Folded Alcove Walks and their Applications

Elizabeth Milićević
Haverford College

Joint with Kaisa Taipale,
Petra Schwer & Anne Thomas
April 29, 2023

Women in Algebra and Combinatorics
Northeast Conference Celebrating
AWM: 50 Years & Counting

The Philosophy

To study flag varieties, we very often:

- carve them into orbits of convenient subgroups,

Overview

The Philosophy

To study flag varieties, we very often:

- carve them into orbits of convenient subgroups, and then
- describe the structure of these orbits; e.g. how they meet.

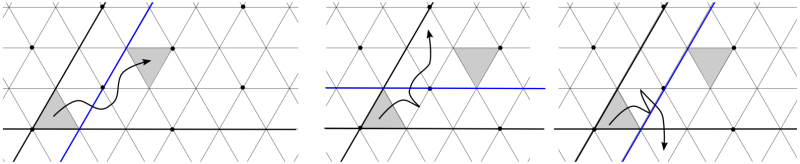
Overview

The Philosophy

To study flag varieties, we very often:

- carve them into orbits of convenient subgroups, and then
- describe the structure of these orbits; e.g. how they meet.

PUNCHLINE! Such orbits are usually ***folded alcove walks***.



Flag Varieties: Finite Flags

Notation:

- G split connected reductive group over $\mathbb{C}$
- Fix a Borel containing a split maximal torus $G \supset B \supset T$
- The opposite Borel subgroup is B^-
- W is the finite Weyl group $N_G(T)/T$

Example ($G = \mathrm{GL}_3$)

B is upper-triangular matrices and T is the diagonal matrices:

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \supset T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$$

$$B^- = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W = S_3$$

Flag Varieties: Finite Flags

Notation:

- G split connected reductive group over $\mathbb{C}$
- Fix a Borel containing a split maximal torus $G \supset B \supset T$
- The opposite Borel subgroup is B^-
- W is the finite Weyl group $N_G(T)/T$

Definition

The *flag variety* is the quotient $G(\mathbb{C})/B$.

Flag Varieties: Finite Flags

Notation:

- G split connected reductive group over $\mathbb{C}$
- Fix a Borel containing a split maximal torus $G \supset B \supset T$
- The opposite Borel subgroup is B^-
- W is the finite Weyl group $N_G(T)/T$

Definition

The *flag variety* is the quotient $G(\mathbb{C})/B$.

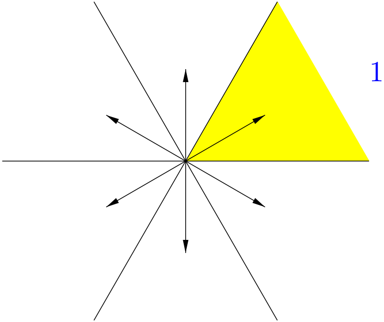
Fact (Bruhat Decomposition)

The flag variety G/B decomposes into various *Schubert cells* via

$$G(\mathbb{C}) = \bigsqcup_{u \in W} BuB = \bigsqcup_{v \in W} B^- vB.$$

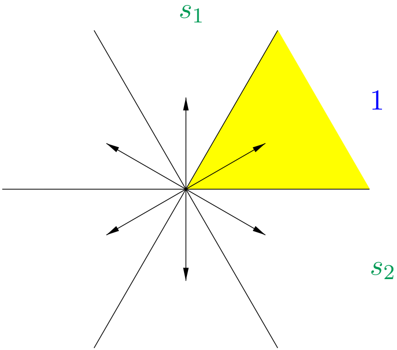
Finite Weyl Group:

For $G = \mathrm{SL}_3$, the group $W \cong S_3$



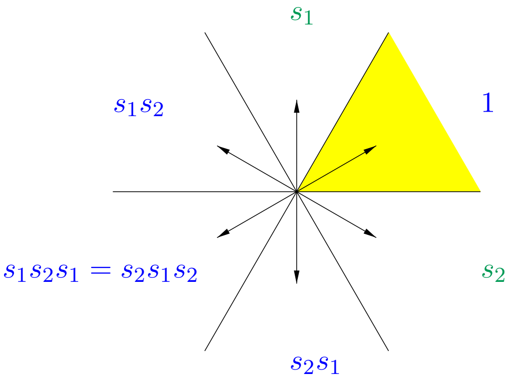
Finite Weyl Group:

For $G = \mathrm{SL}_3$, the group $W \cong S_3$ is generated by two reflections.



Finite Weyl Group:

For $G = \mathrm{SL}_3$, the group $W \cong S_3$ is generated by two reflections.



Affine Setting:

- $F = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]] \supset \mathcal{O}^\times = \mathbb{C}^\times + t\mathbb{C}[[t]]$

Affine Setting:

- $F = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]] \supset \mathcal{O}^\times = \mathbb{C}^\times + t\mathbb{C}[[t]]$
- Iwahori subgroup I of $G(F)$ is the preimage of $B(\mathbb{C})$ under the projection $G(\mathcal{O}) \to G(\mathbb{C})$ by $t \mapsto 0$

Flag Varieties: Affine Flags

Affine Setting:

- $F = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]] \supset \mathcal{O}^\times = \mathbb{C}^\times + t\mathbb{C}[[t]]$
- Iwahori subgroup I of $G(F)$ is the preimage of $B(\mathbb{C})$ under the projection $G(\mathcal{O}) \to G(\mathbb{C})$ by $t \mapsto 0$

Example ($G = \mathrm{SL}_2$)

If B is upper-triangular matrices, the Iwahori subgroup equals

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \middle| \, b \in \mathcal{O};\ a, d \in \mathcal{O}^\times;\ c \in t\mathcal{O} \right\}$$

Affine Setting:

- $F = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]] \supset \mathcal{O}^\times = \mathbb{C}^\times + t\mathbb{C}[[t]]$
- Iwahori subgroup I of $G(F)$ is the preimage of $B(\mathbb{C})$ under the projection $G(\mathcal{O}) \to G(\mathbb{C})$ by $t \mapsto 0$

Example ($G = \mathrm{SL}_2$)

If B is upper-triangular matrices, the Iwahori subgroup equals

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \,\middle|\, b \in \mathcal{O};\ a, d \in \mathcal{O}^\times;\ c \in t\mathcal{O} \right\}$$

Definition

The *affine flag variety* is the quotient $G(F)/I$.

Definition

The *affine flag variety* is the quotient $G(F)/I$.

We can also study the affine flag variety by carving it up using an appropriate analog of the Bruhat decomposition.

Flag Varieties: Affine Flags

Definition

The *affine flag variety* is the quotient $G(F)/I$.

We can also study the affine flag variety by carving it up using an appropriate analog of the Bruhat decomposition.

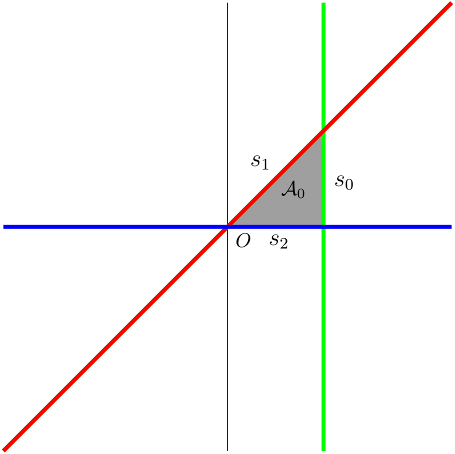
Fact (Affine Bruhat Decomposition)

$G(F)/I$ decomposes into various *affine Schubert cells*

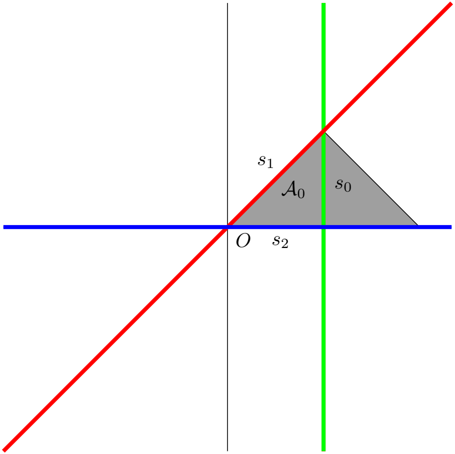
$$G(F) = \bigsqcup_{x \in \widetilde{W}} IxI = \bigsqcup_{y \in \widetilde{W}} I^- y I,$$

where $\widetilde{W}$ is the *affine Weyl group*.

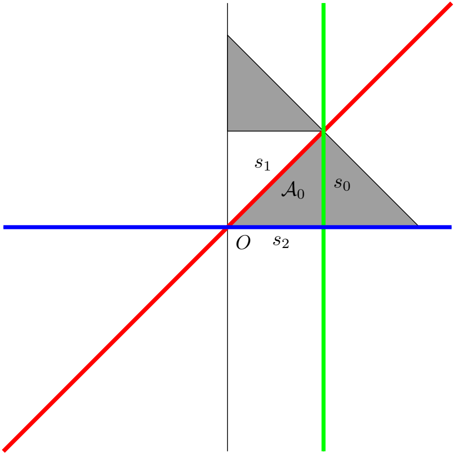
Affine Weyl Group: Let $G = \mathrm{Sp}_4$



The additional generator s_0 is an affine transformation.

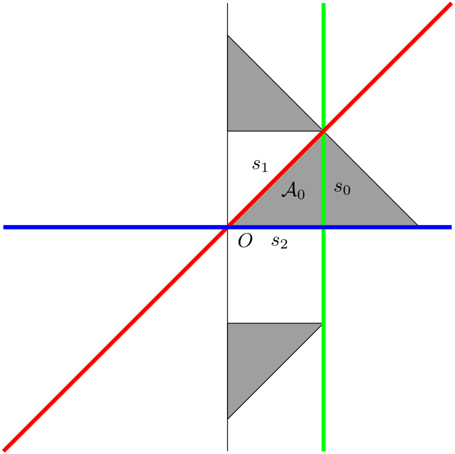


The result of applying s_0 to the base alcove $\mathcal{A}_\circ \longleftrightarrow 1$.



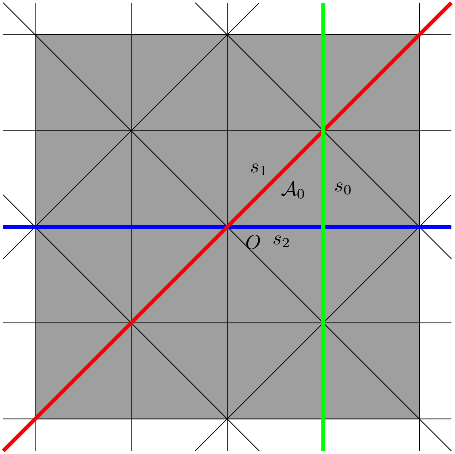
The result of applying s_1 to $s_0(\mathcal{A}_\circ)$ is $s_1 s_0(\mathcal{A}_\circ)$.

Flag Varieties: Affine Flags



The result of applying s_2 to $s_1 s_0(\mathcal{A}_\circ)$ is $s_2 s_1 s_0(\mathcal{A}_\circ)$.

Flag Varieties: Affine Flags



Elements of the affine Weyl group $\widetilde{W}$ correspond to alcoves.

Flag Varieties: Affine Flags

The Philosophy

To study the affine flag variety G/I, we could use:

Flag Varieties: Affine Flags

The Philosophy

To study the affine flag variety G/I, we could use:

- *Richardson stratification* by cells $IxI \cap I^{-}yI$ with $x, y \in \widetilde{W}$.

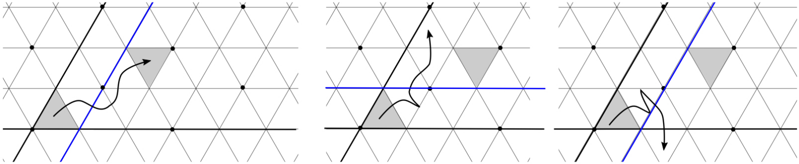
Flag Varieties: Affine Flags

The Philosophy

To study the affine flag variety G/I, we could use:

- *Richardson stratification* by cells $IxI \cap I^{-}yI$ with $x, y \in \widetilde{W}$.

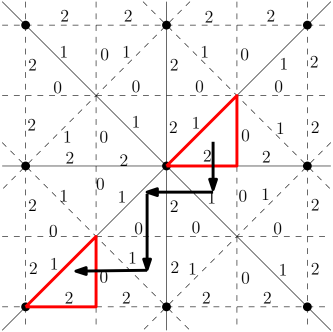
PUNCHLINE! These orbits are labeled ***folded alcove walks***.



Labeled Folded Alcove Walks

Definition

An *alcove walk* is a sequence of moves from an alcove to an adjacent alcove obtained by crossing an affine hyperplane.

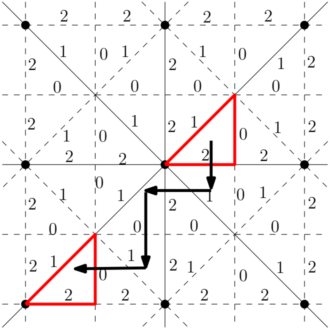


An alcove walk corresponding to the word $s_2 s_1 s_2 s_0 s_1 s_0$.

$$\{\text{alcove walks}\} \longleftrightarrow \left\{\text{words in } \widetilde{W}\right\}$$

Labeled Folded Alcove Walks

Given a minimal walk to x, labeling each crossing by a parameter $c_i \in \mathbb{C}$ gives the full I-orbit of the point xI in G/I.



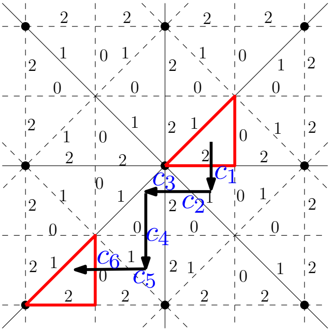
A minimal alcove walk to s_{212010}.

Labeled Folded Alcove Walks

Given a minimal walk to x, labeling each crossing by a parameter $c_i \in \mathbb{C}$ gives the full I-orbit of the point xI in G/I.

Theorem (Steinberg 1967, Parkinson-Ram-Schwer 2009)

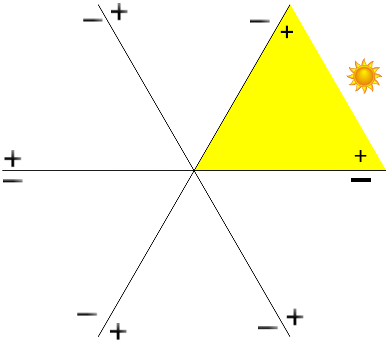
$$\{\textit{labeled alcove walks}\} \longleftrightarrow \{\textit{double cosets } IxI\}$$



All points of Sp_4/I in the affine Schubert cell $Is_{212010}I$.

Labeled Folded Alcove Walks

For each $x \in \widetilde{W}$, the orientation induced by x is defined so that alcove x is on the positive side of every affine hyperplane.



The orientation induced by $\mathcal{A}_\circ$.

Labeled Folded Alcove Walks

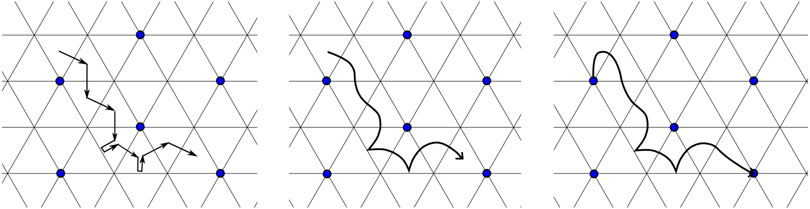
Definition

Orient the hyperplanes so that the identity alcove $\mathcal{A}_\circ$ is on the positive side of every affine hyperplane. A fold is a *positive fold* if it occurs on the positive side of a hyperplane.

Labeled Folded Alcove Walks

Definition

Orient the hyperplanes so that the identity alcove $\mathcal{A}_\circ$ is on the positive side of every affine hyperplane. A fold is a *positive fold* if it occurs on the positive side of a hyperplane.



Several equivalent ways to draw a positively folded alcove walk.

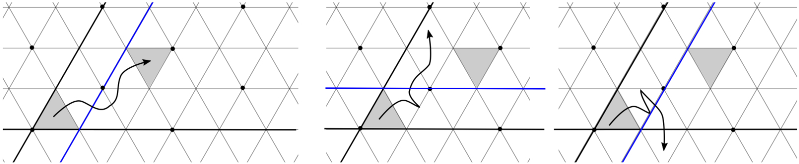
Rules for creating folded alcove walks:

1. Can only do positive folds.
2. Must fold tail-to-tip.

Labeled Folded Alcove Walks

Rules for creating folded alcove walks:

1. Can only do positive folds.
2. Must fold tail-to-tip.



Definition

For $x, y \in \widetilde{W}$, define $\mathcal{A}_\circ(x, y)$ to be the set of all labeled folded alcove walks γ such that:

Labeled Folded Alcove Walks

Definition

For $x, y \in \widetilde{W}$, define $\mathcal{A}_\circ(x, y)$ to be the set of all labeled folded alcove walks γ such that:

- γ is obtained by folding a minimal walk from 1 to x,

Labeled Folded Alcove Walks

Definition

For $x, y \in \widetilde{W}$, define $\mathcal{A}_\circ(x, y)$ to be the set of all labeled folded alcove walks γ such that:

- γ is obtained by folding a minimal walk from 1 to x,
- γ ends in the y alcove, and

Labeled Folded Alcove Walks

Definition

For $x, y \in \widetilde{W}$, define $\mathcal{A}_\circ(x, y)$ to be the set of all labeled folded alcove walks γ such that:

- γ is obtained by folding a minimal walk from 1 to x,
- γ ends in the y alcove, and
- γ is positively folded with respect to the orientation induced by $\mathcal{A}_\circ$.

Labeled Folded Alcove Walks

Definition

For $x, y \in \widetilde{W}$, define $\mathcal{A}_\circ(x, y)$ to be the set of all labeled folded alcove walks γ such that:

- γ is obtained by folding a minimal walk from 1 to x,
- γ ends in the y alcove, and
- γ is positively folded with respect to the orientation induced by $\mathcal{A}_\circ$.

Labeled folded alcove walks parameterize affine Schubert cells.

Theorem (M.–Taipale, arXiv:2303.12170)

$$\mathcal{A}_\circ(x, y) \overset{\sim}{\longleftrightarrow} IxI \cap I^- yI$$

Equivariant Cohomology of Flag Varieties:

Equivariant Cohomology of Flag Varieties:

The inclusion of the $T(F)$-fixed points into G/I induces

$$\begin{array}{rcl} H_T^*(G/I) & \hookrightarrow & H_T^*(G/I)^T \\ [X_v] & \mapsto & \xi^v \end{array}$$

where $\xi^v : \widetilde{W} \to \mathbb{Z}[\alpha_1, \ldots, \alpha_{n-1}, \delta]$ is a polynomial function.

Equivariant Cohomology of Flag Varieties:

The inclusion of the $T(F)$-fixed points into G/I induces

$$\begin{array}{rcl} H_T^*(G/I) & \hookrightarrow & H_T^*(G/I)^T \\ [X_v] & \mapsto & \xi^v \end{array}$$

where $\xi^v : \widetilde{W} \to \mathbb{Z}[\alpha_1, \ldots, \alpha_{n-1}, \delta]$ is a polynomial function.

Theorem (Goresky-Kottwitz-MacPherson; Kostant-Kumar)

The ring structure of $H_T^*(G/I)$ is completely determined by the pointwise product of the functions ξ^v in this image.

Application to GKM Theory

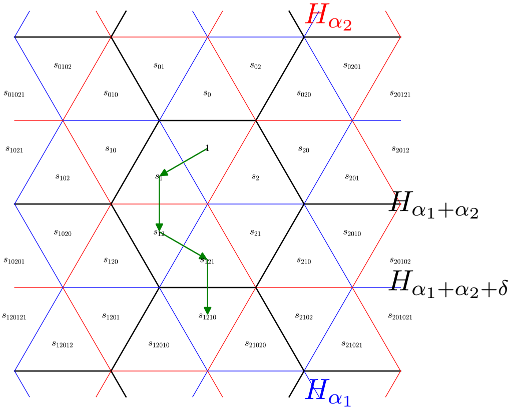
The Philosophy

To compute the function values $\xi^v(w)$ ...

Application to GKM Theory

The Philosophy

To compute the function values $\xi^v(w)$ ... *PUNCHLINE!*
We can use ***folded alcove walks***.



Fix any walk to $w = s_1 s_2 s_1 s_0$, and note the hyperplanes crossed.

Application to GKM Theory

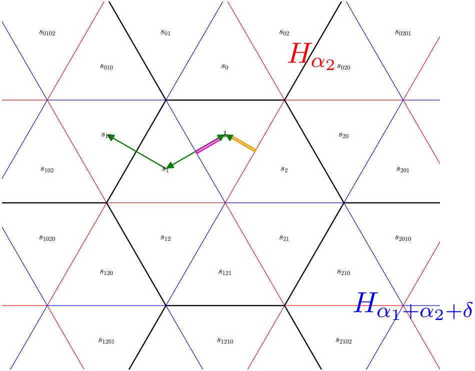
Example

Fix $v = s_1 s_0$. Can form $v \leq w = s_1 s_2 s_1 s_0$ via two subwords:

Example

Fix $v = s_1 s_0$. Can form $v \leq w = s_1 s_2 s_1 s_0$ via two subwords:

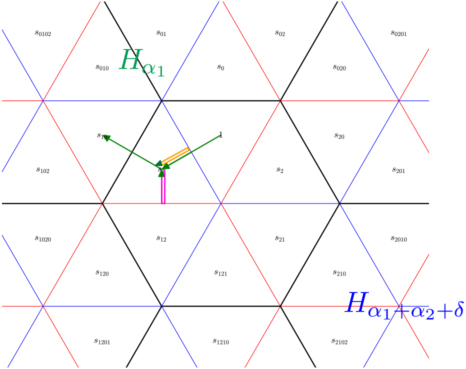
1. $v = \widehat{s_1}\,\widehat{s_2} s_1 s_0 \leftrightarrow$ (fold, fold, step, step) $\leftrightarrow \alpha_2(\alpha_1 + \alpha_2 + \delta)$
2.



Example

Fix $v = s_1 s_0$. Can form $v \leq w = s_1 s_2 s_1 s_0$ via two subwords:

1. $v = \widehat{s_1}\widehat{s_2} s_1 s_0 \leftrightarrow$ (fold, fold, step, step) $\leftrightarrow \alpha_2(\alpha_1 + \alpha_2 + \delta)$
2. $v = s_1 \widehat{s_2}\widehat{s_1} s_0 \leftrightarrow$ (step, fold, fold, step) $\leftrightarrow \alpha_1(\alpha_1 + \alpha_2 + \delta)$



Application to GKM Theory

Theorem (M.–Taipale, arXiv:2303.12170)

Let $w, v \in \widetilde{W}$, and fix any unfolded alcove walk to w. Then

$$\xi^v(w) = \sum_{\substack{folded\ walks \\ ending\ in\ v}} \left(\prod_{steps} (\alpha + k\delta) \right)$$

Application to GKM Theory

Theorem (M.–Taipale, arXiv:2303.12170)

Let $w, v \in \widetilde{W}$, and fix any unfolded alcove walk to w. Then

$$\xi^v(w) = \sum_{\substack{\text{folded walks} \\ \text{ending in } v}} \left(\prod_{\text{steps}} (\alpha + k\delta) \right)$$

Example

Let $w = s_1 s_2 s_1 s_0$. Can form $v = s_1 s_0 \leq w$ via two subwords:

1. $v = \widehat{s_1}\widehat{s_2}s_1 s_0 \leftrightarrow$ (fold, fold, step, step) $\leftrightarrow \alpha_2(\alpha_1 + \alpha_2 + \delta)$
2. $v = s_1\widehat{s_2}\widehat{s_1}s_0 \leftrightarrow$ (step, fold, fold, step) $\leftrightarrow \alpha_1(\alpha_1 + \alpha_2 + \delta)$

Application to GKM Theory

Theorem (M.–Taipale, arXiv:2303.12170)

Let $w, v \in \widetilde{W}$, and fix any unfolded alcove walk to w. Then

$$\xi^v(w) = \sum_{\substack{\text{folded walks} \\ \text{ending in } v}} \left(\prod_{\text{steps}} (\alpha + k\delta) \right)$$

Example

Let $w = s_1 s_2 s_1 s_0$. Can form $v = s_1 s_0 \leq w$ via two subwords:

1. $v = \widehat{s_1}\widehat{s_2} s_1 s_0 \leftrightarrow$ (fold, fold, step, step) $\leftrightarrow \alpha_2(\alpha_1 + \alpha_2 + \delta)$
2. $v = s_1 \widehat{s_2}\widehat{s_1} s_0 \leftrightarrow$ (step, fold, fold, step) $\leftrightarrow \alpha_1(\alpha_1 + \alpha_2 + \delta)$

The theorem says to sum over these two folded alcove walks:

$$\xi^{s_1 s_0}(s_1 s_2 s_1 s_0) = (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \delta).$$

Application to Kazhdan-Lusztig Theory

Kazhdan-Lusztig Polynomials:

1. Carry important representation theoretic information.
2. Are defined recursively in terms of R-polynomials.

Kazhdan-Lusztig Polynomials:

1. Carry important representation theoretic information.
2. Are defined recursively in terms of R-polynomials.

Theorem (Deodhar, M.–Schwer–Thomas)

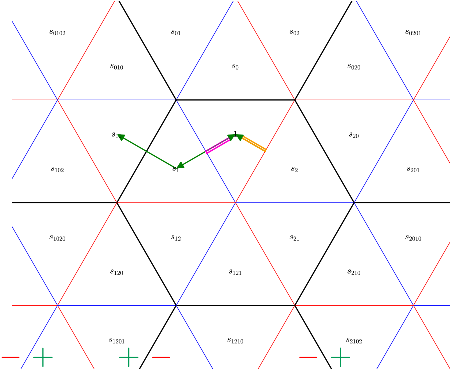
Let $v, w \in \widetilde{W}$, and fix any minimal unfolded walk to w. Then

$$R_{v,w} = \sum_{\substack{\text{positively}\\ \text{folded walks}\\ \text{ending in } v}} (q-1)^{\#folds}\, q^{\#positive\ crossings}$$

Example

Let $w = s_1 s_2 s_1 s_0$. Can form $v = s_1 s_0 \leq w$ via two subwords:

1. $v = \widehat{s_1}\widehat{s_2} s_1 s_0 \leftrightarrow$ (fold, fold, step, step) $\leftrightarrow$ 2 positive folds
2. $v = s_1 \widehat{s_2}\widehat{s_1} s_0 \leftrightarrow$ (step, fold, fold, step)

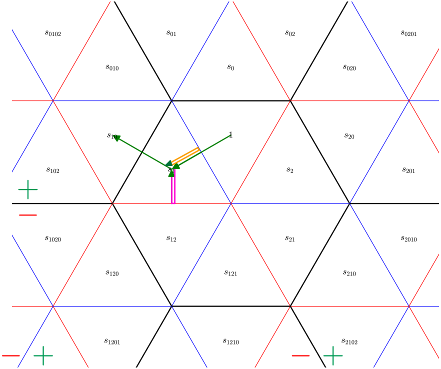


Application to Kazhdan-Lusztig Theory

Example

Let $w = s_1 s_2 s_1 s_0$. Can form $v = s_1 s_0 \leq w$ via two subwords:

1. $v = \widehat{s_1}\widehat{s_2} s_1 s_0 \leftrightarrow$ (fold, fold, step, step) $\leftrightarrow$ 2 positive folds
2. $v = s_1 \widehat{s_2}\widehat{s_1} s_0 \leftrightarrow$ (step, fold, fold, step) $\leftrightarrow$ non-positive fold



Application to Kazhdan-Lusztig Theory

Theorem (Deodhar, M.–Schwer–Thomas)

Let $v, w \in \widetilde{W}$, and fix any minimal unfolded walk to w. Then

$$R_{v,w} = \sum_{\substack{\text{positively} \\ \text{folded walks} \\ \text{ending in } v}} (q-1)^{\#folds}\, q^{\#positive\ crossings}$$

Example

Let $w = s_1 s_2 s_1 s_0$. Can form $v = s_1 s_0 \leq w$ via two subwords:

1. $v = \widehat{s_1}\widehat{s_2}s_1 s_0 \leftrightarrow$ (fold, fold, step, step) $\leftrightarrow (q-1)^2 q^0$
2. $v = s_1 \widehat{s_2}\widehat{s_1} s_0 \leftrightarrow$ (step, fold fold, step) $\leftrightarrow 0$

Application to Kazhdan-Lusztig Theory

Theorem (Deodhar, M.–Schwer–Thomas)

Let $v, w \in \widetilde{W}$, and fix any minimal unfolded walk to w. Then

$$R_{v,w} = \sum_{\substack{\text{positively} \\ \text{folded walks} \\ \text{ending in } v}} (q-1)^{\#folds}\, q^{\#positive\ crossings}$$

Example

Let $w = s_1 s_2 s_1 s_0$. Can form $v = s_1 s_0 \leq w$ via two subwords:

1. $v = \widehat{s_1}\widehat{s_2} s_1 s_0 \leftrightarrow$ (fold, fold, step, step) $\leftrightarrow (q-1)^2 q^0$
2. $v = s_1 \widehat{s_2}\widehat{s_1} s_0 \leftrightarrow$ (step, fold fold, step) $\leftrightarrow 0$

The theorem says to sum the statistics on these alcove walks:

$$R_{s_1 s_0, s_1 s_2 s_1 s_0} = (q-1)^2 q^0 + 0 = (q-1)^2$$

Thank you!