Braid group actions, crystals, and cacti
(joint work with T. Licata, I. Losev, O. Yacobi)

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Let $\mathfrak{g}$ be a simply-laced Kac–Moody Lie algebra.
Categorical $g$–action I

Warm-up case: $g = \mathfrak{sl}_2$

Suppose $\mathfrak{sl}_2 = \mathbb{C}\{e, f, h\} \curvearrowright V = \bigoplus_{n \in \mathbb{Z}} V_n$, an integrable $\mathfrak{sl}_2$–rep.

$$e : V_n \to V_{n+2}, \quad f : V_n \to V_{n-2}, \quad (ef - fe)|_{V_n} = n \text{Id}_{V_n} \quad \forall n \in \mathbb{Z}.$$

Categorified $\mathfrak{sl}_2$–action (Chuang–Rouquier, Khovanov–Lauda):

- an abelian category $\mathcal{C} = \bigoplus_n C_n$ (with $K_0(C_n) = V_n$)
- exact endofunctors $E, F$ of $\mathcal{C}$, $E : C_n \to C_{n+2}, F : C_n \to C_{n-2}$
- natural transformations $\epsilon : EF \to I, \eta : I \to FE$ (unit and counit of adjunction)

$$X : E \to E, T : E^2 \to E^2$$

such that...
Categorical $g$–action II

- For $n \geq 0$ (analogously for $n < 0$), we have an isomorphism:

$$(\sigma, \epsilon, \epsilon \circ Xl_F, \ldots, \epsilon \circ X^{n-1}l_F) : EF|_{C_n} \xrightarrow{\cong} FE|_{C_n} \oplus l_{C_n}^{\oplus n},$$

where $\sigma$ is composed of $\eta$, $T$, and $\epsilon$.

- The natural transformations $X$, $T$ give an action of the nil affine Hecke algebra $H_n$ on $E^n$.

Example The adjoint representation of $\mathfrak{sl}_2$:

$$(C_2 = \mathbb{C} - \text{mod}) \xrightarrow{\text{Ind}} (C_0 = \mathbb{C}[x]/x^2 - \text{mod}) \xleftarrow{\text{Res}} (C_{-2} = \mathbb{C} - \text{mod})$$
Categorical $g$–action III

More generally:

The 2–category $\mathcal{U}_g$ categorifies (Lusztig’s idempotent form) $\dot{\mathcal{U}}_g$:

- objects are elements $\lambda$ of the $g$–weight lattice.
- 1-morphisms are generated by $E_i : \lambda \to \lambda + \alpha_i$, $F_i : \lambda \to \lambda - \alpha_i$.
- 2-morphisms are generated by
  \[
  X_i = \begin{pmatrix} 1 \end{pmatrix}_i : E_i \to E_i,
  X_i = \begin{pmatrix} 1 \end{pmatrix}_i : F_i \to F_i,
  T_{ij} = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} : E_i E_j \to E_j E_i,
  T_{ij} = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} : F_i F_j \to F_j F_i
  \]
  \[
  \bigwedge^i : E_i F_i \to I,
  \bigvee^i : F_i E_i \to I,
  \bigcup^i : I \to F_i E_i,
  \bigcup^i : I \to E_i F_i
  \]
  + KLR algebra and further relations.

A categorical $g$–representation is a 2–functor $\mathcal{U}_g \to \mathcal{K}$ to an appropriate 2-category.

Note: A graded version, $\mathcal{U}_{qg}$, categorifies $\dot{\mathcal{U}}_{qg}$. 
The Rickard complex I

\( g = \mathfrak{sl}_2: \) Let \( s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \exp(-f) \exp(e) \exp(-f) \in SL_2. \)

Then \( s \) restricts to an isomorphism on weight spaces, of the form

\[
s|_{V_{-n}} = \sum_k (-1)^k e^{(n+k)} f(k)
\]

Rickard complex: Consider the complex of functors \( \Theta = \bigoplus_n \Theta_n, \)

\[
\Theta_n : \text{Comp}(C_{-n}) \to \text{Comp}(C_n)
\]

\[
\Theta_n = (\ldots \to E^{(n+2)} F(2) \to E^{(n+1)} F(1) \to E^{(n)})
\]

- \( E^{(n)} \subseteq E^n, F^{(n)} \subseteq F^n \) defined using the \( H_n \)-action
- \( E^{(n+k)} F(k) \to E^{(n+k-1)} F(k-1) \) comes from adjunction
The Rickard complex II

**Theorem (Chuang-Rouquier ’08)**
\( \Theta \) induces a self-equivalence on \( D^b(C) \) and, by restriction, an equivalence \( D^b(C-\mathfrak{n}) \cong D^b(C_\mathfrak{n}) \). Furthermore, \( [\Theta] = s \).

**Example:** Let \( R = \mathbb{C}[x]/x^2 \). For the adjoint \( \mathfrak{sl}_2 \)-representation and \( N \in C_0 = R - \text{mod} \),

\[ \Theta_0 : D^b(R - \text{mod}) \to D^b(R - \text{mod}) \]
\[ N \mapsto (R \otimes N \xrightarrow{\text{act}} N) \]
Perverse equivalences

Suppose that $\mathcal{A}, \mathcal{A}'$ are abelian categories, $F : D^b(\mathcal{A}) \xrightarrow{\cong} D^b(\mathcal{A}')$, we have filtrations $\mathcal{A}_\bullet, \mathcal{A}'_\bullet$ by Serre subcategories

$$0 = \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_r = \mathcal{A}$$

$$0 = \mathcal{A}'_{-1} \subset \mathcal{A}'_0 \subset \mathcal{A}'_1 \subset \ldots \subset \mathcal{A}'_r = \mathcal{A}'$$

and a perversity function $p : \{0, \ldots, r\} \to \mathbb{Z}$.

Definition

$F$ is perverse with respect to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$ if:

1. $F[-p(i)]$ restricts to an equivalence $D^b_{\mathcal{A}_i}(\mathcal{A}) \xrightarrow{\cong} D^b_{\mathcal{A}_i}(\mathcal{A}')$.

2. The induced $D^b_{\mathcal{A}_i}(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \xrightarrow{\cong} D^b_{\mathcal{A}_i}(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$ equivalence induces an equivalence

$$\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\cong} \mathcal{A}'_i/\mathcal{A}'_{i-1}.$$
Perversity of the Rickard complexes

Consider $\mathcal{C}$ endowed with an $\mathfrak{sl}_2$-categorical action. Let $S$ be the set of simple objects, and consider the filtrations:

$$S_i = \{ V \in S : F^{i+1} V = 0 \} \quad \text{and} \quad S'_i = \{ V \in S : E^{i+1} V = 0 \}.$$

**Proposition (Chuang–Rouquier)**

The equivalence $\Theta : D^b(\mathcal{C}) \xrightarrow{\sim} D^b(\mathcal{C})$ is perverse with respect to $(S_\bullet, S'_\bullet, \rho = \text{Id})$.

For general $g$, take a reduced word $w = s_{i_1} \ldots s_{i_k}$ and weight $\mu$, consider the composition $\Theta^\mu_w = \Theta^\mu_{s_{i_2} \ldots s_{i_k}} \circ \ldots \circ \Theta^\mu_{s_{i_1}}$.

**Theorem 1 (H–Licata–Losev–Yacobi)**

Let $w_0 \in W$ be the longest element, and $\mu$ a weight of $\mathcal{C}$. Then $\Theta^\mu_{w_0} : D^b(\mathcal{C}_\mu) \to D^b(\mathcal{C}_{w_0(\mu)})$ is a perverse equivalence.
Construction details

Set $\mathcal{A} = C_\mu$ and $\mathcal{A}' = C_{w_0(\mu)}$. Consider $\Theta^\mu_{w_0} : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$. $V = [C]_C = \bigoplus_{\lambda \in X_+} \text{Iso}_\lambda(V)$ is an integrable $g$-rep.

Theorem (Jordan–Hölder filtration)

There is a filtration of $\mathcal{C}$ by Serre subcategories

$$0 = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_r = C,$$

such that

1. $C_i$ is a subrepresentation of $\mathcal{C}$,
2. $C_i/C_{i-1} \cong \mathcal{L}(\lambda_i) \otimes A_i$ is a simple rep., with $\mathcal{L}(\lambda_i)$ a minimal categorification and $A_i$ an abelian category.

There are induced filtrations on $\mathcal{A}$ and $\mathcal{A}'$.

Let $\lambda_0, \ldots, \lambda_r$ be the dominant weights that appear. The perversity function $p : \{0, \ldots, r\} \to \mathbb{Z}$ can be defined as $p(i) = ht(\mu - w_0(\lambda_i))$. 
Braid and cactus groups I

The **braid group** $Br_g$ is generated by $\sigma_i, i \in I$, with relations:

\[
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } i \text{ and } j \text{ are connected},
\]

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } i \text{ and } j \text{ are not connected}.
\]

The **cactus group** $C_g$ is generated by $c_J$, for connected $J \subseteq I$, with:

\[
c_J^2 = 1 \quad \forall J \subseteq I
\]

\[
c_J c_K = c_K c_J \quad \forall J \cup K \subseteq I \text{ not connected}
\]

\[
c_J c_K = c_{\tau_J(K)} c_J \quad \forall K \subseteq J \subseteq I
\]

Where $\forall j \in J$, $\alpha_j$ simple root, $\alpha_{\tau_J(j)} = -w_0^j \alpha_j$.

**Proposition (Cautis–Kamnitzer ’10)**

*The Rickard complexes satisfy the braid relations.*
Braid and cactus groups II

We have $PBr_n \to Br_n \to S_n$ and $PC_n \to C_n \to S_n$. Topologically,

$$PBr_n \cong \pi_1 \left( \text{Configurations of } n \text{ distinct points in the plane} \right) \cong \pi_1(\mathbb{C}^n \setminus \bigcup_{i \neq j} \{z_i = z_j\})$$

$$PC_n \cong \pi_1(\overline{M_{0,n+1}}(\mathbb{R}))$$

The real locus of the Deligne–Mumford moduli space of curves with $n + 1$ marked points

Example. Consider $\overline{M_{0,4}}(\mathbb{R})$.

Figure: Opuntia cactus.

$$\pi_1(\overline{M_{0,4}}(\mathbb{R})) \cong \mathbb{Z} \cong PC_3 = \langle c_{12}c_1c_2c_1 \rangle$$
A cactus group action I

**Combinatorial realization**: $C_g \curvearrowright B = \bigsqcup \lambda B_\lambda$ on any $\mathfrak{g}$-crystal $B$ via generalized Schützenberger involutions.

Let $\xi_\lambda : B_\lambda \to B_\lambda$ be the unique map:

1. $e_i \cdot \xi_\lambda(b) = \xi_\lambda(f_{\tau(i)} \cdot b)$
2. $f_i \cdot \xi_\lambda(b) = \xi_\lambda(e_{\tau(i)} \cdot b)$
3. $\text{wt}(\xi_\lambda(b)) = w_0 \cdot \text{wt}(b)$

where $e_i, f_i$ are the Kashiwara operators on $B_\lambda$.

$\xi : B \to B$ applies $\xi_\lambda$ to each connected $B_\lambda$.

**Cactus action**: $c_J(b) = \xi_{B_J}(b)$ for all $J \subset I$ and $b \in B$, where $B_J = B$ restricted to $J$. 
Example

Consider $\mathfrak{sl}_3$ and the adjoint rep. crystal $B_{\alpha_1 + \alpha_2}$.

$C_{\mathfrak{sl}_3} = \langle c_1, c_2, c_{12} \mid c_1^2 = c_2^2 = c_{12}^2 = 1, c_1 c_{12} = c_{12} c_2 \rangle$

The $c_{12}$ action.

The $c_1$ action.

The $c_2$ action.
A cactus group action II

Theorem 2 (H–Licata–Losev–Yacobi)

Let $\theta_J : \text{Irr}(\mathcal{C}) \to \text{Irr}(\mathcal{C})$ denote the bijection induced from $\Theta_{w_0}$. Then the map $c_J \mapsto \theta_J$ defines a cactus group action $C_g \acts \text{Irr}(\mathcal{C})$ which coincides with the combinatorial action of the cactus group on crystals.
The End

Thank you!