Quantum walks on Cayley graphs

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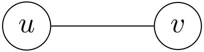
NEWAC50AWM

November, 2021

Outline

Example - K_2



K_2

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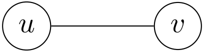

K_2

It has adjacency matrix

$$A(K_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Quantum Walks

Definition

Let A be the adjacency matrix of a graph X. The *continuous-time quantum walk* on X is given by the matrix

$$U(t) := e^{itA} = \sum_{n\geq 0} \frac{(it)^n}{n!} A^n, \quad t \in \mathbb{R}.$$

The matrix $U(t)$ is called the *transition matrix* of the walk.

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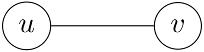
For more, see S. Bose [2] and M. Christandl et al, [3].

Example - K_2 again



K_2

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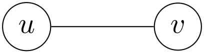

K_2

We have

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{so} \quad A^{2n} = I, \text{ and } A^{2n+1} = A,$$

for all n.

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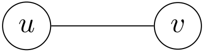

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for all n. Therefore

$$\begin{aligned} e^{itA} &= I + itA - \frac{t^2}{2} I - \frac{it^3}{6} A + \frac{t^4}{24} I + \cdots \\ &= \cos(t) I + i \sin(t) A. \end{aligned}$$

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Perfect State Transfer

Definition

For distinct vertices, u and v of X, we say that we have *perfect state transfer (PST)* from u to v at time t if

$$U(t)\mathbf{e}_u = \gamma \mathbf{e}_v,$$

for some scalar γ with $|\gamma| = 1$.

Periodicity

Definition

We say that a vertex u is *periodic* at time t if

$$U(t)\mathbf{e}_u = \gamma \mathbf{e}_u,$$

for some scalar γ with $|\gamma| = 1$. We say that a graph X is periodic at time t if $U(t)$ is diagonal.

Example - K_2 yet again

Recall that for K_2,

$$U(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}.$$

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Example - K_2 yet again

Recall that for K_2,

$$U(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}.$$

We see that

$$U(\pi/2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad U(\pi) = -I.$$

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So K_2 has PST at time $\pi/2$ and is periodic at time π.

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If there is PST from u to v at time t, then there is PST from v to u at time t. In this case, u and v are periodic at time $2t$.

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Theorem (A. Kay, [5])

If there is perfect state transfer from u to v in X and from u to w, then $v = w$.

Outline

Cayley graphs

Definition

Let G be a group and $\mathcal{C} \subseteq G \backslash e$ a subset with $\mathcal{C}^{-1} = \mathcal{C}$. The *Cayley graph*, $X := X(G, \mathcal{C})$, has vertex set $V(X) := G$ and

$$g \sim h \quad \text{if} \quad hg^{-1} \in \mathcal{C}.$$

We call $\mathcal{C}$ the *connection set* of the Cayley graph.

The adjacency matrix

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- Each element, g of G is therefore a permutation on the vertices of X, so we can think of it as a permutation matrix P_g. In fact, since this action is regular, the map $g \to P_g$ is an isomorphism from G to a group of permutation matrices.

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- Each element, g of G is therefore a permutation on the vertices of X, so we can think of it as a permutation matrix P_g. In fact, since this action is regular, the map $g \to P_g$ is an isomorphism from G to a group of permutation matrices.
- It can be shown that if $\mathcal{C} = \{g_1, \ldots, g_k\}$, then

$$A(X) = P_{g_1} + \cdots + P_{g_k}.$$

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Cubelike graphs

Definition

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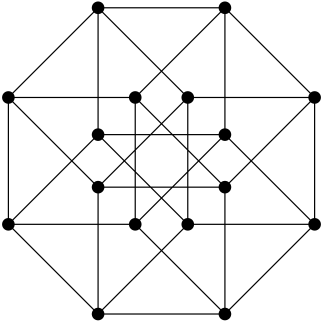

Figure: *The hypercubes are cubelike*

Cubelike graphs

Theorem (A. Bernasconi et al, [1])

Let $X = X(\mathbb{Z}_2^n, \mathcal{C})$ be a cubelike graph, and define

$$c := \sum_{x \in \mathcal{C}} x.$$

If $c \neq 0$, then X has PST at time $\pi/2$.

Proof.

We can write $A = A(X)$ as a sum of permutation matrices

$$A = P_1 + \cdots + P_k, \qquad P_r^2 = I, \quad P_r P_s = P_s P_r.$$

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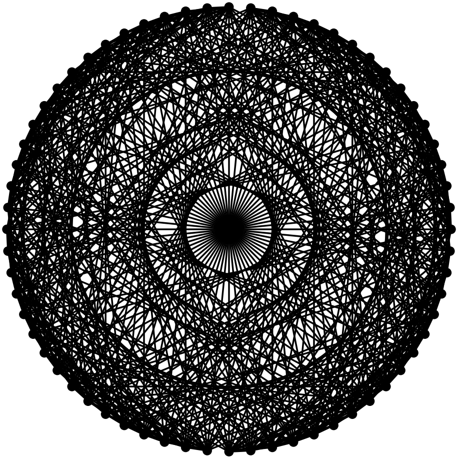
Also, $e^{itP_r} = \cos(t)I + i\sin(t)P_r$. Therefore

$$U(t) = \prod_{r=1}^{k} \left(\cos(t)I + i\sin(t)P_r\right),$$

thus $U(\pi/2) = i^k \prod_r P_r$. □

Thank you for listening



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[1] Anna Bernasconi, Chris Godsil, and Simone Severini. Quantum networks on cubelike graphs. *Phys. Rev. A (3)*, 78(5):052320, 5, 2008.

[2] Sougato Bose. Quantum communication through an unmodulated spin chain. *Physical Review Letters*, 91(20), Nov 2003.

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[5] Alastair Kay. Basics of perfect communication through quantum networks. *Physical Review A*, 84(2), Aug 2011.