomial part of

$$\sum t^{\text{area}(\beta/\alpha)} \sigma \left( \frac{(w_0(F_{\beta}(z; q) \overline{E_{\alpha}(z; q)}))}{\prod_{i < j} (1 - q z_i / z_j)} \right)$$

# Series for diagonal harmonics

$$\nabla s_{1^n}$$

$$= \sum$$



*(X/H/H/H/H/H)*

$$\textcolor{violet}{H} \left( \frac{z_1 \cdots z_n \prod_{i+1 < j} (1 - q t z_i / z_j)}{\prod_{i < j} (\textcolor{magenta}{X/H/H/H/H}) (1 - t z_i / z_j)} \right)$$

$$= \sum_{\pi} t^{\text{area}(\pi)} LLT_{\textcolor{violet}{\pi}}(z; q)$$

*(X/H/H/H/H/H)*

$$\sum t^{\text{area}(\beta/\alpha)} \textcolor{violet}{H} \left( \frac{(w_0(F_{\beta}(z; q) \overline{E_{\alpha}(z; q)})}{\prod_{i < j} (\textcolor{magenta}{X/H/H/H/H})} \right)$$

# Unstraightened Series Identity

## Theorem

$$z_1 \cdots z_l \frac{\prod_{i+1 < j} (1 - q t z_i / z_j)}{\prod_{i < j} (1 - t z_i / z_j)} = \sum_{a_1, \dots, a_{l-1} \geq 0} t^{|a|} w_0(F_\beta(z; q) \overline{E_\alpha(z; q)})$$

where  $\beta = (1, a_{l-1} + 1, \dots, a_1 + 1)$  and  $\alpha = (a_1, \dots, a_{l-1}, 0)$ .

Proof: Cauchy Identity

$$\frac{\prod_{i < j} (1 - tqz_i y_j)}{\prod_{i \leq j} (1 - tz_i y_j)} = \sum_a t^{|a|} E_a(z; q^{-1}) F_a(y; q)$$

implies Shuffle Theorem of [Carlsson-Mellit'15] by symmetrizing and taking the polynomial part

# Series for an entire $q, t$ -Basis?

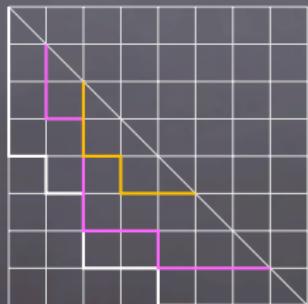
Example.  $\nabla s_{1^n}$  is the polynomial part of

$$\sigma \left( \frac{z_1 \cdots z_n \prod_{i+1 < j} (1 - q t z_i / z_j)}{\prod_{i < j} (1 - q z_i / z_j)(1 - t z_i / z_j)} \right)$$

## Loehr-Warrington Conjecture [LW'08]

$$\nabla s_\lambda = \sum_{\substack{\text{nest of paths } \pi \\ \text{determined by } \lambda}} t^{\text{area}(\pi)} LLT_\pi(z; q)$$

$P =$



# Catalanimals

---

Example.  $\nabla s_{1^n}$  is the polynomial part of

$$\sigma \left( \frac{z_1 \cdots z_n \prod_{i+1 < j} (1 - q t z_i / z_j)}{\prod_{i < j} (1 - q z_i / z_j) (1 - t z_i / z_j)} \right)$$

Def. For 3 sets of roots  $R_{qt}, R_q, R_t$  and an integer vector  $\lambda$ ,

$$H(R_{qt}, R_q, R_t, \lambda) = \sigma \left( \frac{z^\lambda \prod_{(i,j) \in R_{qt}} (1 - q t z_i / z_j)}{\prod_{(i,j) \in R_q} (1 - q z_i / z_j) \prod_{(i,j) \in R_t} (1 - t z_i / z_j)} \right)$$

even monomial positivity not guaranteed

$$\sigma(z_1 z_2 z_3 (1 - q t z_1 / z_3)) = \sigma(z_1 z_2 z_3 - q t z_1^2 z_2) = (1 - 2q t) z_1 z_2 z_3 - q t z_1^2 z_2 + \dots$$

# Catalanimals

---

Example.  $\nabla s_{1^n}$  is the polynomial part of

$$\sigma \left( \frac{z_1 \cdots z_n \prod_{i+1 < j} (1 - q t z_i / z_j)}{\prod_{i < j} (1 - q z_i / z_j)(1 - t z_i / z_j)} \right)$$

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is there a catalanimal  $H$  so  $\text{pol}(H) = \nabla s_\mu$ ?

# Loehr-Warrington Conjecture

Schur catalanimal

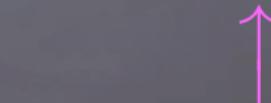
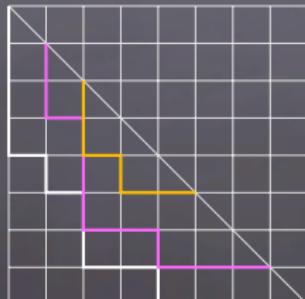
$$\sigma \left( \frac{z^\mu \prod_{(i,j) \in R_{qt}} (1 - q t z_i / z_j)}{\prod_{(i,j) \in R_q} (1 - q z_i / z_j) \prod_{(i,j) \in R_t} (1 - t z_i / z_j)} \right)$$



Theorem (Blasiak, Haiman, M., Pun, Seelinger)

$$\nabla s_\lambda = \sum_{\substack{\text{nested paths } \pi \\ \text{of shape } \lambda}} t^{\text{area}(\pi)} LLT_\pi(z; q)$$

$P =$



LLT Series

$$\sigma \left( \frac{(w_0(F_\beta(z; q) \overline{E_\alpha(z; q)}))}{\prod_{i < j} (1 - q z_i / z_j)} \right)$$

# LLT Catalanimals

$$H = H(R_{qt}, R_q, R_t; \lambda) \text{ where } \text{pol}(H) = \nabla(LLT_\nu(z; q))$$

- $R_+ \setminus R_q$  = pairs of boxes in the same diagonal,
- $R_q \setminus R_t$  = the attacking pairs,
- $R_t \setminus R_{qt}$  = pairs on adjacent diagonals,
- $R_{qt}$  = all other pairs,



$\lambda$ , as a filling of  $\nu$



# Macanimals

LLT catalanimals

$H = H(R_{qt}, R_q, R_t; \lambda)$  where  $\text{pol}(H) = \nabla(LLT_\nu(\mathbf{z}; q))$

$$H_\mu(\mathbf{z}; q, t) = \sum_{\nu} t^{\text{maj}(\nu)} LLT_\nu(\mathbf{z}; q)$$

↓ apply  $\nabla$

$$q^* t^* H_\mu(\mathbf{z}; q, t) = \sum_{\nu} t^{\text{maj}(\nu)} \nabla(LLT_\nu(\mathbf{z}; q))$$

Theorem (Blasiak-Haiman-M.-Pun-Seelinger)

$$H_\mu = \text{pol} \sigma \left( z_1 \cdots z_n \frac{\prod_{(i,j) \in R^*} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j)}{\prod_{i < j} (1 - q z_i / z_j) \prod_{(i,j) \in R_t} (1 - t z_i / z_j)} \right)$$

# What now?

---

- ▶ revisit  $q, t$ -symmetric theory with macanimals and Schur catalanimals in hand
- ▶ Schur positivity
- ▶ explore new families of symmetric functions arising from catalanimals
- ▶ develop theory around Schur positive series

## Conjecture

For any partition  $\mu$  and integer  $s > 0$ ,

$$H_\mu^s = \text{pol} \sigma \left( (z_1 \cdots z_n)^s \frac{\prod_{(i,j) \in R^*} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)}{\prod_{i < j} (1 - q z_i/z_j) \prod_{(i,j) \in R_t} (1 - t z_i/z_j)} \right)$$

is Schur positive.

thank you!