Math 220 -Diagonal Matrices – Eigenvalues – Eigenvectors – Eigenspaces - Fall 2000

Let A be an $n \times n$ matrix. An n–coordinates vector $v \neq 0$ is an eigenvector (of A) with eigenvalue λ if and only if $Av = \lambda v$. The subspace $E_\lambda = \{x \in \mathbb{R}^n : Ax = \lambda x\}$ is called the eigenspace corresponding to the eigenvalue λ.

How to find eigenvalues, eigenvectors and eigenspaces: Find all the values for which $A - \lambda I$ is not invertible.

1. Find $p(\lambda) = \det(A - \lambda I)$. This is called the *characteristic polynomial.*
2. Find the values for which $p(\lambda) = 0$.
3. For each λ in (2), find Null$(A - \lambda I)$. This gives all the eigenvectors with eigenvalue λ. That is, this Null space IS E_λ
4. Write the answers of (3) in vector form. This gives a basis for E_λ.

Problem 1. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $v1 = (1, -1)$ and $v2 = (1, 1)$.

(a) Show that $A * v1 = 2\, v1$ and $A * v2 = 4\, v2$.

(b) Find Q, the change of basis matrix from the basis $B = \{v1, v2\}$ to the standard basis.

(c) Show that $D = Q^{-1} * A * Q$ is the matrix of the transformation $T(x, y) = A(x, y)^t$ in the basis B.

Note: Whenever we find the matrix in the basis given by the eigen vectors, we obtain a diagonal matrix (the one with the eigenvalues in the diagonal). This process is called the *diagonalization* of A. It may not always be possible to diagonalize a matrix. Sometimes, the eigenvalues do not span the Domain space.

Problem 2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. Diagonalize A. (That is, find a diagonal matrix D, and a matrix Q such that $A = Q * D * Q^{-1}$).

Problem 3. Let $A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 4 & 2 & 2 \end{bmatrix}$.

(a) Find all the characteristic polynomial. The characteristic polynomial is a product of terms of the form $(\lambda - a)^m$, where a is an eigen value. The power m is called the *multiplicity* of a.
Find the eigenvalues of A and their multiplicity.

(b) Find the corresponding eigenspaces. The dimension of the eigenspace is called the *geometric multiplicity* of the corresponding eigenvalue.

Find a basis of each eigenspace, and the corresponding geometric multiplicity.

(c) Diagonalize A if possible.

Problem 4. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$. Repeat problem 3.

Problem 5. Let $A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$. Repeat problem 3.

Problem 6. (Complex eigenvalues) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Repeat problem 3.

Problem 7. $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 2 & 1 \end{bmatrix}$. Diagonalize A if possible.

Note: to factor the characteristic polynomial you can use the command "factor" in maple.
Read sections 7.1, 7.2. Do problems Sec. 7.1: 17, 19, 21, ,23 ,27, 30, 31, 32, 34, 35, 37, 48, 49. Sec 7.2: 1, 3, 5, 17, 27.
Miniproject due Friday: Problem A, page 499.