Braid group actions, crystals, and cacti

(joint work with T. Licata, I. Losev, O. Yacobi)

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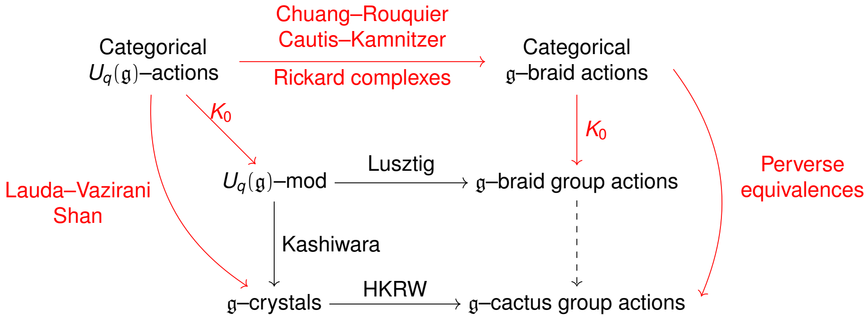
Overview

Let $\mathfrak{g}$ be a simply-laced Kac–Moody Lie algebra.



Categorical $\mathfrak{g}$–action I

Warm-up case: $\mathfrak{g} = \mathfrak{sl}_2$

Suppose $\mathfrak{sl}_2 = \mathbb{C}\{e, f, h\} \curvearrowright V = \bigoplus_{n\in\mathbb{Z}} V_n$, an integrable $\mathfrak{sl}_2$–rep.

$$e : V_n \to V_{n+2}, \quad f : V_n \to V_{n-2}, \quad (ef - fe)|_{V_n} = n\,\mathrm{Id}_{V_n} \quad \forall n \in \mathbb{Z}.$$

Categorified $\mathfrak{sl}_2$–action (Chuang–Rouquier, Khovanov–Lauda):

- an abelian category $\mathcal{C} = \bigoplus_n \mathcal{C}_n$ (with $K_0(\mathcal{C}_n) = V_n$)
- exact endofunctors $E,\ F$ of $\mathcal{C}$, $E : \mathcal{C}_n \to \mathcal{C}_{n+2}$, $F : \mathcal{C}_n \to \mathcal{C}_{n-2}$
- natural transformations

$$\epsilon : EF \to I,\ \eta : I \to FE \qquad \text{(unit and counit of adjunction)}$$
$$X : E \to E,\ T : E^2 \to E^2$$

such that...

Categorical $\mathfrak{g}$–action II

- For $n \geq 0$ (analogously for $n < 0$), we have an isomorphism:

$$(\sigma, \epsilon, \epsilon \circ XI_F, \ldots, \epsilon \circ X^{n-1} I_F) : EF|_{\mathcal{C}_n} \xrightarrow{\cong} FE|_{\mathcal{C}_n} \oplus I_{\mathcal{C}_n}^{\oplus n},$$

  where σ is composed of η, T, and ϵ.

- The natural transformations X, T give an action of the nil affine Hecke algebra H_n on E^n.

Example The adjoint representation of $\mathfrak{sl}_2$:

$$(\mathcal{C}_2 = \mathbb{C} - mod) \underset{Res}{\overset{Ind}{\rightleftarrows}} (\mathcal{C}_0 = \mathbb{C}[x]/x^2 - mod) \underset{Ind}{\overset{Res}{\rightleftarrows}} (\mathcal{C}_{-2} = \mathbb{C} - mod)$$

Categorical $\mathfrak{g}$–action III

More generally:

The 2**–category** $\mathcal{U}\mathfrak{g}$ categorifies (Lusztig's idempotent form) $\dot{U}\mathfrak{g}$:

- objects are elements λ of the $\mathfrak{g}$–weight lattice.
- 1-morphisms are generated by $E_i : \lambda \to \lambda + \alpha_i$, $F_i : \lambda \to \lambda - \alpha_i$.
- 2-morphisms are generated by

$X_i =$  $_i : E_i \to E_i$, $X_i =$  $_i : F_i \to F_i$,

$T_{ij} =$ $_i$  $_j : E_iE_j \to E_jE_i$, $T_{ij} =$ $_i$  $_j : F_iF_j \to F_jF_i$

 $^i : E_iF_i \to I$,  $^i : F_iE_i \to I$,  $^i : I \to F_iE_i$,  $^i : I \to E_iF_i$

$+KLR$ algebra and further relations.

A **categorical $\mathfrak{g}$–representation** is a 2–functor $\mathcal{U}\mathfrak{g} \to \mathcal{K}$ to an appropriate 2-category.
Note: A graded version, $\mathcal{U}_q\mathfrak{g}$, categorifies $\dot{U}_q\mathfrak{g}$.

The Rickard complex I

$\underline{\mathfrak{g} = \mathfrak{sl}_2}$: Let $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = exp(-f)exp(e)exp(-f) \in SL_2$.

Then s restricts to an isomorphism on weight spaces, of the form

$$s|_{V_{-n}} = \sum_k (-1)^k e^{(n+k)} f^{(k)}$$

Rickard complex: Consider the complex of functors $\Theta = \bigoplus_n \Theta_n$,

$$\Theta_n : Comp(\mathcal{C}_{-n}) \to Comp(\mathcal{C}_n)$$

$$\Theta_n = (\ldots \to E^{(n+2)}F^{(2)} \to E^{(n+1)}F^{(1)} \to \underline{E^{(n)}})$$

- $E^{(n)} \subseteq E^n, F^{(n)} \subseteq F^n$ defined using the H_n–action
- $E^{(n+k)}F^{(k)} \to E^{(n+k-1)}F^{(k-1)}$ comes from adjunction

The Rickard complex II

Theorem (Chuang-Rouquier '08)

Θ induces a self-equivalence on $D^b(\mathcal{C})$ and, by restriction, an equivalence $D^b(\mathcal{C}_{-n}) \xrightarrow{\cong} D^b(\mathcal{C}_n)$. Furthermore, $[\Theta] = s$.

Example: Let $R = \mathbb{C}[x]/x^2$. For the adjoint $\mathfrak{sl}_2$–representation and $N \in \mathcal{C}_0 = R - mod$,

$$\Theta_0 : D^b(R - mod) \to D^b(R - mod)$$
$$N \mapsto (R \otimes N \xrightarrow{\text{act}} \underline{N})$$

Perverse equivalences

Suppose that $\mathcal{A}, \mathcal{A}'$ are abelian categories, $F : D^b(\mathcal{A}) \xrightarrow{\cong} D^b(\mathcal{A}')$, we have filtrations $\mathcal{A}_\bullet, \mathcal{A}'_\bullet$ by Serre subcategories

$$0 = \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_r = \mathcal{A}$$
$$0 = \mathcal{A}'_{-1} \subset \mathcal{A}'_0 \subset \mathcal{A}'_1 \subset \ldots \subset \mathcal{A}'_r = \mathcal{A}'$$

and a perversity function $p : \{0, \ldots, r\} \to \mathbb{Z}$.

Definition

F is **perverse** with respect to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$ if:

1. $F[-p(i)]$ restricts to an equivalence $D^b_{\mathcal{A}_i}(\mathcal{A}) \xrightarrow{\cong} D^b_{\mathcal{A}'_i}(\mathcal{A}')$.
2. The induced $D^b_{\mathcal{A}_i}(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \xrightarrow{\cong} D^b_{\mathcal{A}'_i}(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$ equivalence induces an equivalence

$$\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\cong} \mathcal{A}'_i/\mathcal{A}'_{i-1}.$$

Perversity of the Rickard complexes

Consider $\mathcal{C}$ endowed with an $\mathfrak{sl}_2$-categorical action. Let S be the set of simple objects, and consider the filtrations:

$$S_i = \{V \in S \ : \ F^{i+1}V = 0\} \quad \text{and} \quad S'_i = \{V \in S \ : \ E^{i+1}V = 0\}.$$

Proposition (Chuang–Rouquier)

The equivalence $\Theta : D^b(\mathcal{C}) \xrightarrow{\cong} D^b(\mathcal{C})$ is perverse with respect to $(S_\bullet, S'_\bullet, p = \mathrm{Id})$.

For general $\mathfrak{g}$, take a reduced word $w = s_{i_1} \dots s_{i_k}$ and weight μ, consider the composition $\Theta^\mu_w = \Theta^{s_{i_2}\dots s_{i_k}(\mu)}_{s_{i_1}} \circ \dots \circ \Theta^\mu_{s_{i_k}}$.

Theorem 1 (H–Licata–Losev–Yacobi)

Let $w_0 \in W$ be the longest element, and μ a weight of $\mathcal{C}$. Then $\Theta^\mu_{w_0} : D^b(\mathcal{C}_\mu) \to D^b(\mathcal{C}_{w_0(\mu)})$ is a perverse equivalence.

Construction details

Set $\mathcal{A} = \mathcal{C}_\mu$ and $\mathcal{A}' = \mathcal{C}_{w_0(\mu)}$. Consider $\Theta^\mu_{w_0} : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$.
$V = [\mathcal{C}]_\mathbb{C} = \bigoplus_{\lambda \in X_+} \text{Iso}_\lambda(V)$ is an integrable $\mathfrak{g}$-rep.

Theorem (Jordan–Hölder filtration)

There is a filtration of $\mathcal{C}$ by Serre subcategories

$$0 = \mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \ldots \subseteq \mathcal{C}_r = \mathcal{C}, \quad \textit{such that}$$

1. *$\mathcal{C}_i$ is a subrepresentation of $\mathcal{C}$,*
2. *$\mathcal{C}_i/\mathcal{C}_{i-1} \cong \mathcal{L}(\lambda_i) \otimes \mathcal{A}_i$ is a simple rep., with $\mathcal{L}(\lambda_i)$ a minimal categorification and $\mathcal{A}_i$ an abelian category.*

There are induced filtrations on $\mathcal{A}$ and $\mathcal{A}'$.

Let $\lambda_0, ..., \lambda_r$ be the dominant weights that appear. The perversity function $p : \{0, ..., r\} \to \mathbb{Z}$ can be defined as $p(i) = ht(\mu - w_0(\lambda_i))$.

Braid and cactus groups I

The **braid group** $Br_{\mathfrak{g}}$ is generated by $\sigma_i, i \in I$, with relations:

$$\begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } i \text{ and } j \text{ are connected,} \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } i \text{ and } j \text{ are not connected.} \end{array}$$

The **cactus group** $C_{\mathfrak{g}}$ is generated by c_J, for connected $J \subseteq I$, with:

$$\begin{array}{ll} c_J^2 = 1 & \forall\, J \subseteq I \\ c_J c_K = c_K c_J & \forall\, J \cup K \subseteq I \text{ not connected} \\ c_J c_K = c_{\tau_J(K)} c_J & \forall\, K \subseteq J \subseteq I \end{array}$$

Where $\forall\, j \in J, \alpha_j$ simple root, $\alpha_{\tau_J(j)} = -w_0^J \alpha_j$.

Proposition (Cautis–Kamnitzer ’10)

The Rickard complexes satisfy the braid relations.

Braid and cactus groups II

We have $PBr_n \to Br_n \to S_n$ and $PC_n \to C_n \to S_n$. Topologically,

$$PBr_n \cong \pi_1 \left(\begin{array}{c} \text{Configurations of } n \text{ distinct} \\ \text{points in the plane} \end{array} \right) \cong \pi_1(\mathbb{C}^n \setminus \bigcup_{i \neq j} \{z_i = z_j\})$$

$$PC_n \cong \pi_1(\overline{M_{0,n+1}}(\mathbb{R}))$$

The real locus of the Deligne–Mumford moduli space of curves with $n+1$ marked points

Example. Consider $\overline{M_{0,4}}(\mathbb{R})$.

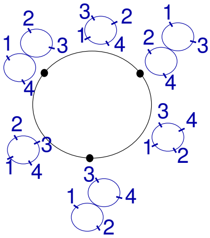


Figure: Opuntia cactus.

$$\pi_1(\overline{M_{0,4}}(\mathbb{R})) \cong \mathbb{Z} \cong PC_3 = \langle c_{12} c_1 c_2 c_1 \rangle$$

A cactus group action I

Combinatorial realization: $C_{\mathfrak{g}} \curvearrowright B = \sqcup_\lambda B_\lambda$ on any $\mathfrak{g}$-crystal B via

generalized Schützenberger involutions.

Let $\xi_\lambda : B_\lambda \to B_\lambda$ be the unique map:

1. $e_i \cdot \xi_\lambda(b) = \xi_\lambda(f_{\tau(i)} \cdot b)$
2. $f_i \cdot \xi_\lambda(b) = \xi_\lambda(e_{\tau(i)} \cdot b)$
3. $\mathrm{wt}(\xi_\lambda(b)) = w_0 \cdot \mathrm{wt}(b)$

where e_i, f_i are the Kashiwara operators on B_λ.

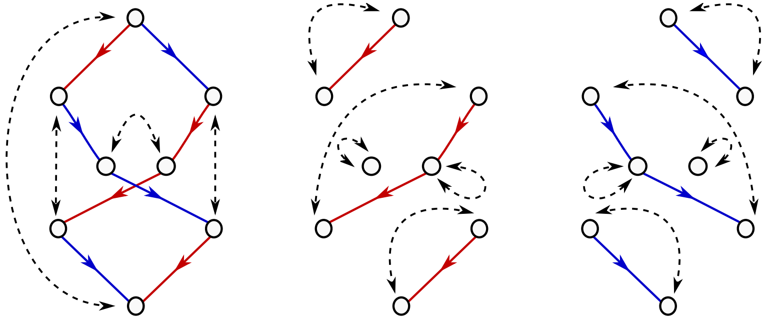
$\xi : B \to B$ applies ξ_λ to each connected B_λ.

Cactus action: $c_J(b) = \xi_{B_J}(b)$ for all $J \subset I$ and $b \in B$,
where $B_J = B$ restricted to J.

Example

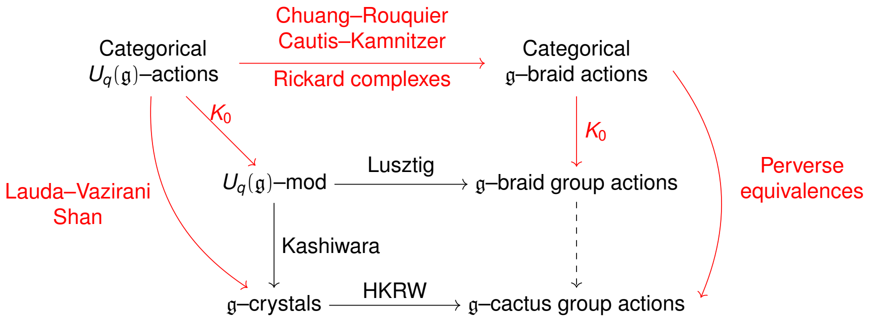
Consider $\mathfrak{sl}_3$ and the adjoint rep. crystal $B_{\alpha_1+\alpha_2}$.

$$C_{\mathfrak{sl}_3} = \left\langle \; c_1, c_2, c_{12} \;\middle|\; c_1^2 = c_2^2 = c_{12}^2 = 1, c_1 c_{12} = c_{12} c_2 \; \right\rangle$$



The c_{12} action. The c_1 action. The c_2 action.

A cactus group action II



Theorem 2 (H–Licata–Losev–Yacobi)

Let $\theta_J : Irr(\mathcal{C}) \to Irr(\mathcal{C})$ denote the bijection induced from $\Theta_{w_0^J}$. Then the map $c_J \mapsto \theta_J$ defines a cactus group action $C_{\mathfrak{g}} \curvearrowright Irr(\mathcal{C})$ which coincides with the combinatorial action of the cactus group on crystals.

The End

Thank you!