Dimension formula of generalized splines of degree 2

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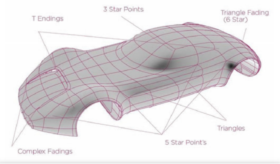
based on joint work with Shaheen Nazir (Lahore University)
and Julianna Tymoczko (Smith College)



Women in Algebra and Combinatorics
SUNY Albany
April 30, 2023

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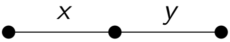
The collection of all splines on (G, α) forms a ring and an R-module with vertex-wise addition, multiplication, and scaling.

What are splines? (example)

Consider the labeled graph



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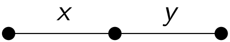

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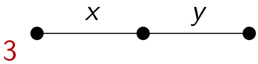

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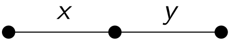

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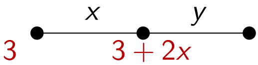

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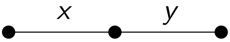

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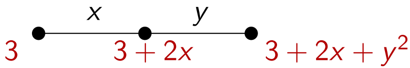

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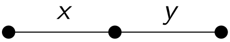

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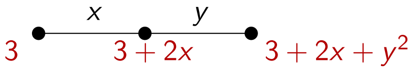

Basis for splines:

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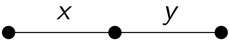

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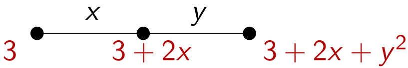

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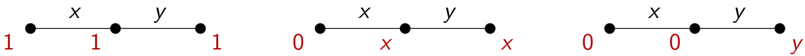

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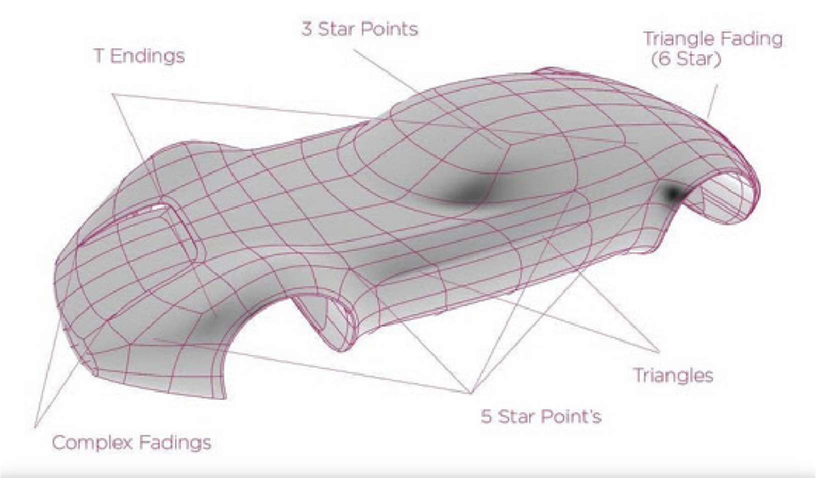

Splines for Meshes with Irregularities, J. Peters, *SMAI journal of computational mathematics*, 2019.

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- Our definition coincides with a combinatorial construction of equivariant cohomology called GKM theory. GKM theory gives conditions on a variety X with the action of a torus T so that
  - the T-fixed points and one-dimensional T-orbits form a graph G_X
  - when the edges of G_X are labeled with the T-weights then

$$H_T^*(X) \cong \text{ splines on } (G_X, \alpha_X)$$

Our problem: the upper-bound conjecture in (classical) splines

The upper-bound conjecture asks for a dimension formula for splines of degree at most d and smoothness r on triangulated regions in the plane.

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- and **completely unknown** when $d = 2$.

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- Smoothness 1 means that edges of the dual graph are labeled with $(ax + by + c)^2$
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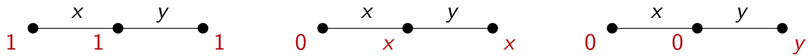
We want a dimension formula for splines of degree 2

Splines on trees and cycles

Splines on trees have a straightforward basis:



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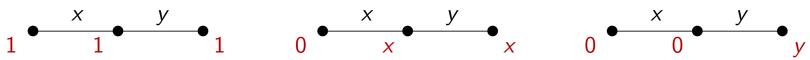

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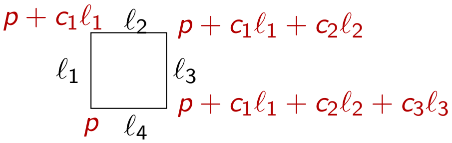

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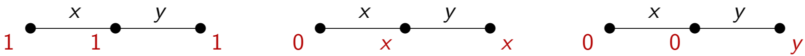

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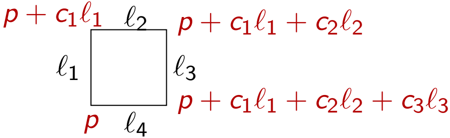

We get an equation $c_1\ell_1 + c_2\ell_2 + c_3\ell_3 + c_4\ell_4 = 0$ for each cycle

Cycle bases

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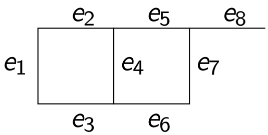

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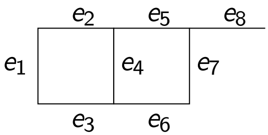
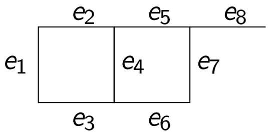


An important question in topological graph theory is which cycles are independent in a reasonable sense, namely form a cycle basis.

Face cycle basis matrix

Order the edges $E = \{e_1, e_2, \ldots, e_{e_G}\}$ and the (bounded) faces $\mathcal{F} = \{F_1, F_2, \ldots, F_{f_G}\}$. The face cycle basis matrix M is the $f_G \times e_G$ matrix that is zero except 1 in row r and column c if edge e_c is an edge on face cycle of F_r.



$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Extended face cycle basis matrix

In our case, edge-labels have the form $\ell_i = (a_i x + y)^2$.

The equation

$$c_1\ell_1 + c_2\ell_2 + c_3\ell_3 + c_4\ell_4 = 0$$

becomes

$$c_1(a_1x+y)^2 + c_2(a_2x+y)^2 + c_3(a_3x+y)^2 + c_4(a_4x+y)^2 = 0$$

Rearranging gives

$$(c_1a_1^2 + c_2a_2^2 + c_3a_3^2 + c_4a_4^2)x^2 + (c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4)2xy \\ +(c_1+c_2+c_3+c_4)y^2 = 0$$

This vanishes as a **polynomial** so each red coefficient is 0.

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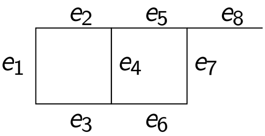

$$M^{\text{ext}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & a_4 & a_5 & a_6 & a_7 & 0 \\ 0 & 0 & 0 & a_4^2 & a_5^2 & a_6^2 & a_7^2 & 0 \end{pmatrix}$$

The dimension of degree 2 splines

Theorem (Nazir, S., Tymoczko 2023)

Let $\mathrm{Spl}_2(G,\ell)$ *be collection of degree two splines associated to the labelled, finite, planar graph* (G,ℓ)*. Then*

$$\dim \mathrm{Spl}_2(G,\ell) = e_G - \mathrm{rank}\ M^{\mathrm{ext}}$$

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When M^{ext} is full rank, the dimension of degree two splines is

$$e_G - 3f_G.$$

When does this happen?

Dimension of degree 2 splines

Note the 3×3 Vandermonde matrices with determinant

$$(a_i - a_j)(a_j - a_k)(a_i - a_k)$$

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$$(a_i - a_j)(a_j - a_k)(a_i - a_k)$$

- Square submatrices of M^{ext} have determinants related to these Vandermonde determinants.
- Generically when $a_i \neq a_j \neq a_k$ these determinants are nonzero.
- Three is a special number for these splines.

Dimension of degree 2 splines

An edge-injective function φ assigns to each face F in a planar graph G up to three (unordered) edges on the boundary of F so that no edge is assigned to more than one face. The **size** of an edge-injective function is the total number of edges in its image.

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Theorem (Nazir, S., Tymoczko 2023)

N is any square $3k \times 3k$ submatrix of M^{ext}. Then

$$\det N = \sum_{\substack{\varphi\colon \mathcal{F} \to \mathcal{E}_3 \\ \text{edge-injective}}} \prod_{F \in \mathcal{F}} \det N_{F,\varphi(F)},$$

$N_{F,\varphi(F)}$ = submatrix of N with the 3 rows corresponding to F and columns indexed by $\varphi(F)$.

Algorithm: Dimension of degree 2 splines

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- **Case** $e_G \leqslant 3f_G$**:** $\dim \mathrm{Spl}_2(G, \ell) = 0$.
- **Case** G **no leaves and no subgraph** G' **with** $e_{G'} \leqslant 3f_{G'}$**:** $\dim \mathrm{Spl}_2(G, \ell) = e_G - 3f_G$.

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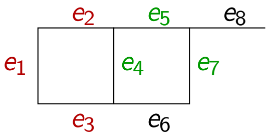
If G has contractible subgraph G' with $e_{G'} \leqslant 3f_{G'}$, contract G'.

Examples: The dimension of degree 2 splines



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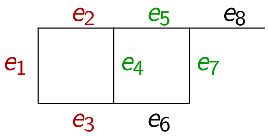

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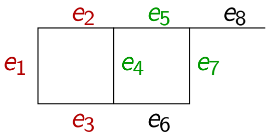

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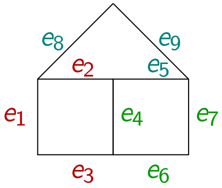

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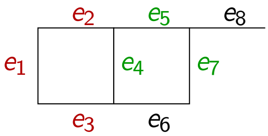

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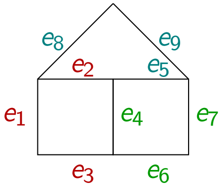

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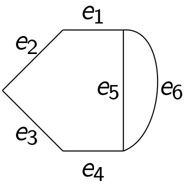
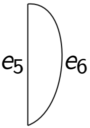


Subgraph



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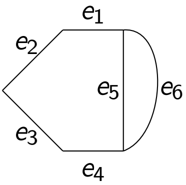

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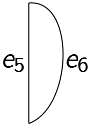

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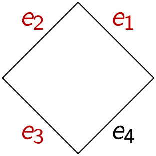
Contracted graph



has dimension 1.

Existence of edge-injective functions

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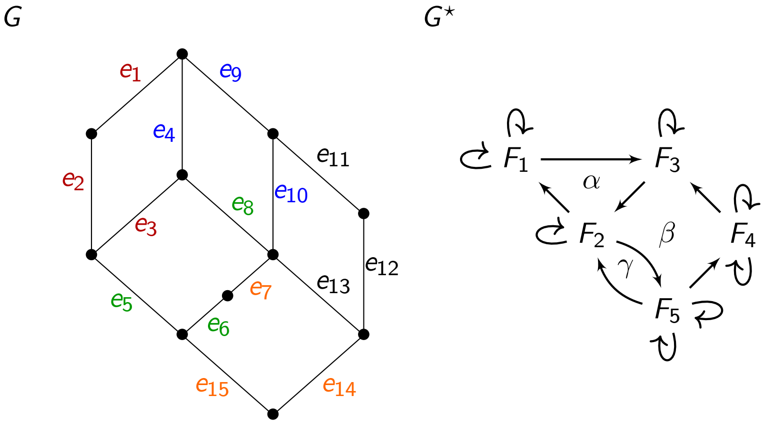
Theorem (Nazir, S., Tymoczko 2023)

If G is a finite, planar graph and contains no proper contractible subset of faces, there is an edge-injective function.

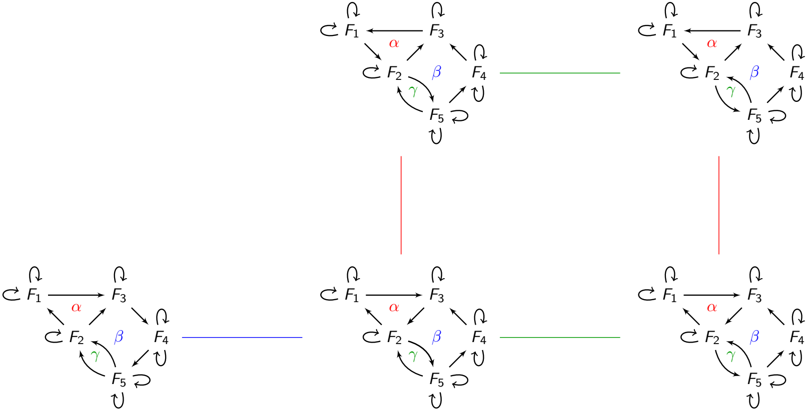
Conjecture (Nazir, S., Tymoczko 2023)

If G is a finite, planar graph, contains no proper contractible subset of faces, and no two faces share more than three edges, there exists a generic edge labeling.

Edge-injective function on dual graph



Construction of all edge-injective functions



Takeaway

- Computing the dimension of splines has a combinatorial aspect (existence of certain kinds of Euler paths/edge-injective functions/coloring of edges)
- ....and an algebraic aspect (whether certain determinants vanish or not).
- The combinatorial aspect governs the generic case, where we obtain a formula for degree 2 splines.
- The algebraic aspect determines the non-generic case.

Thank you!