

MOTIVATION & BACKGROUND

THE GENERAL IDEA THERE IS THAT IN THE CLASSICAL NERVE THEOREM WE CAN RECOVER THE HOMOTOPY TYPE OF A SPACE

BY LOOKING AT THE NERVE OF A "NICE" ENDOWED COVER OF THE SPACE.

WE CAN THEN REPLACE "SPACE" WITH "PERSISTENT SPACE" & OBTAIN AN ANALOGOUS RESULT.

THE GOAL OF THIS TALK IS TO TRY TO UNDERSTAND A FURTHER GENERALIZATION IN WHICH WE CAN LOSEN THE CONDITIONS ON HOW "NICE" THE COVER IS.

DEFN. A SIMPLICIAL COMPLEX $K = (V, S)$ CONSISTS OF A SET V OF VERTICES & A SET S OF FINITE, NON-EMPTY SUBSETS OF V . A SET $\sigma \in S$ WITH $k+1$ ELEMENTS IS CALLED A k -SIMPLEX OF K . WE REQUIRE

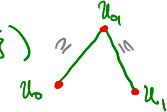
$$1. \{v\} \in S \quad \forall v \in V$$

$$2. \text{ IF } \sigma \in S \quad \text{ & } \phi \neq \tau \subseteq \sigma \text{ THEN } \tau \in S$$

Ex LET $U := \{U_j : j \in J\}$ BE A COVERING OF X . FOR A FINITE $\emptyset \neq E \subseteq J$, LET $U_E := \bigcap_{j \in E} U_j$ & LET $S(J) = \{\emptyset \neq E \subseteq J : U_E \neq \emptyset\}$. THEN $N(U) := (J, S(J))$ IS THE NERVE OF THE COVERING U .



say $X := U_0 \cup U_1$ & $U = \{U_0, U_1\}$; THEN $N(U) = (\{0, 1\}, \{\{0\}, \{1\}, \{0, 1\}\})$

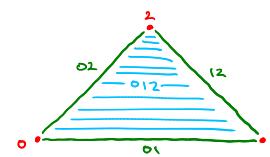


Ex IF (P, \leq) IS A POSET, (P, S_p) IS A SIMPLICIAL COMPLEX WHERE S_p CONSISTS OF THE TOTALLY ORDERED FINITE CHAINS IN P . e.g., A k -SIMPLEX $\sigma = \alpha_0 < \dots < \alpha_k$ IN P .

Ex IF $K = (V, S)$ A SIMPLICIAL COMPLEX, S HAS A NATURAL POSET STRUCTURE: $s \leq t \Leftrightarrow s \subseteq t$.

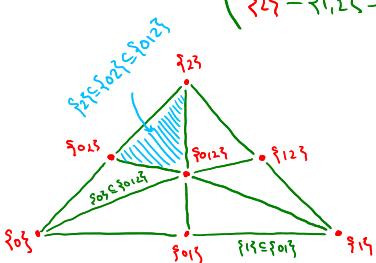
WE SAY $S(K) = (S, \leq)$ IS THE BARYCENTRIC SUBDIVISION OF K .

TAKE $P = [2] = \{0 < 1 < 2\} \rightsquigarrow K := (P, S_p) = (\{0, 1, 2\}, \{03 \xrightarrow{03=013} 013, 13 \xrightarrow{13=013} 013, 23 \xrightarrow{23=013} 013\})$



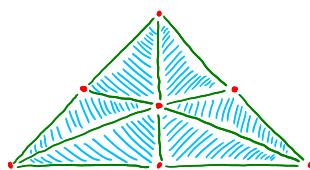
S_p GIVES RISE TO

$$S(K) = \left(\begin{array}{c} 03 \xrightarrow{03=013} 013 \\ 13 \xrightarrow{13=013} 013 \\ 23 \xrightarrow{23=013} 013 \end{array} \right)$$



$$\begin{aligned} & \{03, 13, 23, 013, 123, 0123, 123\} \quad 0\text{-simplices (7)} \\ & \{03 \subseteq 013, 13 \subseteq 013, 23 \subseteq 013, 013 \subseteq 0123\} \\ & \{03 \subseteq 023, 13 \subseteq 023, 23 \subseteq 023, 023 \subseteq 0123\} \\ & \{013 \subseteq 0123, 023 \subseteq 0123, 123 \subseteq 0123\} \\ & \{03 \subseteq 0, 13 \subseteq 0, 23 \subseteq 0\} \\ & \{03 \subseteq 0, 23 \subseteq 0, 13 \subseteq 0\} \\ & \{23 \subseteq 0, 023 \subseteq 0, 0123 \subseteq 0\} \quad 1\text{-simplices (12)} \end{aligned}$$

$$\begin{aligned} & \{03 \subseteq 0, 13 \subseteq 0, 23 \subseteq 0\} \\ & \{03 \subseteq 0, 23 \subseteq 0, 13 \subseteq 0\} \\ & \{23 \subseteq 0, 023 \subseteq 0, 0123 \subseteq 0\} \quad 2\text{-simplices (6)} \end{aligned}$$



DEFN: [GEOMETRIC REALIZATION] IF $K = (V, S)$ A SIMPLICIAL COMPLEX,

WRITE $|K| = \{x: V \rightarrow I^V\} : (1) \{v \in V : x(v) > 0\} \in S \text{ & } (2) \sum_{v \in V} x(v) = 1 \text{ NOTE } |K| \subseteq I^V = \text{Top}(V, I)$

IF $\sigma \in S$, LET $\Delta(\sigma)$ BE THE STANDARD SIMPLEX $\{(t_v) \in |K| : t_v = 0 \text{ FOR } v \notin \sigma\}$.

SO $|K| = \bigcup_{\sigma \in S} \Delta(\sigma)$ WE GIVE IT THE QUOTIENT TOPOLOGY DEFINED BY CANONICAL $\coprod_{\sigma \in S} \Delta(\sigma) \rightarrow |K|$.

DEFN: [GEOMETRIC REALIZATION] THE GEOMETRIC REALIZATION OF A SIMPLICIAL SET $X: \Delta \rightarrow \text{SET}$ IS:

$$|X| := \coprod_{n=0}^{\infty} X_n \times \Delta^n / \sim \quad \text{WHERE}$$

$$(x, d_i(t)) \sim (d_i(x), t), \quad x \in X_{n+1}, \quad t \in \Delta^n$$

$$(x, s_i(x), t) \sim (s_i(x), t), \quad x \in X_{n-1}, \quad t \in \Delta^n$$

COFACE

$$d_i: [n] \rightarrow [n+1]$$

FACE

$$d_i: X_{n+1} \rightarrow X_n$$

CODEGENERACY

$$s_i: [n] \rightarrow [n-1]$$

DEGENERACY

$$s_i: X_{n-1} \rightarrow X_n$$

DEFN: [SIMPLICIAL REPLACEMENT] GIVEN A FUNCTOR $F: \mathcal{T} \rightarrow \text{Top}$, ITS SIMPLICIAL REPLACEMENT

IS THE SIMPLICIAL SET $SREP_n F: \Delta^n \rightarrow \text{Top}$ WHERE $SREP_n F := \coprod F([i])$. WHERE

$$d_i: SREP_n F \rightarrow SREP_{n-1} F \left[\begin{array}{c} F(j_0) \xrightarrow{F(c_0)} F(j_1) \\ j_0 \rightarrow \dots \rightarrow j_n \end{array} \right. \begin{array}{l} \text{IF } i=0 \text{ OR} \\ \text{IDENTITY OTHERWISE} \end{array} \left. \begin{array}{c} j_0 \rightarrow \dots \rightarrow j_n \\ j_1 \rightarrow \dots \rightarrow j_n \end{array} \right]$$

$$j_0 \rightarrow \dots \rightarrow j_n$$

$$j_1 \rightarrow \dots \rightarrow j_n$$

$$S_i: SREP_n F \rightarrow SREP_{n-1} F \left[\begin{array}{c} F(j_0) \xrightarrow{F(c_0)} F(j_1) \\ j_0 \rightarrow \dots \rightarrow j_n \end{array} \right. \begin{array}{l} \xrightarrow{F(c_0)} \\ j_0 \rightarrow \dots \rightarrow j_n \end{array} \left. \begin{array}{c} F(j_0) \xrightarrow{F(c_0)} F(j_1) \dots \xrightarrow{F(c_{n-1})} F(j_n) \\ j_0 \rightarrow \dots \rightarrow j_n \end{array} \right]$$

DEFN: [HOMOTOPY COHOMIT] THE HOMOTOPY COHOMIT OF $F: \mathcal{T} \rightarrow \text{Top}$ IS DEFINED TO BE $|SREP.F|$, i.e.,

$$\text{Hocolim } F := \coprod_{n=0}^{\infty} \coprod_{j_0 \rightarrow \dots \rightarrow j_n} F([i]) / \sim$$

THM: [CLASSICAL NERVE THEOREM] IF $\mathcal{U} = \{U_j : j \in J\}$ IS A COVER OF A PARACOMPACT SPACE X SUCH THAT ALL NON-EMPTY FINITE INTERSECTIONS ARE CONTRACTIBLE, THEN $|N(\mathcal{U})| \cong X$.

PF. [SKETCH] DEFINE $K := N(\mathcal{U})$; OBSERVE THAT EVERY ASC HAS A NATURAL CATEGORY ASSOCIATED TO IT:

THE OBJECTS ARE THE SIMPLICES OF K ; THE MORPHISMS ARE INCLUSIONS OF SIMPLICES.

DEFINE TWO FUNCTORS $F: K \xrightarrow{\Phi} \text{Top} \quad ; \quad G: K \xrightarrow{\Psi} \text{Top}$

$$\begin{aligned} \sigma &\mapsto U_\sigma \\ \sqcup &\mapsto \sqcup \\ \sqcap &\mapsto \sqcap_U \end{aligned} \quad \begin{aligned} \sigma &\mapsto \{*\} \\ \sqcup &\mapsto \sqcup \\ \sqcap &\mapsto \sqcap \end{aligned}$$

DEFINE A NATURAL TRANSFORMATION $\tau: F \Rightarrow G$ BY $\tau_\sigma: F(\sigma) \rightarrow G(\sigma) = \text{CONST}_*$.

By hypothesis this is a pointwise homotopy equivalence.

τ induces $\tau_f: \text{Hocolim } F \rightarrow \text{Hocolim } G$ WHICH IS A HOMOTOPY EQUIVALENCE BY RESULT OF DUGGER.

$$1. \text{ Hocolim } F := |SREP.F| = \coprod_{n \geq 0} \coprod_{\sigma_0 \leq \dots \leq \sigma_n} U_{\sigma_0} \times \Delta^n / \sim \stackrel{\text{SUBDIVISION TRICK}}{\cong} \coprod_{k \geq 0} \coprod_{j_0 \leq \dots \leq j_k} U_{j_0 \dots j_k} \times \text{so } \Delta^k \cong X$$

By PARACOMPACTNESS ARGUMENT

$$2. \text{ Hocolim } G := |SREP.G| = |N(K)| \stackrel{\text{DEFN}}{\cong} |SS.(K)| \stackrel{\text{SUBDIVISION TRICK}}{\cong} |N(\mathcal{U})|$$

NERVE OF CATEGORY K , AS SET
NATURAL SET ASSOCIATED TO EVERY ASC

Finally $X \cong \text{Hocolim } F \xrightarrow{\cong} \text{Hocolim } G \cong |N(\mathcal{U})|$

A NATURALITY RESULT DUE TO CHAZAL, QUOTED THAT WE'LL REQUIRE IS:

LEMMA 1 LET $X \subseteq X'$ BE TWO FINITE SIMPLICIAL COMPLEXES WITH GOOD COVERS $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$, $\mathcal{V}' = \{V'_\alpha\}_{\alpha \in A}$:

$V_\alpha \subseteq V'_\alpha \quad \forall \alpha \in A$. THERE EXIST HOMOTOPY EQUIVALENCES $|N(\mathcal{V})| \xrightarrow{\sim} X \xrightarrow{\sim} |N(\mathcal{V}')| \xrightarrow{\sim} X'$:

$$\begin{array}{ccc} |N(\mathcal{V})| & \longrightarrow & |N(\mathcal{V}')| \\ \downarrow \sim & \circlearrowleft & \downarrow \sim \\ X & \xrightarrow{\sim} & X' \end{array}$$

REMARK. I'D LIKE TO SEE THIS AS ARISING FROM SOME TYPE OF NATURALITY CONDITION USING THE LANGUAGE FROM ABOVE.

SOMETHING LIKE $\text{Hocolim}(|N(\mathcal{V})| \rightarrow \text{Top}) \longrightarrow \text{Hocolim}(|N(\mathcal{V}')| \rightarrow \text{Top})$, BUT THE \longrightarrow DOES NOT SEEM OBVIOUS

$$\begin{array}{ccc} & \downarrow & \downarrow \\ X & \xrightarrow{\sim} & X' \end{array}$$

SINCE THE HOLOLIMS ARE BEING TAKEN OVER FUNCTORS WITH DIFFERENT DOMAINS.

IT'S POSSIBLE THAT $V_\alpha \subseteq V'_\alpha \quad \forall \alpha$ INDUCES A FUNCTOR $N(\mathcal{V}) \rightarrow N(\mathcal{V}')$ WHICH INDUCES DESIRED \longrightarrow MAKING DIAGRAM \circlearrowleft .

(I HAVEN'T THOUGHT ABOUT THIS ENOUGH, BUT I THINK THERE MIGHT BE PROBLEMS WITH THIS APPROACH.)

THEOREM 2 [PERSISTENCE EQUIVALENCE] TWO FILTRATIONS $F = (F^\alpha)_{\alpha \geq 0} \dagger$, $G = (G^\alpha)_{\alpha \geq 0}$ WITH p.W.F.O PERSISTENCE

NOTES: $H_*(F) \cong H_*(G)$ $\Rightarrow \text{DGM } F = \text{DGM } G$
 $\exists \text{ NATURAL } H_\alpha F^\alpha \cong H_\alpha G^\alpha \quad \forall \alpha \geq 0$

DEFN A δ -INTERLEAVING OF $F, G : R \rightarrow \mathcal{D}$ CONSISTS OF NATURAL TRANSFORMATIONS $F \xrightarrow{\varphi} GT_\delta \dagger$; $G \xrightarrow{\psi} FT_\delta$

WHERE $T_\delta : R \rightarrow R$: $R \xrightarrow{T_\delta} R \xrightarrow{T_\delta} R$
 $a \mapsto a + \delta$ $F \downarrow \xrightarrow{\varphi} G \downarrow \xrightarrow{\psi} \downarrow$ $\rho = \rho = \rho$

$$(\varphi T_\delta)^\varphi = F \gamma_{2\delta} \quad \text{AND} \quad (\psi T_\delta)^\psi = G \gamma_{2\delta}$$

$$F(a) \xrightarrow{\varphi} F(a+2\delta)$$

$$\begin{array}{ccc} & \varphi(a) \downarrow & \uparrow \psi(a) \\ & \gamma_{2\delta} & \gamma_{2\delta} \\ G(a) & \xrightarrow{\psi} & G(a+2\delta) \end{array}$$

$$G(a) \xrightarrow{\psi} G(a+2\delta)$$

$$\begin{array}{ccc} & \psi(a) \downarrow & \uparrow \varphi(a) \\ & \gamma_{2\delta} & \gamma_{2\delta} \\ F(a) & \xrightarrow{\varphi} & F(a+2\delta) \end{array}$$

$$\begin{array}{ccc} F \xrightarrow{\varphi} GT_\delta & & G \xrightarrow{\psi} FT_\delta \\ F(a) \xrightarrow{\varphi} F(b) & & G(a) \xrightarrow{\psi} G(b) \\ \varphi(a) \downarrow \cup & & \psi(a) \downarrow \cup \\ G(a+\delta) \xrightarrow{\psi} G(b+\delta) & & F(a+\delta) \xrightarrow{\varphi} F(b+\delta) \\ G(a+\delta \leq b+\delta) & & F(a+\delta \leq b+\delta) \end{array}$$

THEOREM 3 [ALGEBRAIC STABILITY] GIVEN $F = (F^\alpha)_{\alpha \geq 0} \dagger$; $G = (G^\alpha)_{\alpha \geq 0} \dagger$; $\forall \alpha \geq 0$, $\dim H_* F^\alpha, \dim H_* G^\alpha < \infty$,

IF $H_* F \dagger H_* G$ ARE δ -INTERLEAVED THEN $d_B(\text{DGM}_K F, \text{DGM}_K G) \leq \delta + K$

LET $\mathcal{U} = \{U_0, \dots, U_n\}$ BE A COLLECTION OF SIMPLICIAL FILTRATIONS, WHERE $U_i := (U_i^\alpha)_{\alpha \geq 0} \dagger$; U_i^α IS A FINITE SIMPLICIAL COMPLEX OF SOME AMBIENT SIMPLICIAL COMPLEX.

LET $\mathcal{U}^\alpha := \{U_0^\alpha, \dots, U_n^\alpha\}$. $\forall \alpha \neq \beta \in [n]$, $U_\beta^\alpha := \bigcap_{i \in \beta} U_i^\alpha$ YIELDING SIMPLICIAL FILTRATION $U_\beta := (U_\beta^\alpha)_{\alpha \geq 0}$.

THE COLLECTION OF FILTRATIONS \mathcal{U} GIVES RISE TO NERVE FILTRATION $N(\mathcal{U}) := (N(\mathcal{U}^\alpha))_{\alpha \geq 0}$

DEFINE $\forall \alpha \geq 0$, $W^\alpha := \bigcup_{i=0}^n U_i^\alpha \dagger$; $W := (W^\alpha)_{\alpha \geq 0}$. $\forall \alpha$, \mathcal{U}^α IS A COVER OF W^α .

WE SAY \mathcal{U} IS A GOOD COVER FILTRATION OF W IF \mathcal{U}^α IS A GOOD COVER OF W^α $\forall \alpha$.

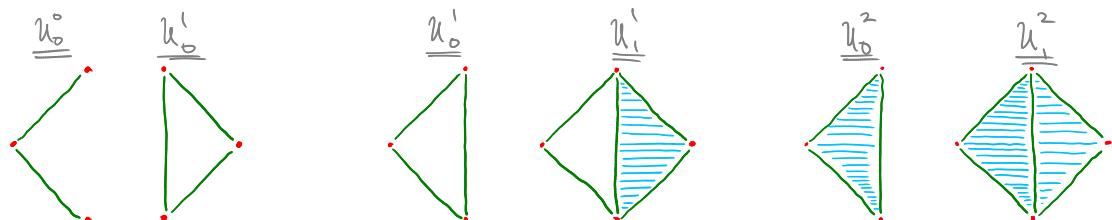
THEOREM 5 [PERSISTENT NERVE LEMMA] GIVEN A COLLECTION OF FINITE SIMPLICIAL FILTRATIONS \mathcal{U} WHERE \mathcal{U} IS A GOOD COVER FILTRATION OF \mathcal{W} , THEN $\text{dgm}(\text{N}(\mathcal{U})) = \text{dgm}(\mathcal{W})$.

WE WANT TO LOOSEN THE CONDITION OF \mathcal{U} A GOOD COVER OF \mathcal{W} SO THAT 'SMALL VARIATIONS' ARE MADE INSIGNIFICANT.

DEFN. A COVER FILTRATION \mathcal{U} IS AN ε -GOOD COVER OF A FILTRATION $\mathcal{W} = (\bigcup_{i=0}^n \mathcal{U}_i^\varepsilon)_{\varepsilon > 0}$ IF FOR ALL $\phi \neq \psi \subseteq [n]$,

$$\exists \delta > 0, \tilde{H}_*(\mathcal{U}_\phi^\varepsilon \hookrightarrow \mathcal{U}_\psi^{\varepsilon+\delta}) = 0\text{-MAP}.$$

EXAMPLE [1-GOOD COVER THAT IS NOT GOOD]

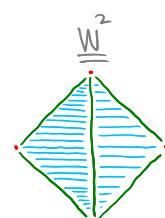
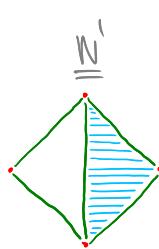
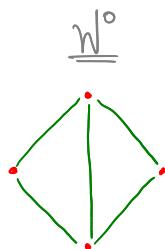
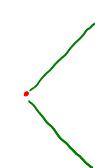


$$\underline{\mathcal{U}_0^\circ} = \mathcal{U}_0^\circ \cap \mathcal{U}_1^\circ$$

$$\underline{\mathcal{U}_0^1} = \mathcal{U}_0^1 \cap \mathcal{U}_1^1$$

$$\underline{\mathcal{U}_0^2} = \mathcal{U}_0^2 \cap \mathcal{U}_1^2$$

NOT GOOD
SINCE NOT
CONTRACTIBLE



$$\tilde{H}_*(\mathcal{U}_0^\circ \hookrightarrow \mathcal{U}_0^1)$$

$$\tilde{H}_*(\mathcal{U}_0^1 \hookrightarrow \mathcal{U}_0^2)$$

$$\tilde{H}_*\left(\begin{array}{c} \cdot \\ \vdots \end{array} \hookrightarrow \begin{array}{c} \cdot \\ \swarrow \searrow \end{array} \right)$$

$$\tilde{H}_*\left(\begin{array}{c} \cdot \\ \swarrow \searrow \end{array} \hookrightarrow \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right)$$

$$\left. \begin{array}{l} O \hookrightarrow \mathbb{Z} \\ O \mapsto O \end{array} \right\} O\text{-MAP}$$

$$\left. \begin{array}{l} \mathbb{Z} \hookrightarrow O \\ n \mapsto O \end{array} \right\} O\text{-MAP}$$

★ THEOREM 7 [GENERALIZED PERSISTENT NERVE THEOREM] GIVEN A FINITE COLLECTION OF FINITE SIMPLICIAL FILTRATIONS $\mathcal{U} = \{\mathcal{U}_0, \dots, \mathcal{U}_n\}$, WHERE $\mathcal{U}_i := (\mathcal{U}_i^\varepsilon)_{\varepsilon > 0}$; ALL $\mathcal{U}_i^\varepsilon$ ARE SUBCOMPLEXES OF A SUFFICIENTLY LARGE SIMPLICIAL COMPLEX, IF \mathcal{U} IS AN ε -GOOD COVER FILTRATION OF $\mathcal{W} = (\bigcup_{i=0}^n \mathcal{U}_i^\varepsilon)_{\varepsilon > 0}$, THEN $d_B(\text{dgm}_K(\mathcal{W}), \text{dgm}_K(\text{N}(\mathcal{U}))) \leq (k+1)\varepsilon$.

Fix a dimension K for rest of proof. $C_*(x) := (C_k(x))_{k \in K}$

IDEA. 1. Construct diagram of chain complexes ; chain maps that yield a $(K+1)\varepsilon$ -interleaving between

the filtered chain complexes $C_{\leq k}(W)$; $C_k(Nu)$ $\forall k \leq K$

2. Apply homology to chain maps to yield $(K+1)\varepsilon$ -interleaving between $H_k(W)$; $H_k(Nu)$ $\forall k \leq K$.

3. Apply Algebraic Stability Theorem.

WE WRITE Nu^{α} FOR THE NERVE OF $U = \{U_0, \dots, U_n\}$ AS AN ASC $\vdash N^{\alpha} := |\text{iso } Nu^{\alpha}|$; DEFINE $N := (N^{\alpha})_{\alpha \geq 0}$.
; NOTE $N^{\alpha} \cong |Nu|$.

VIEWING Nu^{α} AS A CATEGORY WITH OBJECTS THE SIMPLICES ; Morphisms inclusions of simplices, DEFINE

$$D^{\alpha} : (Nu^{\alpha})^{\text{op}} \rightarrow \text{Top} \quad \text{By} \quad V \mapsto V' \mapsto U_{V'}^{\alpha} \hookrightarrow U_V^{\alpha}$$

LEMMA: $B^{\alpha} := \text{Hocolim } D^{\alpha} = \bigcup_{\substack{U_0 \hookrightarrow \dots \hookrightarrow U_K \\ k \geq 0}} U_{U_k}^{\alpha} \times \Delta^k \subseteq N^{\alpha} \times N^{\alpha}$

$$\begin{aligned} \text{PROOF. } \text{Hocolim } D^{\alpha} &= |\text{SRep}_0 D^{\alpha}| = \coprod_{k \geq 0} \text{SRep}_k D^{\alpha} \times \Delta^k / \sim & d^i : [k] \rightarrow [k+1] & s^i : [k] \rightarrow [k-1] \\ &= \coprod_{k \geq 0} \coprod_{V_0 \subseteq \dots \subseteq V_k} D^{\alpha}(V_k) \times \Delta^k / \sim & d_i : \text{SRep}_{k+1} D^{\alpha} \rightarrow \text{SRep}_k D^{\alpha} & s_i : \text{SRep}_{k-1} D^{\alpha} \rightarrow \text{SRep}_k D^{\alpha} \\ &\text{where } (x, d^i(t)) \sim (d_i(x), t), \quad x \in \text{SRep}_{k+1} D^{\alpha}, \quad t \in \Delta^k & x \mapsto d_i(x) & x \mapsto s_i(x) \end{aligned}$$

$$(x, s^i(t)) \sim (s_i(x), t), \quad x \in \text{SRep}_{k-1} D^{\alpha}, \quad t \in \Delta^k$$

WE NEED only focus on the nondegenerate simplices. $\coprod_{k \geq 0} \coprod_{V_0 \subseteq \dots \subseteq V_k} D^{\alpha}(V_k) \times \Delta^k / \sim, \quad (x, d^i(t)) \sim (d_i(x), t)$

$$x \in \text{SRep}_{k+1} D^{\alpha}, \quad t \in \Delta^k$$

EXPLICITLY WE'RE GLUING $(x \in U_{V_{k+1}}^{\alpha}, d^i(t) \in \Delta^{k+1}) \sim (d_i(x) \in U_{V_{d^i(V_{k+1})}}^{\alpha}, t \in \Delta^k)$

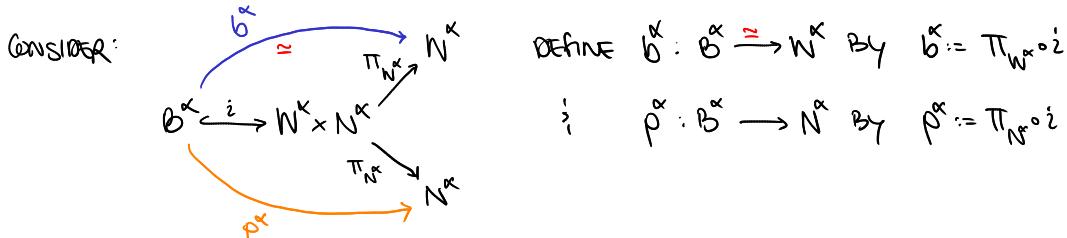
$$i < k+1 : \underbrace{(x \in U_{V_{k+1}}^{\alpha}, d^i(t) \in \Delta^{k+1})}_{V_0 \subseteq \dots \subseteq V_{k+1}} \sim \underbrace{(x \in U_{V_{k+1}}^{\alpha}, t \in \Delta^k)}_{V_0 \subseteq \dots \subseteq V_i \subseteq \dots \subseteq V_{k+1}} \quad \text{"glues to itself"}$$

$$i = k+1 : \underbrace{(x \in U_{V_{k+1}}^{\alpha}, d^i(t) \in \Delta^{k+1})}_{V_0 \subseteq \dots \subseteq V_{k+1}} \sim \underbrace{(D^{\alpha}(V_k \subseteq V_{k+1})(x) \in U_{V_k}^{\alpha}, t \in \Delta^k)}_{V_0 \subseteq \dots \subseteq V_k} \quad \text{"subsets glue into their supersets"}$$

$$\text{Conclude } B^{\alpha} = \text{Hocolim } D^{\alpha} = \bigcup_{k \geq 0} \bigcup_{V_0 \subseteq \dots \subseteq V_k} D^{\alpha}(V_k) \times \Delta^k / \sim \stackrel{*}{=} \bigcup_{k \geq 0} \bigcup_{V_0 \subseteq \dots \subseteq V_k} U_{V_k} \times \Delta^k.$$

THE ASSOCIATED FILTRATION IS $B := (B^{\alpha})_{\alpha \geq 0}$ WHERE $B^{\alpha} \subseteq N^{\alpha} \times N^{\alpha}$.

THE GOAL NOW IS TO CONSTRUCT A $(K+1)\varepsilon$ -INTERLEAVING BETWEEN $H_k(B)$; $H_k(Nu)$.



A STANDARD SHOWS b^* IS A HOMOTOPY EQUIVALENCE FOR COVERS OF PARACOMPACT SPACES, WHICH FINITE SIMPLICIAL COMPLEXES ARE, SO $H_*^{CW}(B^*) \cong H_{*p}(N^*) \cong H_*(N^*)$

TWO PROJECTIONS $b^* \wedge b^*$ COMMUTE WITH INCLUSIONS $i_B^{k\beta} : B^* \hookrightarrow B^\beta \wedge i_N^{*\beta} : N^* \hookrightarrow N^\beta$ SO $\forall 0 \leq \alpha \leq \beta$,

$$\begin{array}{ccc} H_*(N^*) & \xrightarrow{i_N^{k\beta}} & H_*(B^\beta) \\ b^* \downarrow \cong & \curvearrowright & \downarrow b^* \\ H_*^{CW}(B^*) & \xrightarrow{i_B^{k\beta}} & H_*^{CW}(B^\beta) \end{array}$$

WHICH GIVES US THE NATURALITY REQUIRED BY LEMMA 2, TO CONCLUDE
 $DGM B = DGM N$.

NOTE: p^* IS A HOMOTOPY EQUIVALENCE FOR GOOD COVERS, BUT HERE IS PRECISELY WHERE WE'VE LOOSENED OUR ASSUMPTIONS.

GOAL: MAKE UP FOR THIS DEFICIENCY BY CONSTRUCTING A $(k+1)\varepsilon$ -INTERLEAVING BETWEEN $H_k(N)$ \wedge $H_k^{CW}(B)$.

FOR A k -SIMPLEX $\sigma := v_0 \rightarrow \dots \rightarrow v_k \in S(N^*)$, WRITE $\sigma_i := |\sigma_{[v_0, \dots, v_i]}| \wedge \bar{\sigma}_i := |\sigma_{[v_i, \dots, v_k]}|$

MOREOVER, FOR $v \in \sigma$, WRITE $\sigma \setminus v := |\sigma_{[v_0, \dots, \hat{v}, \dots, v_k]}|$

FOR EVERY $\emptyset \neq V \subseteq [n]$, PICK A VERTEX $x_V \in U_V^{*\alpha} \subseteq N^*$ WHERE $\alpha := \min \{x \geq 0 : U_V^x \neq \emptyset\}$

\wedge NOTE $x_V \in U_V^\beta \subseteq N^* \wedge \beta \geq \alpha$.

CONSIDER THE CONSTANT FUNCTION $x_v : U_v^* \rightarrow U_v^{*\varepsilon}$, $x \mapsto x_v$. THIS YIELDS A CHAIN MAP x_v :

$$\dots \rightarrow C_{k+1}^{CW} U_v^* \rightarrow C_k^{CW} U_v^* \rightarrow C_{k-1}^{CW} U_v^* \rightarrow \dots \rightarrow C_1^{CW} U_v^* \rightarrow C_0^{CW} U_v^* \rightarrow 0$$

$$\downarrow \circ \quad \downarrow \circ$$

$$\dots \rightarrow C_{k+1}^{CW} U_v^{*\varepsilon} \rightarrow C_k^{CW} U_v^{*\varepsilon} \rightarrow C_{k-1}^{CW} U_v^{*\varepsilon} \rightarrow \dots \rightarrow C_1^{CW} U_v^{*\varepsilon} \rightarrow C_0^{CW} U_v^{*\varepsilon} \rightarrow 0$$

WE WRITE $x_v : C_*^{CW} U_v^* \rightarrow C_*^{CW} U_v^{*\varepsilon}$ WHERE $\sigma \mapsto \begin{cases} x_v & \text{IF } \dim \sigma = 0, \\ 0 & \text{OTHERWISE.} \end{cases}$

LEMMA 6: Fix $\alpha \geq 0$. Given $\emptyset \neq V \subseteq [n]$, where $v \in N^* U_V^*$ \wedge $x_v \in U_V^*$ is v 's corresponding vertex,

THERE EXISTS A CHAIN HOMOTOPY BETWEEN THE INCLUSION CHAIN MAP $i_V^{*\varepsilon} : C_*^{CW} U_V^* \rightarrow C_*^{CW} U_V^{*\varepsilon}$

\wedge $x_v : C_*^{CW} U_V^* \rightarrow C_*^{CW} U_V^{*\varepsilon}$.

PROOF: WE CONSTRUCT THE CHAIN HOMOTOPY BY INDUCTION ON DIMENSION TO PROVE THAT FOR ALL n ,

$$\exists c_n, c_{n-1} : C_{n-1} \partial_n + \partial_{n+1} = i_V^{*\varepsilon} - x_v.$$

$$\begin{array}{ccccccccc} \dots & \rightarrow & C_{k+1}^{CW} U_V^* & \rightarrow & C_k^{CW} U_V^* & \rightarrow & C_{k-1}^{CW} U_V^* & \rightarrow & \dots \rightarrow C_1^{CW} U_V^* \rightarrow C_0^{CW} U_V^* \rightarrow 0 \rightarrow 0 \\ \downarrow \circ & & \downarrow \circ & & \downarrow \circ & & \downarrow \circ & & \downarrow \circ \\ \dots & \rightarrow & C_{k+1}^{CW} U_V^{*\varepsilon} & \rightarrow & C_k^{CW} U_V^{*\varepsilon} & \rightarrow & C_{k-1}^{CW} U_V^{*\varepsilon} & \rightarrow & \dots \rightarrow C_1^{CW} U_V^{*\varepsilon} \rightarrow C_0^{CW} U_V^{*\varepsilon} \rightarrow 0 \rightarrow 0 \end{array}$$

THE OBVIOUS BASE CASE IS $c_{-1}, c_{-2} = 0$, BUT THIS IS NOT ENLIGHTENING, SO LET'S TRY TO CONSTRUCT c_0 .

$$\begin{array}{ccccc} \rightarrow C_1^{\text{CW}}(U_V^\kappa) & \xrightarrow{\partial_1^\kappa} & C_0^{\text{CW}}(U_V^\kappa) & \xrightarrow{\partial_0^\kappa} & 0 \\ & \downarrow c_0, \overset{\sigma}{\underset{\sim}{\downarrow}}_{\partial_0^\kappa} & \downarrow x_{V,0} & \downarrow c_{-1} = 0 & \\ \rightarrow C_1^{\text{CW}}(U_V^{\kappa+\varepsilon}) & \xrightarrow{\partial_1^{\kappa+\varepsilon}} & C_0^{\text{CW}}(U_V^{\kappa+\varepsilon}) & \rightarrow 0 & \end{array}$$

WE DEFINE c_0 ON THE GENERATING 0-CELLS $\sigma \in U_V^\kappa$ \nmid EXTEND BY LINEARITY.

$$\text{DEFINE } z := -c_0 \partial_0^\kappa(\sigma) + \overset{\sigma}{\underset{\sim}{\partial}}_{V,0}^{\kappa, \kappa+\varepsilon}(\sigma) - x_V(\sigma) = \sigma - x_V \in C_0^{\text{CW}}(U_V^\kappa) \subseteq C_0^{\text{CW}}(U_V^{\kappa+\varepsilon})$$

$$\begin{array}{c} \text{By } \varepsilon\text{-GOODNESS, } (\overset{\kappa, \kappa+\varepsilon}{\partial}_{V,0})_* : \widetilde{H}_0^{\text{CW}}(U_V^\kappa) \rightarrow \widetilde{H}_0^{\text{CW}}(U_V^{\kappa+\varepsilon}) = 0\text{-MAP} \\ \parallel \qquad \qquad \qquad \parallel \\ C_0^{\text{CW}}(U_V^\kappa)/\text{IM } \partial_0^\kappa \rightarrow C_0^{\text{CW}}(U_V^{\kappa+\varepsilon})/\text{IM } \partial_0^{\kappa+\varepsilon} \end{array}$$

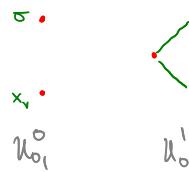
IN PARTICULAR, $z + \text{IM } \partial_1^\kappa \mapsto z + \text{IM } \partial_1^{\kappa+\varepsilon} = 0$ SINCE THIS IS THE 0-MAP

SO $z \in \text{IM } \partial_1^{\kappa+\varepsilon} \Rightarrow \exists b \in C_1^{\text{CW}}(U_V^{\kappa+\varepsilon}) : \partial_1^{\kappa+\varepsilon}(b) = z$. DEFINE $c_0(\sigma) := b$.

CAN WE GUARANTEE $b \in C_1^{\text{CW}}(U_V^\kappa)$?

POSSIBLE COUNTER EXAMPLE:

(FROM BEFORE)



$$z := \sigma - x_V, \quad c_0(\sigma) := b ; \quad \partial_1 b = \sigma - x_V = z$$

BUT $b \notin U_0^0$ SINCE IT ONLY CONTAINS 0-CELLS

ASSUME, FOR THE INDUCTIVE STEP, FOR $k > 0$, $\exists c_{k-1}, c_{k-2} : C_{k-2} \partial_{k-1}^\kappa + \partial_k^{\kappa+\varepsilon} c_{k-1} = \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, k-1} - x_{V, k-1}$.

$$\begin{array}{ccccccc} & & \overset{\sigma}{\bullet} & & & & \\ & & \downarrow & & & & \\ & & C_K^{\text{CW}}(U_V^\kappa) & \xrightarrow{\partial_K^\kappa} & C_{K-1}^{\text{CW}}(U_V^\kappa) & \xrightarrow{\partial_{K-1}^\kappa} & C_{K-2}^{\text{CW}}(U_V^\kappa) \\ & & \downarrow \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K} & & \downarrow \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K-1} & & \downarrow \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K-2} \\ C_{K+1}^{\text{CW}}(U_V^{\kappa+\varepsilon}) & \longrightarrow & C_K^{\text{CW}}(U_V^{\kappa+\varepsilon}) & \xrightarrow{\overset{\kappa, \kappa+\varepsilon}{\partial}_K} & C_{K-1}^{\text{CW}}(U_V^{\kappa+\varepsilon}) & \xrightarrow{\overset{\kappa, \kappa+\varepsilon}{\partial}_{K-1}} & C_{K-2}^{\text{CW}}(U_V^{\kappa+\varepsilon}) \\ z \longleftarrow & & \xrightarrow{\overset{\kappa, \kappa+\varepsilon}{\partial}_K(z) = 0} & & \xrightarrow{\overset{\kappa, \kappa+\varepsilon}{\partial}_{K-1}(z) = 0} & & \end{array}$$

FOR $\sigma \in C_K^{\text{CW}}(U_V^\kappa)$ A k -SIMPLEX, DEFINE $z := -c_{k-1} \partial_k^\kappa(\sigma) + \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K}(\sigma) - x_{V, K}(\sigma) \in C_K^{\text{CW}}(U_V^{\kappa+\varepsilon})$

$$\begin{aligned} \text{OBSERVE: } \partial_K^{\kappa+\varepsilon}(z) &= -\overset{\kappa, \kappa+\varepsilon}{\partial}_K(c_{k-1} \partial_k^\kappa(\sigma)) + \overset{\kappa, \kappa+\varepsilon}{\partial}_K \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K} \sigma - \overset{\kappa, \kappa+\varepsilon}{\partial}_K x_{V, K} \sigma \\ &= -(\overset{\kappa, \kappa+\varepsilon}{\partial}_V - x_V - c_{k-2} \overset{\kappa}{\partial}_{k-1})(\overset{\kappa}{\partial}_K(\sigma)) + \overset{\kappa, \kappa+\varepsilon}{\partial}_K \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K} \sigma - \overset{\kappa, \kappa+\varepsilon}{\partial}_K x_{V, K} \sigma \\ &= -\overset{\kappa, \kappa+\varepsilon}{\partial}_V \overset{\kappa}{\partial}_K \sigma + x_V \overset{\kappa}{\partial}_K \sigma + c_{k-2} \overset{\kappa}{\partial}_{k-1} \overset{\kappa}{\partial}_K \sigma + \overset{\kappa, \kappa+\varepsilon}{\partial}_K \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K} \sigma - \overset{\kappa, \kappa+\varepsilon}{\partial}_K x_{V, K} \sigma = 0 \end{aligned}$$

SO z IS A CYCLE: $z \in \text{KER } \overset{\kappa, \kappa+\varepsilon}{\partial}_K$. NOW ε -GOODNESS TELLS US $(\overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K})_* : \widetilde{H}_K^{\text{CW}}(U_V^\kappa) \rightarrow \widetilde{H}_K^{\text{CW}}(U_V^{\kappa+\varepsilon}) = 0\text{-MAP}$

$$\begin{array}{ccc} & & \frac{\text{KER } \overset{\kappa}{\partial}_K}{\text{IM } \overset{\kappa}{\partial}_{K+1}} \longrightarrow \frac{\text{KER } \overset{\kappa+\varepsilon}{\partial}_K}{\text{IM } \overset{\kappa+\varepsilon}{\partial}_{K+1}} \\ & & \end{array}$$

NOT AT ALL CLEAR $z \in C_K^{\text{CW}}(U_V^\kappa)$

NOW, IF $\overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K}(z) = z$, THEN $0 = \overset{\kappa, \kappa+\varepsilon}{\partial}_K(z) = \overset{\kappa, \kappa+\varepsilon}{\partial}_K \overset{\kappa, \kappa+\varepsilon}{\partial}_{V, K}(z) = \overset{\kappa, \kappa+\varepsilon}{\partial}_V \overset{\kappa}{\partial}_K(z) = \overset{\kappa}{\partial}_K(z)$, SO $\overset{\kappa}{\partial}_K(z) = 0$.

ε -GOODNESS SAYS $z + \text{IM } \overset{\kappa}{\partial}_{K+1} \mapsto z + \text{IM } \overset{\kappa, \kappa+\varepsilon}{\partial}_K = 0$ SO $z \in \text{IM } \overset{\kappa, \kappa+\varepsilon}{\partial}_K \Rightarrow \exists b \in C_{K+1}^{\text{CW}}(U_V^{\kappa+\varepsilon}) : \overset{\kappa, \kappa+\varepsilon}{\partial}_{K+1}(b) = z$.

DEFINE $c_K(\sigma) := b$ \nmid EXTEND BY LINEARITY.

SO, BY LEMMA 8, $\forall \alpha \geq 0$, $\exists \nu \in N\mathbb{N}^\alpha$, $\exists c_\nu^\alpha$ A CHAIN HOMOTOPY BETWEEN $i_V^{\alpha, \alpha+1}, x_V^\alpha : C_*^{CW}(B_V^\alpha) \rightarrow C_*^{CW}(A_V^{\alpha+\epsilon})$

$$\text{By defn: } \partial c_\nu^\alpha + c_\nu^\alpha \delta = i_V^{\alpha, \alpha+1} - x_V^\alpha$$

RECALL. X, Y FINITE CW COMPLEXES, \exists NATURAL ISO $C_*^{CW}(X \times Y) \cong C_*^{CW}(X) \otimes C_*^{CW}(Y) \cong \bigoplus_{p+q=k} [C_p(X) \otimes C_q(Y)]$

SINCE WE HAVE $B^\alpha \subseteq W^\alpha \times N^\alpha$, IT MAKES SENSE TO VIEW $C_*^{CW}(B^\alpha) \subseteq C_*^{CW}(W^\alpha) \otimes C_*^{CW}(N^\alpha) \cong \bigoplus_{p+q=k} C_p^{CW}(W^\alpha) \otimes C_q^{CW}(N^\alpha)$

DEFINE $t := (k+1)\epsilon$. FOR $k \leq K$, DEFINE $\zeta^{\alpha} : C_k^{CW}(B^\alpha) \rightarrow C_{k+1}^{CW}(B^{\alpha+t})$ "EXTEND c_ν^α 'S TO ALL B^α ".

$$\begin{array}{c} \text{P-CELL} \quad \text{Q-CELL} \\ \sim \overbrace{I \otimes \sigma} \sim \rightarrow C_\sigma^\alpha(t) := (c_{V_0}^{\alpha+q\epsilon} \cdots c_{V_n}^{\alpha}) (t) \\ \sigma = V_0 \rightarrow \cdots \rightarrow V_n \end{array}$$

FOR $I \otimes \sigma \in C_k^{CW}(B^\alpha)$, DEFINE $b^\alpha : C_*^{CW}(B^\alpha) \rightarrow C_k^{CW}(N^\alpha)$ BY ; $p^\alpha : C_*^{CW}(B^\alpha) \rightarrow C_*^{CW}(N^\alpha)$ BY

$$I \otimes \sigma \mapsto \begin{cases} I \text{ IF } \dim \sigma = 0 \\ 0 \text{ OTHERWISE} \end{cases} \quad I \otimes \sigma \mapsto \begin{cases} \sigma \text{ IF } \dim I = 0 \\ 0 \text{ OTHERWISE} \end{cases}$$

DEFINE THE CHAIN MAP $\varphi^\alpha : C_*^{CW}(N^\alpha) \rightarrow C_*^{CW}(N^{\alpha+t})$ FOR A BASIS k -SIMPLEX $\sigma \in C_k(N^\alpha)$ TO BE

$$\sigma \mapsto \begin{cases} c^\alpha(x_{V_0} \otimes (\sigma \setminus V_0)) & \text{IF } \dim \sigma \geq 1 \\ x_{V_0} & \text{IF } \dim \sigma = 0 \end{cases}$$

LEMMA 9. THE MAP $\varphi^\alpha : C_*^{CW}(N^\alpha) \rightarrow C_*^{CW}(N^{\alpha+t})$ AS DEFINED ABOVE, WHERE $t := (k+1)\epsilon$ IS A CHAIN MAP $\forall k \leq K$.

DEFINE $\beta^\alpha : C_*^{CW}(B^\alpha) \rightarrow C_*^{CW}(N^{\alpha+t})$ TO BE $\varphi^\alpha \circ b^\alpha$.

$$I \otimes \sigma \mapsto \begin{cases} c^\alpha(x_{V_0} \otimes (\sigma \setminus V_0)) & \text{IF } \dim \sigma \geq 1 \text{ & } \dim I = 0 \\ x_{V_0} & \text{IF } \dim \sigma = 0 \text{ & } \dim I = 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

SO FAR, WE HAVE

$$\begin{array}{ccc} C_k^{CW}(W^\alpha) & \xrightarrow{i_W^{\alpha, \alpha+t}} & C_k^{CW}(W^{\alpha+t}) \\ b^\alpha \uparrow & \nearrow \zeta^{\alpha, \alpha+t} & \downarrow \varphi^\alpha \\ C_k^{CW}(B^\alpha) & \xrightarrow{i_B^{\alpha, \alpha+t}} & C_k^{CW}(B^{\alpha+t}) \\ p^\alpha \downarrow & \searrow \zeta^{\alpha, \alpha+t} & \downarrow p^{\alpha+t} \\ C_k^{CW}(N^\alpha) & \xrightarrow{i_N^{\alpha, \alpha+t}} & C_k^{CW}(N^{\alpha+t}) \end{array}$$

LEMMA 10. ζ^{α} IS A CHAIN HOMOTOPY BETWEEN THE CHAIN MAPS $i_W^{\alpha, \alpha+t}, i_B^{\alpha, \alpha+t} : C_k^{CW}(B^\alpha) \rightarrow C_k^{CW}(N^{\alpha+t}) \quad \forall k \leq K$.

[INTERLEAVING TOOLS] LET X, Y, Z BE SIMPLICIAL COMPLEXES

THE ALEXANDER-WHITNEY UNIQUENESS APPROXIMATION CHAIN MAP $\Delta_* : C_*^{CW}(X) \rightarrow C_*^{CW}(X) \otimes C_*^{CW}(X)$

$$\underline{\sigma} \mapsto \Delta(\sigma) = \sum_{i=0}^k \sigma_i \otimes \bar{\sigma}_i$$

GIVEN A CHAIN MAP $f : C_*^{CW}(X) \otimes C_*^{CW}(Y) \rightarrow C_*^{CW}(Z)$, THE CHAIN MAP $\hat{f} : C_*^{CW}(X) \otimes C_*^{CW}(Y) \rightarrow C_*^{CW}(Z) \otimes C_*^{CW}(Y)$

IS DEFINED BY $\hat{f} := (ff \otimes id_Y) \circ (id_X \otimes \Delta_*)$:

$$C_*^{CW}(X) \otimes C_*^{CW}(Y) \xrightarrow{id_X \otimes \Delta_*} C_*^{CW}(X) \otimes C_*^{CW}(Y) \otimes C_*^{CW}(Y) \xrightarrow{f \otimes id_Y} C_*^{CW}(Z) \otimes C_*^{CW}(Y)$$

WE CALL \hat{f} THE LIFT OF f

WE CAN LIFT THE FOLLOWING DIAGRAM :

$$\begin{array}{ccc}
 C_*^{CW}(W^\alpha) & \xrightarrow{i_W^{\alpha, \alpha+t}} & C_*^{CW}(W^{\alpha+t}) \\
 \downarrow b^\alpha & \nearrow \hat{q}^\alpha & \downarrow b^{\alpha+t} \\
 C_*^{CW}(B^\alpha) & \xrightarrow{i_B^{\alpha, \alpha+t}} & C_*^{CW}(B^{\alpha+t}) \\
 \downarrow p^\alpha & \nearrow \hat{q}^\alpha & \downarrow p^{\alpha+t} \\
 C_*^{CW}(N^\alpha) & \xrightarrow{i_N^{\alpha, \alpha+t}} & C_*^{CW}(N^{\alpha+t})
 \end{array}
 \quad
 \begin{array}{ccc}
 C_*^{CW}(W^\alpha) & \longrightarrow & C_*^{CW}(W^{\alpha+t}) \\
 \downarrow b^\alpha & & \downarrow b^{\alpha+t} \\
 C_*^{CW}(B^\alpha) & \xrightarrow{i_B^{\alpha, \alpha+t} \circ b^\alpha} & C_*^{CW}(B^{\alpha+t}) \\
 \downarrow p^\alpha & \nearrow \hat{q}^\alpha & \downarrow p^{\alpha+t} \\
 C_*^{CW}(N^\alpha) & \longrightarrow & C_*^{CW}(N^{\alpha+t})
 \end{array}$$

IT TAKES SOME JUSTIFICATION

$$\text{TO SEE } \hat{q}^\alpha \subseteq C_*^{CW}(B^{\alpha+t})$$

$$\text{NOTE: } \hat{q}^\alpha = \hat{q}^\alpha p^\alpha$$

$$\hat{q}^{\alpha+t} = p^{\alpha+t} \hat{q}^\alpha$$

$$\hat{q}^{\alpha+t} b^\alpha = p^{\alpha+t} \hat{q}^\alpha$$

LEMMA 11: IF $f, g : C_*^{CW}(X) \otimes C_*^{CW}(Y) \rightarrow C_*^{CW}(Z)$ ARE CHAIN HOMOTOPIC, THEN $\hat{f} \circ \hat{g}$ ARE CHAIN HOMOTOPIC.

★ THEOREM 12: [GENERALIZED PERSISTENT NERVE THEOREM] GIVEN A FINITE COLLECTION OF FINITE

SIMPLICIAL FILTRATIONS $\mathcal{U} = \{U_0, \dots, U_n\}$, WHERE $U_i := (U_i^\alpha)_{\alpha \geq 0}$; ALL U_i^α ARE SUBCOMPLEXES

OF A SUFFICIENTLY LARGE SIMPLICIAL COMPLEX, IF \mathcal{U} IS AN ε -GOOD COVER FILTRATION OF

$W = (\bigcup_{i=0}^n U_i^\alpha)_{\alpha \geq 0}$, THEN $d_B(DGM_K(W), DGM_K(N^{\mathcal{U}})) \leq (k+1)\varepsilon$.

PROOF. SINCE b^α IS A HOMOTOPY EQUIVALENCE \Rightarrow , WE HAVE NATURAL ISOMORPHISMS $H_*^{CW}(B^\alpha) \cong H_*^{CW}(W^\alpha)$ so $DGM(B) = DGM(W)$.

Fix $k \geq 0$. WE CONSTRUCT A $(k+1)\varepsilon$ -INTERLEAVING BETWEEN $H_K^{CW}(B) \circlearrowleft H_K^{CW}(W)$. DEFINE $t = (k+1)\varepsilon$. CONSIDER

THE CHAIN MAPS $p := (p^{\alpha+t} i_B^{\alpha, \alpha+t})_{\alpha \geq 0} : C_*^{CW}(B) \rightarrow C_*^{CW}(N^{\alpha+t})$; $\hat{q} := (\hat{q}^\alpha)_{\alpha \geq 0} : C_*^{CW}(N^\alpha) \rightarrow C_*^{CW}(B^{\alpha+t})$

BY LEMMA 10, $\hat{q}^\alpha \simeq i_N^{\alpha, \alpha+t} b^\alpha \Rightarrow$ BY LEMMA 11, $\hat{q}^\alpha \simeq \widehat{i_B^{\alpha, \alpha+t} b^\alpha} = i_B^{\alpha, \alpha+t}$

SO, FOR $k \geq 0$, $\hat{q}^{\alpha+t} i_B^{\alpha, \alpha+t} = \hat{q}^\alpha i_B^{\alpha, \alpha+t} \simeq i_B^{\alpha+t, \alpha+t} i_B^{\alpha, \alpha+t} = i_B^{\alpha, \alpha+2t}$

$\therefore p^{\alpha+2t} i_B^{\alpha+2t, \alpha+2t} \hat{q}^\alpha = i_N^{\alpha+2t, \alpha+2t} p^{\alpha+2t} \hat{q}^\alpha = i_N^{\alpha+2t, \alpha+2t} i_B^{\alpha, \alpha+2t} = i_N^{\alpha, \alpha+2t}$

APPLYING K -DIMENSIONAL CELLULAR HOMOLOGY TO $p \circ \hat{q}$, WE HAVE

$$H_K^{CW}(\hat{q}^{\alpha+t}) H_K^{CW}(p^{\alpha+t} i_B^{\alpha, \alpha+t}) = H_K^{CW}(i_B^{\alpha, \alpha+2t})$$

$$\therefore H_K^{CW}(p^{\alpha+2t} i_B^{\alpha+2t, \alpha+2t}) H_K^{CW}(\hat{q}^\alpha) = H_K^{CW}(i_B^{\alpha, \alpha+2t})$$

SO $\forall \alpha \geq 0$, $H_K^{CW}(p) \circ \hat{q}$ FORM A $(k+1)\varepsilon$ -INTERLEAVING BETWEEN $H_K^{CW}(B) \circlearrowleft H_K^{CW}(N)$

NOTE $N^\alpha \cong |N^{\mathcal{U}}|$ so by THEOREM 2, $DGM(N) = DGM(N^{\mathcal{U}})$.

Also, $H_K^{CW}(N) \cong H_K(N)$ SINCE CELLULAR \circlearrowleft SIMPLICIAL HOMOLOGY AGREE FOR SIMPLICIAL COMPLEXES

FINALLY, SINCE $H_K^{CW}(B) \circlearrowleft H_K^{CW}(N) \cong H_K(N)$ ARE $(k+1)\varepsilon$ -INTERLEAVED, BY THE STABILITY THEOREM,

$$d_B(DGM_K(B), DGM_K(N^{\mathcal{U}})) < (k+1)\varepsilon$$

$$d_B(DGM_K(N), DGM_K(N^{\mathcal{U}})) < (k+1)\varepsilon$$

WE CAN ALSO CONCLUDE $d_B(DGM(N), DGM(N^{\mathcal{U}})) \leq (D+1)\varepsilon$ WHERE D IS THE MAXIMUM DIMENSION OF $N^{\mathcal{U}}$.