Mode transition algebras and higher-level Zhu algebras

Chiara Damiolini

April 30, 2023

Women in Algebra and Combinatorics
Northeast Conference Celebrating the Association for Women in Mathematics:
50 Years and Counting

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jt. with Angela Gibney and Danny Krashen

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The algebras A_d are not easy to compute!

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Today's goal: understand the relation between A_d and $\mathfrak{A}_d = \mathfrak{A}_{d,-d}$

Understanding the Mode Transition Algebras $\mathfrak{A}_d$

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$$\Rightarrow \quad \mathfrak{A}_{d,e} \star \mathfrak{A}_{e,f} \subseteq \mathfrak{A}_{d,f} \quad \text{and} \quad \mathfrak{A}_{d,e} \star \mathfrak{A}_{d',e'} = 0 \text{ whenever } e \neq d'$$

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$(\mathfrak{A}_d, +, \star)$ is an associative algebra and $(\mathfrak{A}_0, +, \star) = (\mathsf{A}_0, +, \cdot)$

Example: Heisenberg Lie algebra $\mathfrak{h} = H\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}k$

$[Ht^n + \alpha k, Ht^m + \beta k] = \delta_{n+m=0}k$ $\qquad$ $\deg(Ht^n) = -n$ $\qquad$ $\mathscr{U} := U(\mathfrak{h})/(k=1)$

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$\mathsf{A}_1 \cong \mathsf{A}_0 \oplus \mathbb{C}[x]$ [Barron–Vander Werf–Yang]

$\mathsf{A}_2 \cong \mathsf{A}_1 \oplus \mathrm{Mat}_2(\mathbb{C}[x])$ [Addabbo–Barron]

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Conjecture [Addabbo–Barron]: $\mathsf{A}_d \cong \mathsf{A}_{d-1} \oplus \mathrm{Mat}_{p(d)}(\mathbb{C}[x])$.

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$d = 7 \qquad \mathfrak{l} = [2|5] \qquad \mathfrak{r} = [1|3|3]$

$\epsilon_{\mathfrak{l},\mathfrak{r}} = Ht^{-5} \cdot Ht^{-2} \otimes 1 \otimes Ht^1 \cdot Ht^3 \cdot Ht^3$

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$(\mathfrak{A}_d, +, \star)$ is isomorphic to the algebra of matrices $\mathrm{Mat}_{p(d)}(\mathbb{C}[x])$

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