Spectral Moore Bounds

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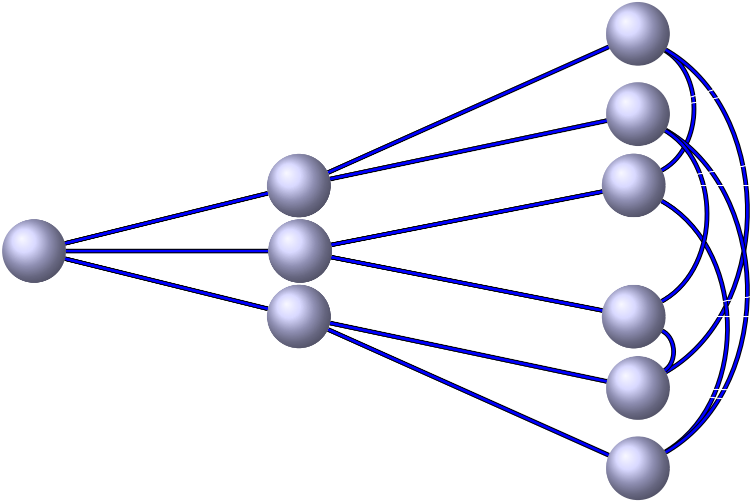
Overview

Moore Bound



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Let G be a k-regular graph with diameter d. Then the number of vertices in G is at most

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For $i > 0$, there is a single vertex adjacent to w and at distance $i - 1$ from v.
If $i < d$, there are no vertices adjacent to w and at distance i from v. If $i = d$, there are $k - 1$ vertices adjacent to w and at distance i from v.

Definition

A graph is **distance-regular** *if, for any choice of vertices v, w, the number of vertices adjacent to w and at distance i from v depends only on $d(v, w)$.*

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Define the ith distance matrix A_i to be the matrix indexed by the vertices where

$$(A_i)_{vw} = \begin{cases} 1 & d(v, w) = i \\ 0 & \text{otherwise.} \end{cases}$$

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A graph G of diameter d is distance-regular if and only if for all $0 \leq i \leq d$, the ith distance matrix is polynomial in A_1.

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For $d = 2$, Hoffman and Singleton (1960) showed that the graphs meeting the Moore bound are:

- Petersen Graph
- Hoffman-Singleton Graph
- 57-regular graph on 3250 vertices?

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Damerell (1973) and Bannai and Ito (1973) showed that there are no Moore graphs of diameter at least three.

Spectral Moore Bound

Theorem (Alon-Boppana (1986))

Let $k \geq 3$, and let $(G_m)_{m\geq 0}$ be a sequence of finite k-regular graphs with $V(G_m) \to \infty$ as $m \to \infty$. Then

$$\liminf_{m\to\infty} \lambda_2 (G_m) \geq 2\sqrt{k-1}.$$

Theorem (Cioabă, Koolen, Nozaki, and Vermette (2017))

For $k \geq 3$ and $\lambda < 2\sqrt{k-1}$, there exist t, c such that the number of vertices of a k-regular graph with second-largest eigenvalue at most λ is at most

$$1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}.$$

If the bound is tight, the graph is distance-regular.

Semiregular Moore Bound

Let G be a (k,ℓ)-semiregular bipartite graph with diameter $2d+1$. Then the number of vertices of valency k in G is at most

$$1+\sum_{i=1}^{d} k(k-1)^{i-1}(\ell-1)^{i}.$$

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$$\sum_{i=0}^{d-1} \ell(k-1)^{i}(\ell - 1)^{i}.$$

Semiregular Spectral Moore Bound

Theorem (Lato, 2021+)

Let G be a (k,ℓ)-semiregular graph with $k,\ell \geq 2$ and $k \neq \ell$. If the second-largest eigenvalue $\lambda < \sqrt{k-1}+\sqrt{\ell-1}$, then there exist t, c such that the number of vertices of valency k is bounded above by either

$$1 + k\sum_{i=1}^{t-2} (\ell-1)^i (k-1)^{i-1} + \frac{k(\ell-1)^{t-1}(k-1)^{t-2}}{c}$$

or

$$\ell\sum_{i=0}^{t-2} (\ell-1)^i (k-1)^i + \frac{\ell(\ell-1)^{t-1}(k-1)^{t-1}}{c}.$$

Distance-Biregular Graphs

Definition

A bipartite graph is **distance-biregular** *if, for any choice of vertices v, w, the number of vertices adjacent to w and at distance i from v depends only on $d(v, w)$ and the set of the partition that v lies in.*

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A bipartite graph is **distance-biregular** *if, for any choice of vertices v, w, the number of vertices adjacent to w and at distance i from v depends only on $d(v, w)$ and the set of the partition that v lies in.*

Theorem (Godsil and Shawe-Taylor (1985))

Let G be a graph and suppose that, for any vertex v, the number of vertices adjacent to w and at distance i from v depends only on $d(v, w)$ and the choice of v. Then G is either distance-regular or distance-biregular.

Theorem (Lato, 2021+)

Let G be a (k,ℓ)-semiregular graph with $k,\ell \geq 2$ and $k \neq \ell$. If the second-largest eigenvalue $\lambda < \sqrt{k-1} + \sqrt{\ell-1}$, then there exist t, c such that the number of vertices of valency k is bounded above by either

$$1 + k\sum_{i=1}^{t-2} (\ell-1)^i (k-1)^{i-1} + \frac{k(\ell-1)^{t-1}(k-1)^{t-2}}{c}$$

or

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If the bound is tight, the graph is distance-biregular.

Thank You!