

DIAGONAL ARGUMENTS $\hat{=}$ CARTESIAN CLOSED CATEGORIES [LAWVERE, 69] \odot

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RECALL: $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ ADJUNCTION

$$\rho(Fc, d) \cong \mathcal{C}(c, Gd)$$

NATURAL IN $c; d$ \perp

1. CARTESIAN CLOSED CATEGORIES.

C.C.C.'s CAN BE COMPLETELY DESCRIBED BY ADJUNCTIONS:

DEFN: \mathcal{C} IS A C.C.C. PROVIDED

① \mathcal{C} HAS A TERMINAL 1 .

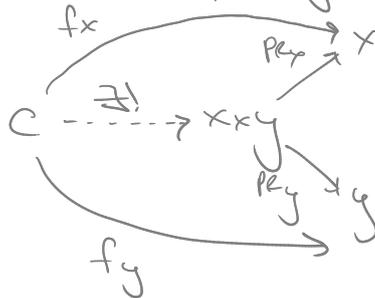
THE UNIQUE $\mathcal{C} \rightarrow 1$ HAS A RIGHT ADJOINT $1 \rightarrow \mathcal{C}$:

$$* \cong 1(1(\mathcal{C}), *) \cong \mathcal{C}(c, T^*)$$

② \mathcal{C} HAS FINITE PRODUCTS

THE DIAGONAL $\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C}$ HAS A RIGHT ADJOINT $\mathcal{C} \times \mathcal{C} \xrightarrow{x} \mathcal{C}$:

$$\mathcal{C} \times \mathcal{C}(\Delta(c) = (c, c), (x, y)) \cong \mathcal{C}(c, x \times y)$$



$$\begin{aligned} \mathcal{C} \times \mathcal{C}(\Delta(c), (x, y)) &\xrightarrow{\cong} \mathcal{C}(c, x \times y) \\ (f_x, f_y) &\xrightarrow{\exists!} z: \mathcal{C} \rightarrow x \times y \end{aligned}$$

$$\begin{aligned} \mathcal{C} \times \mathcal{C}(\Delta(x \times y), (x, y)) &\xrightarrow{\cong} \mathcal{C}(x \times y, x \times y) \\ (pr_x, pr_y) &\xleftarrow{\text{id}_{x \times y}} \end{aligned}$$

$$\begin{aligned} \mathcal{C} \times \mathcal{C}((x \times y, x \times y), (x, y)) &\xrightarrow{\cong} \mathcal{C}(x \times y, x \times y) \\ \downarrow z^* & \quad \quad \quad \downarrow z^* \\ \mathcal{C} \times \mathcal{C}((c, c), (x, y)) &\xrightarrow{\cong} \mathcal{C}(c, x \times y) \end{aligned}$$

NATURALITY + $c \xrightarrow{z} x \times y$

$$\begin{aligned} (pr_x z, pr_y z) &\xrightarrow{*} z \\ \text{BUT } z \xrightarrow{*} (f_x, f_y) &\xrightarrow{*} (pr_x z, pr_y z) = (f_x, f_y). \end{aligned}$$

② \mathcal{C} HAS EXPONENTIAL OBJECTS

②

$\forall B \in \mathcal{C}$, THE FUNCTOR $- \times B : \mathcal{C} \rightarrow \mathcal{C}$ HAS A RIGHT ADJOINT $-^B : \mathcal{C} \rightarrow \mathcal{C}$

$$\lambda \quad \mathcal{C}(A \times B, C) \cong \mathcal{C}(A, C^B)$$

UNIT: $\eta : \text{id}_{\mathcal{C}} \Rightarrow (- \times B)^B : \mathcal{C} \rightarrow \mathcal{C}$

CURRY = $\zeta_A : A \rightarrow (A \times B)^B : a \mapsto B \rightarrow A \times B$
 $b \mapsto (a, b)$

COUNIT: $\epsilon : -^B \times B \Rightarrow \text{id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$

APPLY = $\epsilon_A : A \times B \rightarrow A$
 $(f, b) \mapsto f(b)$.

λ -TRANSFORM: $A \times X \xrightarrow{f} Y \rightsquigarrow X \rightarrow Y^*$, $x \mapsto (a \mapsto f(a, x))$

EXAMPLES: ① SETS.

② "NICE" SPACES: $\text{TOP}(X \times Y, Z) \cong \text{TOP}(X, \text{TOP}(Y, Z))$

PROVIDED Y LOCALLY COMPACT. e.g., COMPACTLY GENERATED WEAK-HAUSDORFF SPACES.

③ BOOLEAN ALGEBRAS \rightsquigarrow LATTICE \rightsquigarrow POSET \rightsquigarrow CATEGORY

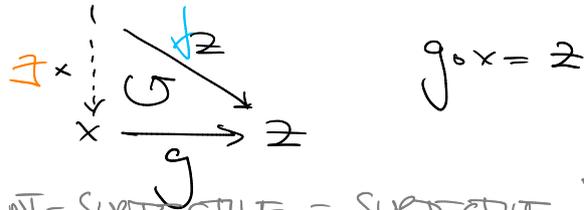
④ ANY TOPOS

2. LAWVERE FIXED POINT THEOREM

(3)

DEFN: A morphism $1 \rightarrow X$ in \mathcal{C} is called a GLOBAL ELEMENT in X . IN SET, $1 \xrightarrow{x} X \iff x \in X$.

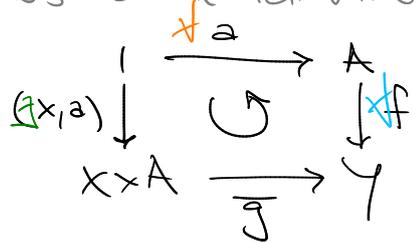
DEFN: A morphism $X \xrightarrow{g} Z$ is POINT-SURJECTIVE \iff FOR EVERY $1 \xrightarrow{x} X$ THERE EXISTS $1 \rightarrow Z$ WITH $g(x) = z$



IN SETS, POINT-SURJECTIVE = SURJECTIVE. NOT IN GENERAL: $[e^{\mathcal{C}}, \text{SETS}]$

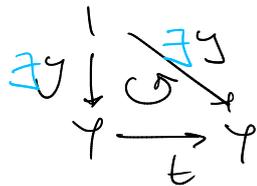
DEFN: A morphism $X \xrightarrow{g} Y^A$ is WEAKLY POINT-SURJECTIVE $\iff \forall A \xrightarrow{f} Y, \exists 1 \xrightarrow{x} X : \exists 1 \rightarrow A, g(x)(a) = f(a)$

USING THE ADJUNCTION $\mathcal{C}(X \times A, Y) \cong \mathcal{C}(X, Y^A)$, THIS CONDITION CORRESPONDS TO THE FOLLOWING DIAGRAM COMMUTING:



THINK OF \bar{g} AS SERIES OF FUNCTIONS $\bar{g}(-, a) = f$
 „EXTENSIONALLY EQUAL“

DEFN: An object $Y \in \mathcal{C}$ HAS THE FIXED POINT PROPERTY \iff FOR EVERY ENDO-MORPHISM $Y \xrightarrow{t} Y$ THERE EXISTS $1 \xrightarrow{y} Y : ty = y$.



THEOREM [LAWVERE]. IN ANY C.C.C. IF THERE EXISTS AN OBJECT A \exists A WEAKLY POINT-SURJECTIVE MORPHISM $A \xrightarrow{j} Y^A$, THEN Y HAS THE FIXED POINT PROPERTY ⑦

PROOF. LET $\bar{g}: A \times A \rightarrow Y$ BE MAP WHOSE λ -TRANSFORM IS g
 LET $Y \xrightarrow{t} Y$ BE ANY ENDMORPHISM. DEFINE $A \xrightarrow{f} Y$ AS FOLLOWS

$$\begin{array}{ccc} A \times A & \xrightarrow{\bar{g}} & Y \\ \Delta \uparrow & & \downarrow t \\ A & \xrightarrow{f} & Y \\ & f := t \bar{g} \Delta & \end{array}$$

SINCE g SWAPS, $\exists 1 \xrightarrow{x} A : \forall 1 \xrightarrow{a} A, \bar{g}(x, a) = f(a)$

$$\begin{array}{ccc} 1 & \xrightarrow{a} & A \\ (x, a) \downarrow & \circlearrowleft & \downarrow f \\ A \times A & \xrightarrow{\bar{g}} & Y \end{array}$$

IN PARTICULAR, IF $a = x$, THEN $\bar{g}(x, a) = \bar{g}(x, x) = f(x) = t \bar{g}(x, x)$
 $\therefore \bar{g}(x, x)$ IS A FIXED POINT OF t . $\therefore Y$ HAS F.P.P. ■

3. MAIN RESULTS

COROLLARY: [CANTOR'S THEOREM] IF $\exists Y \xrightarrow{t} Y : t y \neq y \forall 1 \xrightarrow{j} Y$
 THEN FOR NO A DOES THERE EXIST A POINT-SURJECTIVE MORPHISM $A \rightarrow Y^A$.

IN SET, $Y := 2 = \{ \text{FALSE, TRUE} \}$, THE SUBOBJECT CLASSIFIER OF SET,
 FOR NO A DOES THERE EXIST A SURJECTION $A \rightarrow 2^A = \mathcal{P}(A)$

MOREOVER, IN ANY (NON-DEGENERATE) TOPOS, WE CAN REPLACE Y W/ Ω ,
 ITS SUBOBJECT CLASSIFIER TO GET A "CANTOR'S THEOREM" FOR THE TOPOS.

⑤

WE CAN DO AWAY, IN SOME SENSE, WITH THE REQUIREMENT THAT C BE A C.C.C.

THEOREM: [CANTOR'S THEOREM]. IF THERE EXISTS $Y \xrightarrow{t} Y : \forall 1 \xrightarrow{g} Y, t \circ g \neq g$, THEN THERE DOES NOT EXIST $\bar{g}: A \times A \rightarrow Y : \forall 1 \xrightarrow{f} A, \exists 1 \xrightarrow{x} A : \forall 1 \xrightarrow{a} A, \bar{g}(a, x) = f(a)$.

PF: THE CONTRAPOSITIVE IS PRECISELY THE PROOF OF LAWIÈRE'S THEOREM.

REMARK: IF MOREOVER C IS A C.C.C, THE CONCLUSION ON \bar{g} IS PRECISELY SAYING g IS NOT WEAKLY POINT-SURJECTIVE.

THEOREM: [CANTOR $\mathbb{N} \not\subseteq 2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$]. THERE IS NO SURJECTIVE $\mathbb{N} \rightarrow 2^{\mathbb{N}}$.

PROOF: SUPPOSE THERE IS: $\{S_0, S_1, S_2, \dots\} = 2^{\mathbb{N}}$. WHERE $2 := \{0, 1\}$

CONSIDER THE NEGATION FUNCTION $\neg: 2 \rightarrow 2 : 0 \leftrightarrow 1$. LET

$\mathbb{N} \times \mathbb{N} \xrightarrow{f} 2$ BE DEFINED AS

$$f(n, m) := \begin{cases} 1 & \text{IF } n \in S_m, \\ 0 & \text{IF } n \notin S_m. \end{cases}$$

FOR EACH m , $f(-, m) = \chi_{S_m}$. CONSTRUCT $\mathbb{N} \xrightarrow{g} 2$ AS FOLLOWS

$$\begin{array}{ccc} \mathbb{N} \times \mathbb{N} & \xrightarrow{f} & 2 \\ \Delta \uparrow & & \downarrow \neg \\ \mathbb{N} & \xrightarrow{g := \neg f \Delta} & 2 \end{array}$$

$g = \chi_{G_1}$, WHERE $G_1 := \{n \in \mathbb{N} : n \notin S_n\}$.

FOR ALL m , $\chi_{G_1} = g(-) \neq f(-, m) = \chi_{S_m}$, SINCE \neg HAS NO FIXED POINTS IN PARTICULAR, G_1 IS NOT IN THE ENUMERATION OF $2^{\mathbb{N}}$.

[RUSSELL'S PARADOX]. "THE SET OF ALL SETS THAT ARE NOT MEMBERS OF THEMSELVES IS BOTH A MEMBER OF ITSELF; NOT A MEMBER OF ITSELF."

⑥

LET \mathcal{U} BE SOME UNIVERSE OF SETS. CONSIDER $\neg: 2 \rightarrow 2$, $0 \leftrightarrow 1$.

LET $\mathcal{U} \times \mathcal{U} \xrightarrow{f} 2$ BE DEFINED BY

$$f(x, y) = \begin{cases} 1 & \text{IF } x \in y, \\ 0 & \text{IF } x \notin y. \end{cases}$$

CONSTRUCT $\mathcal{U} \xrightarrow{g} 2$ AS FOLLOWS:

$$\begin{array}{ccc} \mathcal{U} \times \mathcal{U} & \xrightarrow{f} & 2 \\ \Delta \uparrow & & \neg \downarrow \\ \mathcal{U} & \dashrightarrow & 2 \\ & g = \neg f \Delta & \end{array}$$

g IS THE CHARACTERISTIC FUNCTION OF SETS WHICH ARE NOT MEMBERS OF THEMSELVES.

FOR ALL $Y \in \mathcal{U}$, $g(-) \neq f(-, Y)$ SINCE \neg HAS NO FIXED POINTS.

KEY: IN ORDER TO MAKE SURE THERE ARE NO PARADOXES, WE MUST SAY

g IS THE CHARACTERISTIC FUNCTION OF A "COLLECTION" OF SETS, WHICH IS NOT ITSELF A SET!

THEOREM [DIAGONAL THEOREM] IF THERE EXISTS $\bar{g}: A \times A \rightarrow Y$:

$\forall A \xrightarrow{f} Y$, THERE EXISTS $1 \xrightarrow{x} A$: $\forall 1 \xrightarrow{a} A$, $\bar{g}(a, x) = f(a)$,

THEN Y HAS THE FIXED POINT PROPERTY.

PROOF: PRECISELY THE PROOF OF LAWRIER'S THEOREM.

REMARK. OFTEN WE DON'T NEED TO HAVE A C.C.C. ON THE NOSE; A CATEGORY W/ FINITE PRODUCTS SUFFICES. THE DEFN OF WPS DOESN'T REQUIRE EXPONENTIALS. HOWEVER, THEY ARE THE NATURAL HOME FOR THESE CONCEPTS. NO MATTER GIVEN A, Y IN CAT. + FIN. PRODS, WE CAN FORM FULL SUBCAT GEN BY THEM (SMALL); THEN APPLY YONEDA EMBEDDING TO PUSH INTO C.C.C. IN A NATURAL WAY.

4. Gödel's INCOMPLETENESS THEOREM.

⑦

DEFIN [LINDENBAUM-TARSKI, CLASSIFYING CATEGORY] LET T BE A FIRST ORDER THEORY. WE FORM ITS CLASSIFYING CATEGORY $\mathcal{C} := \mathcal{C}(T)$ THUS:
OBJECTS OF \mathcal{C} ARE GENERATED BY A SORT OBJECT A ; A DUMMY OBJECT 2 , BY CLOSURE UNDER PRODUCTS. $\} = A^n \times 2^m \}$

Morphisms

φ	EQUIVALENCE CLASSES OF ...
$A^n \rightarrow 2$	PROBABLY EQUIVALENT WELL-FORMED FORMULAS OF n -VARIABLES
$A^n \rightarrow 2 \times 2$	PROBABLY EQUIVALENT TUPLES OF WFS OF n -VARIABLES
$A^n \rightarrow A$	PROBABLY EQUAL TERMS w/ n FREE VARS
$1 \rightarrow 2$	SENTENCES = CLOSED WFS PROVABLE IN T
$1 \rightarrow A$	DEFINABLE CONSTANT TERMS
TRUE = $T : 1 \rightarrow 2$	SENTENCES PROVABLE IN T
FALSE = $\perp : 1 \rightarrow 2$	SENTENCES WHOSE NEGATION IS PROVABLE IN T
$2^n \rightarrow 2$	All propositional operations, e.g., $\neg : 2 \rightarrow 2$.

DEFIN. A THEORY T IS CONSISTENT IF THERE IS A MORPHISM

$$\neg : 2 \rightarrow 2 : \neg \varphi \neq \varphi \text{ FOR ALL SENTENCES } 1 \xrightarrow{\varphi} 2.$$

REMARK: A THEORY IS CONSISTENT IF $\mathcal{C}(1,2)$ CONTAINS AT LEAST TWO ELEMENTS: TRUE, FALSE.

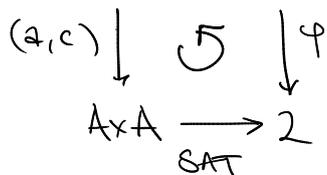
DEFIN: A THEORY IS COMPLETE IF $\mathcal{C}(1,2)$ IS EXACTLY $\{\text{TRUE, FALSE}\}$, I.E.; EVERY SENTENCE IS EITHER PROVABLE OR REFUTABLE.

DEFN. A FORMULA IS SATISFIABLE IF IT IS POSSIBLE TO FIND AN INTERPRETATION OF THE MODEL MAKING THE FORMULA TRUE.

DEFN. SATISFACTION IS DEFINABLE IN $T \iff$ THERE IS A BINARY FORMULA

$SAT : A \times A \rightarrow 2$ IN $\mathcal{C} : \text{FOR EVERY UNARY FORMULA } A \xrightarrow{\varphi} 2,$
 THERE IS A CONSTANT $1 \xrightarrow{c} A : \text{FOR EVERY CONSTANT } 1 \xrightarrow{a} A,$

$1 \xrightarrow{a} A$ IMAGINE $c = \text{Gödel \# FOR A REPR OF } \varphi, c = \ulcorner \varphi \urcorner.$



TRADITIONALLY: $\vdash_T SAT(a, c) \iff \varphi a.$

BUT THIS PRECISELY MEANS THE DIAGRAM COMMUTES IN $\mathcal{C}!$

REMARK: A GÖDEL ENCODING CAN BE DESCRIBED AS FOLLOWS:

$$\ulcorner \cdot \urcorner : \underbrace{\mathcal{C}(A^n, 2)} \longrightarrow \underbrace{\mathcal{C}(1, A)}$$

FORMULAS OF n -VARS DEFINABLE CONSTANT TERMS

COROLLARY: IF SATISFACTION IS DEFINABLE IN THE THEORY, THEN THE THEORY IS NOT CONSISTENT.

PROOF: SUPPOSE, THE SATISFIABILITY PREDICATE IS DEFINABLE IN T . THEN PRECISELY THE CONDITION FOR WEAK POINT-SURJECTIVITY IS SATISFIED! WHICH MEANS THAT EVERY MAP $2 \rightarrow 2$ HAS A FIXED POINT, IN PARTICULAR, \exists SENTENCE $\varphi : 1 \rightarrow 2 : \ulcorner \varphi \urcorner = \varphi \implies T$ IS INCONSISTENT.

DEFN TRUTH IS DEFINABLE IN A THEORY IF THERE IS A FORMULA

$TRUTH : A \rightarrow 2$, CALLED THE TRUTH PREDICATE, IF

$$\mathcal{C}(1, TRUTH) : \underbrace{\mathcal{C}(1, A)}_{\text{CONST TERMS}} \rightarrow \underbrace{\mathcal{C}(1, 2)}_{\text{SENTENCES}}$$

IS A RETRACT OF

$$\ulcorner \cdot \urcorner : \underbrace{\mathcal{C}(1, 2)}_{\text{SENTENCES}} \longrightarrow \underbrace{\mathcal{C}(1, A)}_{\text{CONSTANT TERMS}}$$

i.e., $TRUTH \circ \ulcorner \cdot \urcorner = \text{id}$.

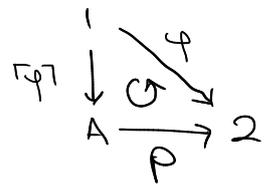
DEFN. A Theory T supports SUBSTITUTION IF THERE IS A BINARY TERM $\textcircled{1}$
 $\text{SUBST}: A \times A \rightarrow A$ SUCH THAT $\vdash_T \text{SUBST}(a, \ulcorner \varphi \urcorner) = \ulcorner \varphi(a) \urcorner$
 FOR ALL $\underbrace{A \xrightarrow{\varphi} 2}_{\text{FORMULA}}$; ALL $\underbrace{1 \xrightarrow{a} A}_{\text{CONSTANT TERM}}$ $\vdash \ulcorner 1 \xrightarrow{a} A \xrightarrow{\varphi} 2 \text{ SENTENCE} \urcorner$
 $\vdash \ulcorner \varphi(a) \urcorner$ ITS GÖDEL #

THEOREM. IF A THEORY T IS CONSISTENT WITH SUBSTITUTION, THEN TRUTH IS NOT DEFINABLE.

PROOF: SUPPOSE TRUTH IS DEFINABLE, $\text{TRUTH}: A \rightarrow 2$.
 DEFINE $\text{SAT} := \text{TRUTH} \circ \text{SUBST}: A \times A \xrightarrow{\text{SUBST}} A \xrightarrow{\text{TRUTH}} 2$
 $\text{SAT}(a, \ulcorner \varphi \urcorner) = \text{TRUTH} \circ \text{SUBST}(a, \ulcorner \varphi \urcorner)$
 $= \text{TRUTH}(\ulcorner \varphi(a) \urcorner)$
 $= \varphi(a)$

BUT THEN SATISFACTION IS DEFINABLE, CONTRADICTION. ■

DEFN. A PROVABILITY PREDICATE IS A PREDICATE $\underbrace{P: A \rightarrow 2}_{\text{FORMULA}}$
 SUCH THAT $\forall \underbrace{1 \xrightarrow{\varphi} 2}_{\text{SENTENCE}}$, $P \circ \ulcorner \varphi \urcorner = \varphi$
 PROVIDED $P \circ \ulcorner \varphi \urcorner$, φ TAKE VALUES IN $\{\text{TRUE}, \text{FALSE}\}$.



EQUIVALENTLY, $\vdash_T P(\ulcorner \varphi \urcorner) \iff \vdash_T \varphi$.

THEOREM: IF A THEORY T IS CONSISTENT WITH SUBSTITUTION, THEN IT IS NOT COMPLETE.

PROOF. IF T IS COMPLETE THEN THE PROVABILITY PREDICATE IS ALSO A TRUTH PREDICATE, CONTRADICTION.