

SPECTRAL SEQUENCES ; TOPOLOGICAL COVERS

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1. A LITTLE INTRODUCTION TO SPECTRAL SEQUENCES

- DIAGRAM CHASES
- BALANCE TEST/TOR LEFT DERIVED FUNCTORS

2. TOPOLOGICAL COVERS ; MAAYER-VIEToris

- RECALL M.V. L.E.S.
- EXTEND THIS TO ARBITRARY COVERS
- SET UP MAAYER-VIEToris DOUBLE COMPLEX
- DESCRIBE M.V. S.S. + „PROOF”.
- CALCULATIONS

3. HOMOTOPY COLIMIT SPECTRAL SEQUENCE

[ADVANCED]



- SIMPLICIAL OBJECTS, BOLD-KAN CORRESPONDENCE
- COLIMITS, COENDS, FUNCTOR TENSOR PRODUCT
- HOMOTOPY COLIMITS
- BousFIELD-KAN SPECTRAL SEQUENCE ; SENAL ; DUGGER ; VOLT.

4. FURTHER INVESTIGATIONS !!

VIEWED AS FUNCTORS \rightsquigarrow FUNCTOR TENSOR PRODUCT

- TENSOR PRODUCTS OF PERSISTENCE MODULES \rightsquigarrow TOR LEFT DERIVED FUNCTORS
- SPECTRAL SEQUENCES OF PERSISTENCE MODULES \rightsquigarrow INVESTIGATE VALUE ; IMPORT ; USEFULNESS.

A LITTLEIFY INTRODUCTION TO SPECTRAL SEQUENCES (A)

①

DEFN: A **BIGRAGED MODULE** IS A DOUBLY INDEXED FAMILY $M = (M_{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$ OF \mathbb{Q} -MODULES [MORE CONCRETELY: VECTOR SPACES; MORE GENERALLY, OBJECTS IN ABELIAN CATEGORY]

WE WRITE $M_{\bullet, \bullet}$:

$$\begin{array}{ccccccc} \dots & \vdots & \vdots & \vdots & \ddots & & \\ \dots & M_{p_1, q+1} & M_{p_1, q+1} & M_{p_1, q+1} & \dots & & \\ & & & & & & \end{array}$$

$$\begin{array}{ccccccc} \dots & M_{p-1, q} & M_{p, q} & M_{p+1, q} & \dots & & \\ & & & & & & \end{array}$$

$$\begin{array}{ccccccc} \dots & M_{p_1, q-1} & M_{p_1, q-1} & M_{p_1, q-1} & \dots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ & \text{BiDegree} & & & & & \end{array}$$

DEFN: M, N BIGRAGED MODULES, $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, A **BIGRAGED MAP OF DEGREE (a, b)** , DENOTED BY $f: M \rightarrow N$, IS A FAMILY OF HOMOMORPHISMS

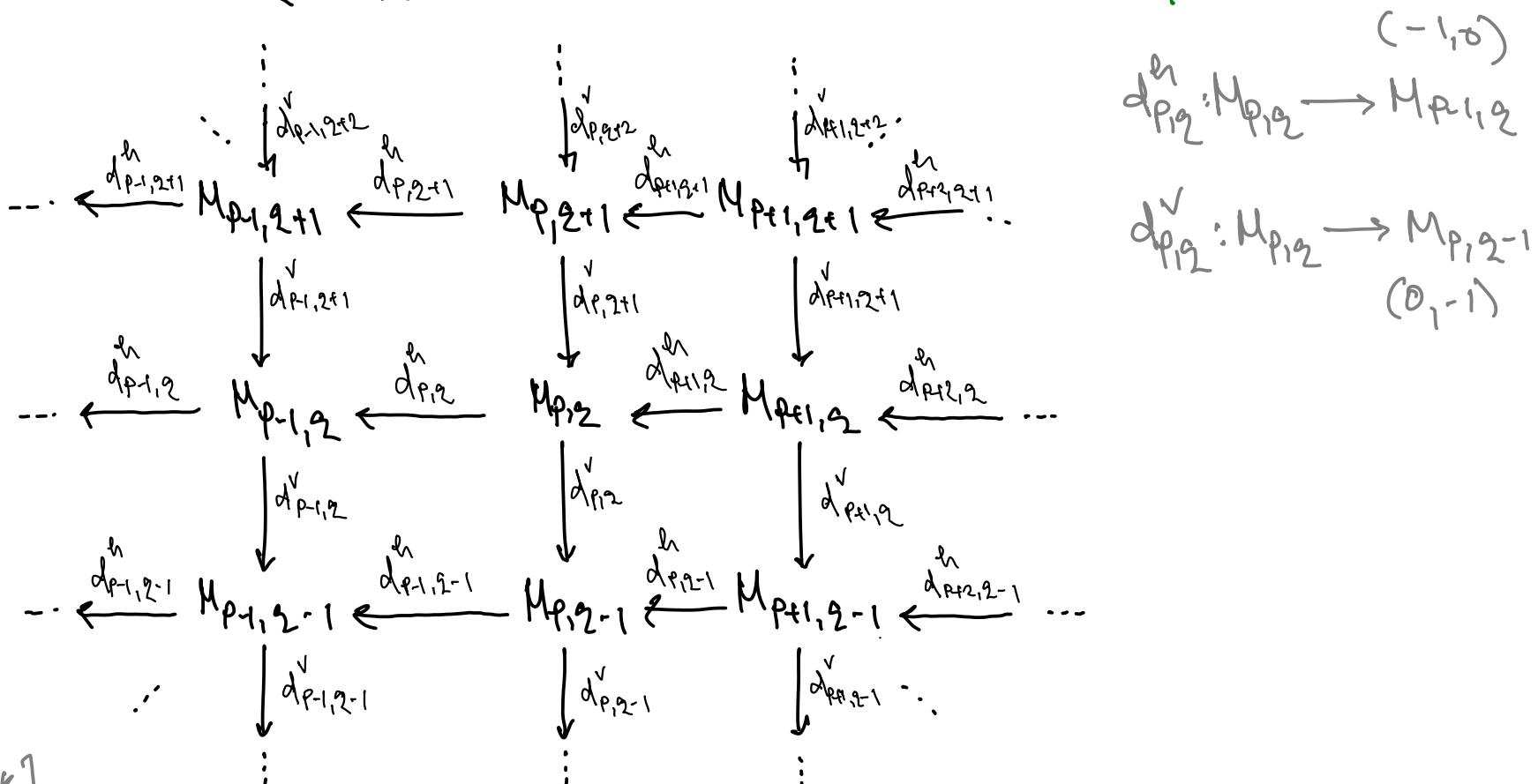
$$f = (f_{p,q}: M_{p,q} \rightarrow N_{p+a, q+b})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}.$$

A LITTLE INTRODUCTION TO SPECTRAL SEQUENCES (B)

(2)

EXAMPLE: A DOUBLE COMPLEX IS AN ORDERED TRIPLE (M, d^h, d^v) WHERE $M = (M_{p,q})$ IS A BIGRADED MODULE, $d^h, d^v : M \rightarrow M$ ARE DIFFERENTIALS OF BI-DEGREE $(-1, 0)$ & $(0, -1)$, RESPECTIVELY :

$$d_{p,q-1}^h d_{p,q}^v + d_{p-1,q}^v d_{p,q}^h = 0 \quad [\text{ANTI-COMMUTATIVITY}]$$



[SIGN CHANGE TRICK]

COMMENT: GIVEN ANY COMMUTATIVE DIAGRAM, WE CAN CONVERT IT INTO A DOUBLE COMPLEX BY CHANGING SIGNS OF COLUMN MORPHISMS APPROPRIATELY.

A LITTLEIFY INTRODUCTION TO SPECTRAL SEQUENCES (C)

(3)

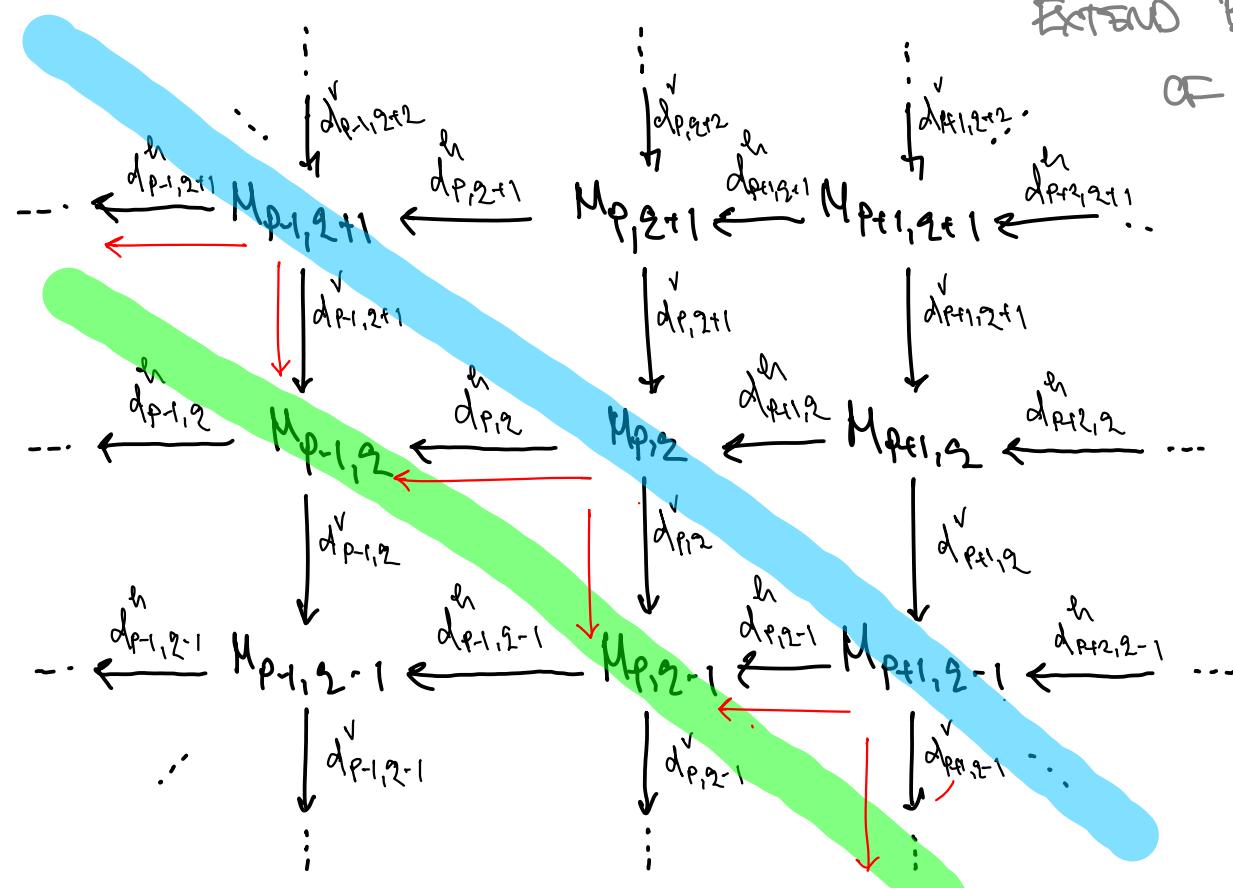
DEFN: ASSOCIATED TO A DOUBLE COMPLEX, THERE IS A CHAIN COMPLEX CALLED THE TOTAL COMPLEX DENOTED BY $\text{TOT}_*(M)$, DEFINED BY

$$\text{TOT}_n(M) := \bigoplus_{p+q=n} M_{p,q}$$

WITH DIFFERENTIALS:

$$D_n: \text{TOT}_n(M) \longrightarrow \text{TOT}_{n-1}(M)$$

$$= \sum_{p+q=n} d_{p,q}^h + d_{p,q}^v$$



Explicitly: ON COMPONENTS,
 $M_{p,q} \xrightarrow{\quad} M_{p-1, q} \oplus M_{p, q-1}$
 $x \longmapsto (d_{p,q}^h(x), d_{p,q}^v(x))$

EXTEND BY UNIVERSAL PROPERTY
 OF COHOMOLOGY

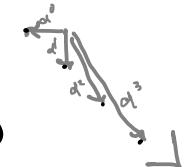
A HESITANT INTRODUCTION TO SPECTRAL SEQUENCES (1)

④

DEFN. A SPECTRAL SEQUENCE IS A SEQUENCE $(E^r, d^r)_{r \geq 0}$ OF DIFFERENTIAL BIGRADED MODULES : $E^{r+1} \cong H(E^r, d^r)$

DUAL NOTION: COHOMOLOGY S.S. : $H^r(E^r, d^r) = (-r, r-1)$

DEFN: WE SAY A HOMOLOGY SPECTRAL SEQUENCE IS ONE WITH $H^r(E^r, d^r) = (-r, r-1)$



E⁰-PAGE

$$\begin{array}{c} \vdots \\ \downarrow \\ \cdots E^0_{p_1, q+1} \\ d^0 \downarrow \\ \cdots E^0_{p_1, q} \\ d^0 \downarrow \\ \cdots E^0_{p_1, q-1} \\ \downarrow \\ \vdots \end{array}$$

E¹-PAGE

$$\begin{array}{c} \vdots \\ \downarrow \\ \cdots E^1_{p+1, q+1} \\ d^1 \downarrow \\ \cdots E^1_{p+1, q} \\ d^1 \downarrow \\ \cdots E^1_{p+1, q-1} \\ \downarrow \\ \vdots \end{array}$$

E²-PAGE

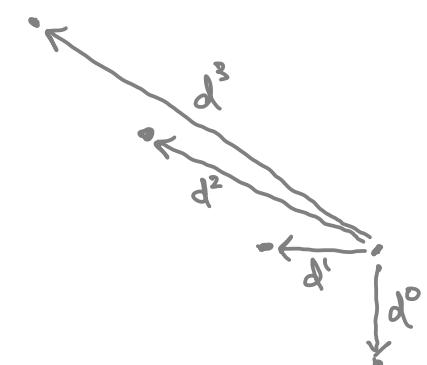
$$\begin{array}{c} \vdots \\ \downarrow \\ \cdots E^2_{p+2, q+1} \\ d^2 \downarrow \\ \cdots E^2_{p+2, q} \\ d^2 \downarrow \\ \cdots E^2_{p+2, q-1} \\ \downarrow \\ \vdots \end{array}$$

To illustrate:

Take Homology

Slope: $(1-r)/r$

$$\begin{array}{ccccc} E^0_{p_1, q+1} & \xrightarrow{d^0_{p_1, q+1}} & E^0_{p_1, q} & \xrightarrow{d^0_{p_1, q}} & E^0_{p_1, q-1} \\ \text{ker } d^0_{p_1, q-1} / \text{im } d^0_{p_1, q+1} & & & & \\ E^1_{p+1, q} & \xrightarrow{d^1_{p+1, q}} & E^1_{p+1, q-1} & & \\ \text{ker } d^1_{p+1, q} / \text{im } d^1_{p+1, q} & & & & \\ E^2_{p+2, q-1} & \xrightarrow{d^2_{p+2, q-1}} & E^2_{p+2, q} & \xrightarrow{d^2_{p+2, q}} & E^2_{p+2, q+1} \end{array}$$



1

A LITTLEIFY INTRODUCTION TO SPECTRAL SEQUENCES (E)

[SLIGHTLY TECHNICAL; DETAILS SKIPPED]

WE WANT A PROPER NOTION OF CONVERGENCE OF A SPECTRAL SEQUENCE.

THERE IS A GRADED MODULE E^∞ , CALLED THE LIMIT TERM, WHERE EACH $E_{p,q}^{\infty}$ CAN BE THOUGHT OF AS THE LIMIT OF REPEATEDLY TAKING QUOTIENTS AT THE (p,q) -COMPONENTS. THAT IS, FIXING p,q , THE "LIMIT" OF THE SEQUENCE $E_{p,q}^0, E_{p,q}^1, E_{p,q}^2, \dots$

DEFN. A SPECTRAL SEQUENCE $(E^r)_{r \geq 0}$ CONVERGES TO A GRADED MODULE H_* , WRITTEN $E_p^{\infty} \Rightarrow H_{p+2}$. IF THERE IS SOME BOUNDED FILTRATION $(\bigoplus_p H_n)$ OF H_* W/ $E_{p,q}^{\infty} \cong \bigoplus_p H_{p+2} / \bigoplus_{p+1} H_{p+2}$.

$$[F_0] = \bigoplus^0 H_{p,q} \subseteq \bigoplus^1 H_{p,q} \subseteq \dots \subseteq \bigoplus^{p+2} H_{p,q} = H_{p+2}]$$

→ WE'LL GIVE A MORE CONCRETE DESCRIPTION OF THIS FOR THE SPECTRAL SEQUENCES DISCUSSED AVOIDING THE TECHNICAL MINETE OF THESE DEFINITIONS ←

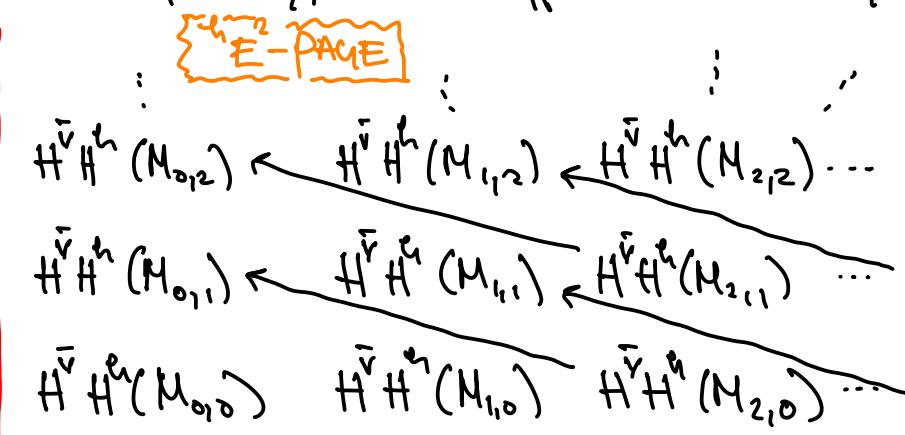
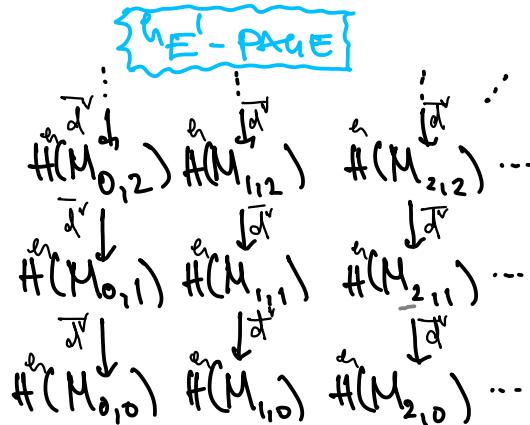
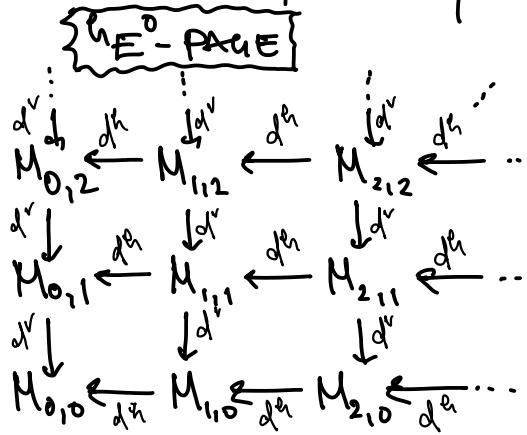
IN PARTICULAR, WE'RE INTERESTED IN TWO SPECTRAL SEQUENCES ARISING FROM DOUBLE COMPLEXES;
HOW THEY ARE RELATED.

A LITTLE HISTORY OF INTRODUCTION TO SPECTRAL SEQUENCES (F)

6

Theorem: If (M, d^p, d^q) is a double complex, then we have two spectral sequences, both converging to $H_*(\text{Tot } M)$. We write: ${}^p E_{p,q}^2 \Rightarrow H_{p+q}(\text{Tot } M)$; ${}^q E_{p,q}^2 \Rightarrow H_{p+q}(\text{Tot } M)$.

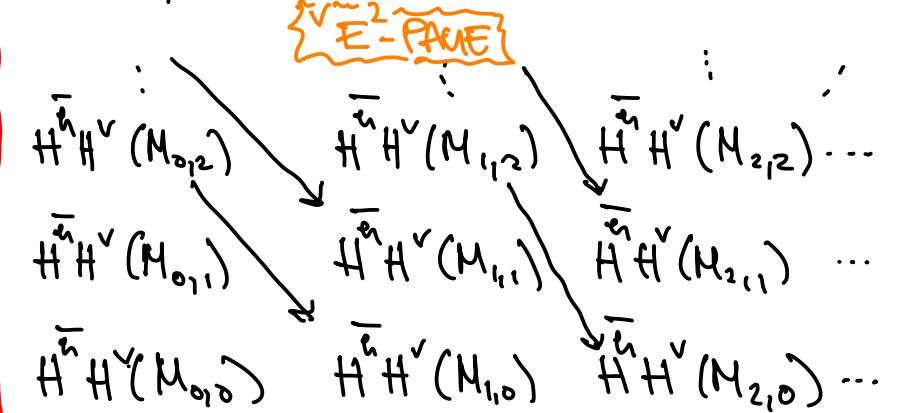
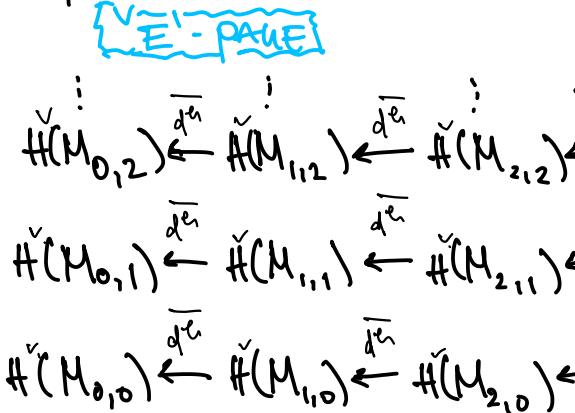
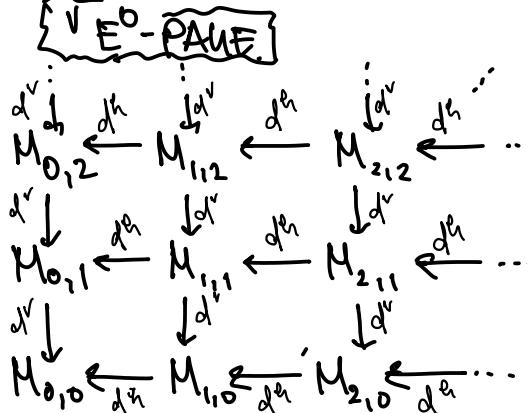
FOR THE SEQUEL, WE ONLY NEED TO DESCRIBE THE FIRST TWO PAGES EXPLICITLY, WHICH HAPPEN TO ARISE NARROWLY.



THE "E"-PAGE OF THE HORIZONTAL SPECTRAL SEQUENCE ARISES FROM FIRST TAKING THE TOTAL HOMOTOPY ; THEN THE d'-DIFFERENTIALS ARE THOSE INDUCED BY THE VERTICAL MAPS.

The "E²-page" then arises from taking vertical homology on these (induced) vertical differentials.

THE d^2 -DIFFERENTIALS ARE SLIGHTLY MORE DIFFICULT TO DESCRIBE, BUT LUCKILY NOT NEEDED FOR THIS DISCUSSION



A LITTLE INTRODUCTION TO SPECTRAL SEQUENCES (6)

(7)

LEMMA [SNAKE LEMMA] Given the following commutative diagram w/ exact rows,

$$\begin{array}{ccccccc} 0 & \rightarrow & M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' \rightarrow 0 \\ & & f \downarrow & & g \downarrow & & h \downarrow \\ 0 & \rightarrow & N' & \xrightarrow{i'} & N & \xrightarrow{p'} & N'' \rightarrow 0 \end{array}$$

THERE IS AN EXACT SEQUENCE

$$0 \rightarrow \ker f \rightarrow \ker g \rightarrow \ker h \rightarrow 0$$

$$\hookrightarrow \text{Coker } f \rightarrow \text{Coker } g \rightarrow \text{Coker } h \rightarrow 0$$

$$\begin{array}{ccccc} 0 & \rightarrow & M_{2,1} & \xrightarrow{d_{2,1}^n} & M_{1,1} & \xrightarrow{d_{1,1}^n} & M_{0,1} & \rightarrow 0 & 0 \rightarrow \ker d_{2,1}^n \rightarrow \ker d_{1,1}^n \rightarrow \ker d_{0,1}^n \rightarrow 0 \\ & & \downarrow d_{2,1}^v & & \downarrow d_{1,1}^v & & \downarrow d_{0,1}^v & & \\ 0 & \rightarrow & M_{2,0} & \xrightarrow{d_{2,0}^n} & M_{1,0} & \xrightarrow{d_{1,0}^n} & M_{0,0} & \rightarrow 0 & \hookrightarrow \text{Coker } d_{2,1}^n \quad \text{Coker } d_{1,1}^n \rightarrow \text{Coker } d_{0,1}^n \rightarrow 0 \end{array}$$

AFTER RELABELING, THIS IS EQUIVALENT TO

WE GET A DOUBLE COMPLEX (w/ SIGN CHANGE) & TWO SPECTRAL SEQUENCES:

$$\begin{array}{ccccc} M_{0,1} & \xleftarrow{d_{0,1}^n} & M_{1,1} & \xleftarrow{d_{1,1}^n} & M_{2,1} \leftarrow 0 \\ \downarrow d_{0,1}^v & -d_{1,1}^v \downarrow & \downarrow d_{1,1}^v & & \downarrow d_{2,1}^v \\ M_{0,0} & \xleftarrow{d_{1,0}^n} & M_{1,0} & \xleftarrow{d_{2,0}^n} & M_{2,0} \leftarrow 0 \end{array}$$

TAKING THE HORIZONTAL SPECTRAL SEQUENCE IMMEDIATELY CONVERGES TO ZERO, SINCE ROWS ARE EXACT.

$\therefore H_*(\text{Tot } M) = 0$. FOR THE VERTICAL SPECTRAL SEQUENCE, WE HAVE,

$$\begin{array}{ccccc} \ker d_{0,1}^n & \xleftarrow{d_{0,1}^v} & \ker d_{1,1}^n & \xleftarrow{d_{1,1}^v} & \ker d_{2,1}^n \\ \downarrow d_{1,0}^n & & \downarrow d_{2,0}^n & & \\ \text{Coker } d_{0,1}^n & \xleftarrow{d_{1,0}^n} & \text{Coker } d_{1,1}^n & \xleftarrow{d_{2,0}^n} & \text{Coker } d_{2,1}^n \end{array}$$

$\check{\Sigma}$ -PAGE

$$\begin{array}{ccccc} \text{Coker } d_{1,1}^n & \xleftarrow{\text{IM } d_{2,1}^n} & \frac{\ker d_{1,1}^n}{\text{IM } d_{2,1}^n} & \xleftarrow{d^2} & \ker d_{2,1}^n \\ \downarrow d_{1,0}^n & & \downarrow d_{2,0}^n & & \downarrow d_{2,0}^n \\ \text{Coker } d_{1,0}^n & & \frac{\ker d_{1,0}^n}{\text{IM } d_{2,0}^n} & & \ker d_{2,0}^n \end{array}$$

$\check{\Sigma}^2$ -PAGE

$$\begin{array}{ccccc} \text{Coker } d_{2,1}^n & & \frac{\ker d_{1,1}^n}{\text{IM } d_{2,1}^n} & & \ker d_{2,1}^n \\ \downarrow d_{1,0}^n & & \downarrow d_{2,0}^n & & \downarrow d_{2,0}^n \\ \text{Coker } d_{1,0}^n & & \frac{\ker d_{1,0}^n}{\text{IM } d_{2,0}^n} & & \ker d_{2,0}^n \end{array}$$

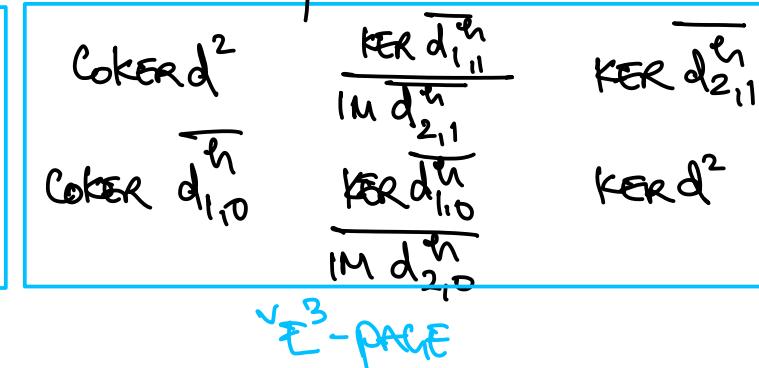
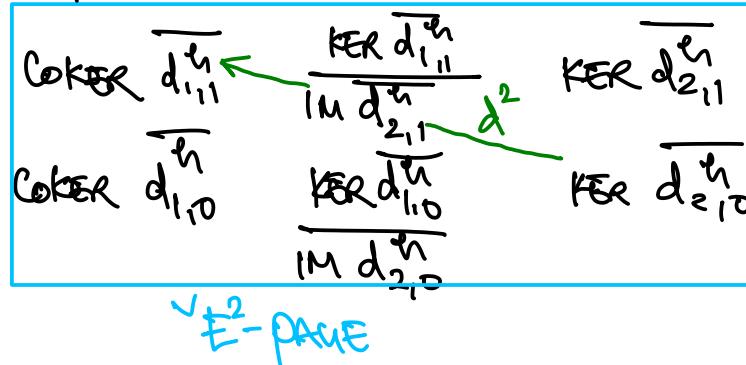
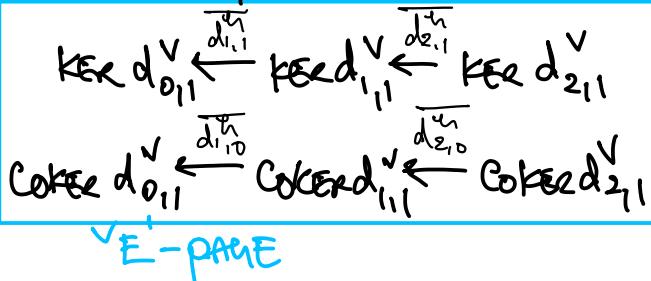
$\check{\Sigma}^3$ -PAGE

d^2 ONLY NONZERO DIFFERENTIAL
(POSSIBLY)

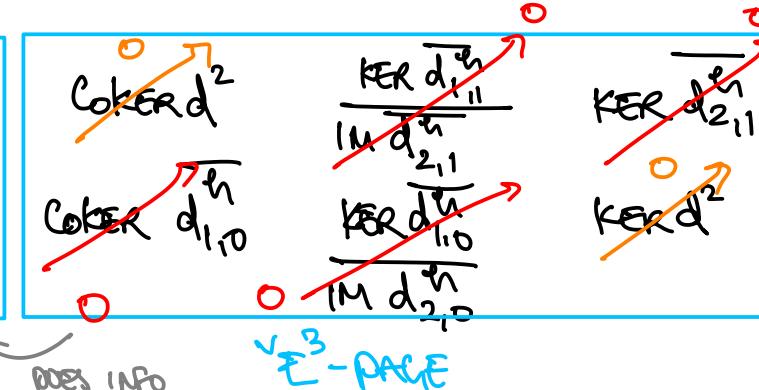
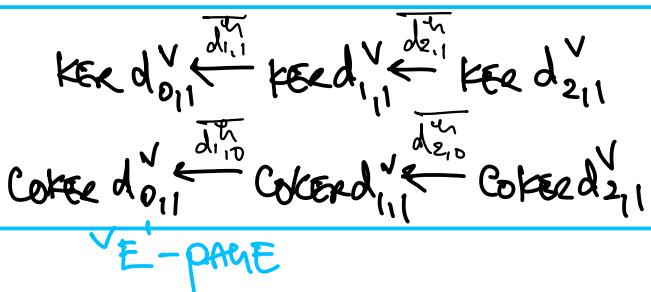
ALREADY COVERED!

A LITTLE INTRODUCTION TO SPECTRAL SEQUENCES (H)

ALREADY CONVERGED! (8)



BUT SINCE WE KNOW THE HORIZONTAL S.S. CONVERGES TO ZERO, SO DOES THIS ONE!



IN PARTICULAR, THE **RED ZERO'S** GIVE US TWO EXACT SEQUENCES:

$$0 \rightarrow \text{KER } d_{2,1}^V \rightarrow \text{KER } d_{1,1}^V \rightarrow \text{KER } d_{0,1}^V \rightarrow \text{Coker } \overline{d_{1,1}^n} \rightarrow 0 ; 0 \rightarrow \text{KER } d_{2,0}^h \rightarrow \text{coker } d_{2,1}^V \rightarrow \text{coker } d_{1,1}^V \rightarrow \text{coker } d_{0,1}^V \rightarrow 0$$

SPLICING THEM TOGETHER ALONG THE ISOMORPHISM \cong WE GET THE DESIRED EXACT SEQUENCE!

RECALL: $\dots \rightarrow A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} K \rightarrow 0 ; 0 \rightarrow K \rightarrow B_0 \xrightarrow{g_0} B_1 \xrightarrow{g_1} \dots$ EXACT SEQUENCES, THEN

$\dots \rightarrow A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0 \circ f_1} B_0 \xrightarrow{g_1} B_1 \rightarrow \dots$ IS EXACT. THE TWO SEQUENCES HAVE BEEN SPLICED TOGETHER.

WHAT DOES INFO
TELL US???

A HESSELEY INTRODUCTION TO SPECTRAL SEQUENCES (I)

(9)

LEMMA: [THE FIVE LEMMA] GIVEN THE FOLLOWING COMMUTATIVE DIAGRAM W/ EXACT ROWS

$$\begin{array}{ccccccc} A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & A_4 & \rightarrow & A_5 \\ h_1 \downarrow & & h_2 \downarrow & & h_3 \downarrow & & h_4 \downarrow & & h_5 \downarrow \\ B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & B_4 & \rightarrow & B_5 \end{array}$$

$$\begin{array}{ccccccc} M_{0,1} & \xleftarrow{d_{1,1}^h} & M_{1,1} & \xleftarrow{d_{2,1}^h} & M_{2,1} & \xleftarrow{d_{3,1}^h} & M_{3,1} & \xleftarrow{d_{4,1}^h} & M_{4,1} \\ d_{0,1}^v \downarrow & & -d_{1,1}^v \downarrow & & d_{2,1}^v \downarrow & & -d_{3,1}^v \downarrow & & d_{4,1}^v \downarrow \\ M_{0,0} & \xleftarrow{d_{1,0}^h} & M_{1,0} & \xleftarrow{d_{2,0}^h} & M_{2,0} & \xleftarrow{d_{3,0}^h} & M_{3,0} & \xleftarrow{d_{4,0}^h} & M_{4,0} \end{array}$$

$$\begin{array}{ccccccc} M_{0,1} & \xleftarrow{d_{1,1}^h} & M_{1,1} & \xleftarrow{d_{2,1}^h} & M_{2,1} & \xleftarrow{d_{3,1}^h} & M_{3,1} & \xleftarrow{d_{4,1}^h} & M_{4,1} \\ d_{0,1}^v \downarrow & & -d_{1,1}^v \downarrow & & d_{2,1}^v \downarrow & & -d_{3,1}^v \downarrow & & d_{4,1}^v \downarrow \\ M_{0,0} & \xleftarrow{d_{1,0}^h} & M_{1,0} & \xleftarrow{d_{2,0}^h} & M_{2,0} & \xleftarrow{d_{3,0}^h} & M_{3,0} & \xleftarrow{d_{4,0}^h} & M_{4,0} \end{array}$$

; THE VERTICAL SPECTRAL SEQUENCE:

IMMEDIATELY CONVERGES AT ${}^V E^1$ -PAGE

${}^H E^1$ -PAGE

$$\begin{array}{ccccc} \text{COKER } d_{1,1}^h & 0 & 0 & 0 & \text{KER } d_{4,1}^h \\ d_{0,1}^v \downarrow & & & & d_{4,1}^v \downarrow \\ \text{COKER } d_{0,0}^h & 0 & 0 & 0 & \text{KER } d_{4,0}^h \end{array}$$

ALREADY CONVERGED!

$$\begin{array}{ccccc} \text{KER } d_{0,1}^v & 0 & 0 & 0 & \text{KER } d_{4,1}^v \\ \text{COKER } d_{0,1}^v & 0 & 0 & 0 & \text{COKER } d_{4,1}^v \end{array}$$

${}^V E^2$ -PAGE

$$\begin{array}{ccccc} 0 & 0 & \text{KER } d_{2,1}^v & 0 & 0 \\ 0 & 0 & \text{COKER } d_{2,1}^v & 0 & 0 \end{array}$$

CONVERGED!

${}^H E^1$ -PAGE

THE IN THE ${}^H E^2$ -PAGE TELLS US $H_2(\text{Tor } M) = 0$, BUT THIS IS $\text{COKER } d_{2,1}^v$, SINCE THEY BOTH CONVERGE TO THE SAME GRADED MODULE $\therefore \text{COKER } d_{2,1}^v = 0 \Rightarrow d_{2,1}^v$ IS SURJECTIVE

THE IN THE ${}^V E^2$ -PAGE TELLS US $H_3(\text{Tor } M) = 0$; SO $\text{KER } d_{2,1}^v = 0 \Rightarrow d_{2,1}^v$ (INJECTIVE!)

THIS PERSPECTIVE ALLOWS US TO SEE WHAT HYPOTHESES ARE NOT NEEDED! NAMELY SURJECTIVITY OF $d_{0,1}^v$; THE INJECTIVITY OF $d_{4,1}^v$!

1

A MEASURABLE INTRODUCTION TO SPECTRAL SEQUENCES (J)

⑩

RECALL: TOR IS A LEFT DERIVED FUNCTOR WHICH MEASURES HOW FAR THE TENSOR PRODUCT DEVIATES FROM EXACTNESS.

ANOTHER QUICK EXAMPLE OF HOW SPECTRAL SEQUENCES CAN BE USEFUL IS IN BALANCING TOR (\mathbb{H}^{ext})

THERE ARE TWO POSSIBLE DEFINITIONS FOR EACH OF THESE; A STANDARD RESULT IN HOMOLOGICAL ALGEBRA IS SHOWING THAT THEY CONCIDE.

DEFN LET A BE A RIGHT R -MODULE; B A LEFT R -MODULE. DEFINE $\text{Tor}_n^R(A, B) : \text{MOD}_R \rightarrow \text{MOD}_R$ AS FOLLOWS. IF $P_0 = \dots \rightarrow P_2 \xrightarrow{d_2^A} P_1 \xrightarrow{d_1^A} P_0 \xrightarrow{\epsilon} A \rightarrow 0$ IS A PROJECTIVE RESOLUTION OF A_R , THEN

$$\text{Tor}_n^R(A, B) = H_n(P_A \otimes_R B) = \ker(d_{n+1}^A \otimes \text{id}_B) / \text{im}(d_{n+1}^A \otimes \text{id}_B)$$

WHERE P_A IS THE RELATED PROJECTIVE RESOLUTION OF A :

$$P_A \otimes B = \dots \rightarrow P_2 \otimes B \xrightarrow{d_2^A \otimes \text{id}_B} P_1 \otimes B \xrightarrow{d_1^A \otimes \text{id}_B} P_0 \otimes B \rightarrow 0$$

SIMILARLY, DEFINE $\text{tor}_n^R(A, B) : {}_R\text{MOD} \rightarrow {}_R\text{MOD}$ AS FOLLOWS. IF $Q_0 \xrightarrow{f} A$ IS A PROJECTIVE RESOLUTION OF R_B , THEN

$$\text{tor}_n^R(A, B) := H_n(A \otimes_R Q_B) = \ker(\text{id}_A \otimes d_n^B) / \text{im}(\text{id}_A \otimes d_{n+1}^B)$$

WE'LL NOW USE SPECTRAL SEQUENCES TO SHOW $\text{Tor}_n^R(A, B) \cong \text{tor}_n^R(A, B)$.

A LITTLE INTRODUCTION TO SPECTRAL SEQUENCES (K)

(1)

$$\text{Prop: } \text{Tor}_n^R(A, B) \cong \text{tor}_n^R(A, B).$$

PF. NOTE THAT THE TWO DELETED PROJECTIVE RESOLUTIONS GIVE RISE TO A DOUBLE COMPLEX:

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & d_{0,3}^v & d_{1,3}^v & d_{2,3}^v & d_{3,2}^v & \dots & \\
 0 \leftarrow P_0 \otimes Q_2 & \xleftarrow{d_{1,2}^v} & P_1 \otimes Q_2 & \xleftarrow{d_{2,1}^v} & P_2 \otimes Q_2 & \xleftarrow{d_{3,1}^v} & \dots \\
 & d_{0,2}^v & d_{1,2}^v & d_{2,1}^v & d_{3,1}^v & \dots & \\
 0 \leftarrow P_0 \otimes Q_1 & \xleftarrow{d_{1,1}^v} & P_1 \otimes Q_1 & \xleftarrow{d_{2,1}^v} & P_2 \otimes Q_1 & \xleftarrow{d_{3,1}^v} & \dots \\
 & d_{0,1}^v & d_{1,1}^v & d_{2,1}^v & d_{3,1}^v & \dots & \\
 0 \leftarrow P_0 \otimes Q_0 & \xleftarrow{d_{1,0}^v} & P_1 \otimes Q_0 & \xleftarrow{d_{2,0}^v} & P_2 \otimes Q_0 & \xleftarrow{d_{3,0}^v} & \dots \\
 & d_{0,0}^v & d_{1,0}^v & d_{2,0}^v & d_{3,0}^v & \dots & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \dots & \\
 0 & 0 & 0 & 0 & 0 & \dots &
 \end{array}$$

WHERE $M_{P+Q} := P_p \otimes Q_q$, $d_{P+Q}^v = d_p^v \otimes d_q^v$; $d_{P+Q}^h = (-1)^p d_p^h \otimes d_q^h$. HERE, THE TOTAL SPACE IS $\text{TOT}_n M = (P \otimes Q)_n := \bigoplus_{P+Q=n} P_p \otimes Q_q$.

THE KEY OBSERVATION IS THAT ALL PROJECTIVE MODULES ARE FLAT!

Recall: A is flat if $A \otimes -$ is an exact functor.

THE FIRST PAGE OF THE HORIZONTAL SPECTRAL SEQUENCE COLLAPSES TO THE y-AXIS, SINCE Q_2 IS PROJ ($\Rightarrow \text{flat}$)

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & d_{0,3}^v & d_{1,2}^v & d_{2,1}^v & d_{3,1}^v & \dots & \\
 A \otimes Q_2 = \text{Coker } d_{1,2}^v & 0 & 0 & 0 & 0 & \dots & \text{By THE SECOND PAGE, WE HAVE ALREADY CONVERGED TO} \\
 & d_{0,2}^v & d_{1,2}^v & d_{2,1}^v & d_{3,1}^v & \dots & H_2(A \otimes Q_0) = 0 \quad \therefore {}^v E_{P+Q}^2 \rightarrow H_{P+Q}(A \otimes Q_0) = \text{tor}_{P+Q}^R(A, B) \\
 A \otimes Q_1 = \text{Coker } d_{1,1}^v & 0 & 0 & 0 & 0 & \dots & H_1(A \otimes Q_0) = 0 \\
 & d_{0,1}^v & d_{1,1}^v & d_{2,1}^v & d_{3,1}^v & \dots & H_0(A \otimes Q_0) = 0 \\
 A \otimes Q_0 = \text{Coker } d_{1,0}^v & 0 & 0 & 0 & 0 & \dots &
 \end{array}$$

SYMMETRIC REASONING SHOWS

$$\begin{aligned}
 {}^v E_{P+Q}^2 &\Rightarrow H_{P+Q}(P \otimes B) \\
 &= \text{Tor}_{P+Q}^R(A, B)
 \end{aligned}$$

BY COLLAPSING TO THE x-AXIS BY " E^1 ";
OBTAINING HOMOLOGY; CONVERGING BY " E^2 "

BY RESULT ABOUT TWO S.S. ARISING FROM A DOUBLE COMPLEX, CONCLUDE DESIRED RESULT. $[\text{Tor}_n^R(A, B) \cong \text{tor}_n^R(A, B)]$

Topological Covers; Mayer-Vietoris (A)

LET X BE A SPACE.

DEFN $\mathcal{U} = \{\mathcal{U}_i\}_{i \in I}$ IS AN OPEN COVER OF X PROVIDED $\bigcup_{i \in I} \mathcal{U}_i = X$.

PROP [MAYER-VIETORIS LONG EXACT SEQUENCE]. IF $\mathcal{U} = \{\mathcal{U}_0, \mathcal{U}_1\}$ IS

AN OPEN COVER OF X , THEN THERE EXISTS A LONG EXACT SEQUENCE

OF SINGULAR HOMOLOGY GROUPS:

$$\cdots \xrightarrow{\alpha_{n+1}} H_n(\mathcal{U}_0 \cap \mathcal{U}_1) \xrightarrow{\alpha_n} H_n(\mathcal{U}_0) \oplus H_n(\mathcal{U}_1) \xrightarrow{\beta_n} H_n(X) \xrightarrow{\alpha_{n-1}} H_{n-1}(\mathcal{U}_0 \cap \mathcal{U}_1) \xrightarrow{\cdots} \xrightarrow{\beta_0} H_0(X) \rightarrow 0$$

PF THIS FOLLOWS FROM OBSERVING THAT WE HAVE THE FOLLOWING SHORT EXACT SEQUENCE OF COMPLEXES:

$$0 \rightarrow S_*(\mathcal{U}_0 \cap \mathcal{U}_1) \xrightarrow{\quad} S_*(\mathcal{U}_0) \oplus S_*(\mathcal{U}_1) \xrightarrow{\quad} S_*^{\mathcal{U}}(X) \rightarrow 0$$

$C_{0,1} \longleftarrow \begin{pmatrix} -C_{0,1}, C_{0,1} \\ C_{0,1}, C_{1,1} \end{pmatrix} \longrightarrow C_0 + C_1$

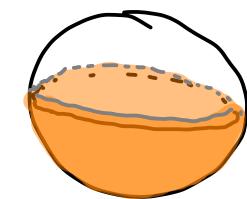
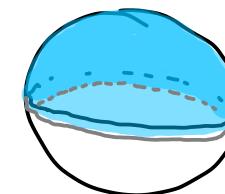
WHERE $S_2(Y) := \mathbb{Z}[\text{Map}(\Delta^2, Y)]$; $S_2^{\mathcal{U}} := \mathbb{Z}[\{\sigma: \Delta^2 \rightarrow X : \exists U_i \in \mathcal{U} \text{ w/ } \text{Im} \sigma \subseteq U_i\}]$ [\mathcal{U} -SIMPLICES]

APPLYING THE LONG EXACT HOMOLOGY SEQUENCE TO THIS CHAIN COMPLEX & NOTING $S_*(X) \rightarrow S_*(X)$ IS A CHAIN HOMOTOPY EQUIVALENCE (\Leftrightarrow THEY HAVE ISOMORPHIC HOMOLOGIES) YIELDS THE RESULT! ■

A NATURAL QUESTION IS WHETHER OR NOT WE CAN EXTEND THIS RESULT TO ARBITRARY COVERS. THE ANSWER IS YES; THIS INFORMATION CAN BE NICELY ENCODED INTO A SPECTRAL SEQUENCE, CALLED THE MAYER-VIETORIS SPECTRAL SEQUENCE.

THE PROOF GIVES US THE LINE OF ATTACK: WE CAN EXPAND THIS SHORT EXACT SEQUENCE OF CHAIN COMPLEX TO A LONG EXACT SEQUENCE THAT TAKES INTO ACCOUNT MORE THAN TWO OPEN SETS!

FOR EXAMPLE, $X = S^2$, $\mathcal{U}^U = \text{UPPER HEMISPHERE}$ (12)
 $\mathcal{U}^L = \text{LOWER HEMISPHERE}$ (01 OVERLAPS)



+

Topological Covers; Mayer-Vietoris (B)

(13)

DEFN. IF $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ AN OPEN COVER OF X , THERE IS AN ASSOCIATED ABSTRACT SIMPLICIAL COMPLEX

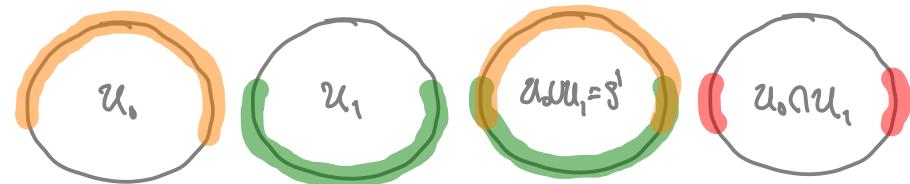
$$K_{\mathcal{U}} = (V_{k_{\mathcal{U}}}, S_{k_{\mathcal{U}}}) \text{ WITH}$$

$$V_{k_{\mathcal{U}}} = \{U_\lambda\}_{\lambda \in \Lambda} : \text{VERTICES ARE THE OPEN SETS OF THE COVER}$$

$$S_{k_{\mathcal{U}}} = \{ \bar{\Gamma} := \{U_\gamma\}_{\gamma \in \Gamma} : \emptyset \neq \Gamma \subseteq \Lambda \text{ FINITE} \mid U_\Gamma = \bigcap_{\gamma \in \Gamma} U_\gamma \neq \emptyset \} \text{ SIMPLICES.}$$

EXAMPLE: IF $\mathcal{U} = \{U_0, U_1\}$, $K_{\mathcal{U}}$:

$$\begin{matrix} U_0 \cap U_1 & \rightarrow & U_1 \\ \downarrow & & \\ U_0 & & \end{matrix}$$



IF $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ IS AN OPEN COVER, THEN THERE EXISTS A LONG EXACT SEQUENCE OF CHAIN COMPLEXES, CALLED THE EXTENDED ORDERED ČECH COMPLEX: ε (AT 2)

$$\dots \xrightarrow{\partial_{2,2}} \bigoplus_{\Gamma \in k_{\mathcal{U}}^{(2)}} S_2(U_\Gamma) \xrightarrow{\partial_{1,2}} \bigoplus_{\Gamma_0} S_2(U_{\Gamma_0}) \rightarrow S_2^{\mathcal{U}}(X) \rightarrow 0$$

PROP: THIS IS AN EXACT SEQUENCE.

Topological Covers; Mayer-Vietoris (c)

DEFN FOR $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ AN OPEN COVER OF X , THE MAYER-VIETORIS DOUBLE COMPLEX, (M, d^h, d^v) IS DEFINED BY $M_{pq} := \bigoplus_{\Gamma \in K_U^{(p)}} S_q(U_\Gamma)$ WITH THE HORIZONTAL DIFFERENTIALS ARISING FROM THE ČECH COMPLEX; THE VERTICAL ONES ARISING FROM THE DIRECT SUM OF THE SINGULAR HOMOLOGY DIFFERENTIALS.

$$\begin{array}{ccccccc}
 & \downarrow & \downarrow & \downarrow & & & \\
 0 & \leftarrow \bigoplus_{\Gamma \in K_U^{(0)}} S_2(U_\Gamma) & \leftarrow \bigoplus_{\Gamma \in K_U^{(1)}} S_2(U_\Gamma) & \leftarrow \bigoplus_{\Gamma \in K_U^{(2)}} S_2(U_\Gamma) & \leftarrow \cdots & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & & & \\
 0 & \leftarrow \bigoplus_{\Gamma \in K_U^{(0)}} S_1(U_\Gamma) & \leftarrow \bigoplus_{\Gamma \in K_U^{(1)}} S_1(U_\Gamma) & \leftarrow \bigoplus_{\Gamma \in K_U^{(2)}} S_1(U_\Gamma) & \leftarrow \cdots & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & & & \\
 0 & \leftarrow \bigoplus_{\Gamma \in K_U^{(0)}} S_0(U_\Gamma) & \leftarrow \bigoplus_{\Gamma \in K_U^{(1)}} S_0(U_\Gamma) & \leftarrow \bigoplus_{\Gamma \in K_U^{(2)}} S_0(U_\Gamma) & \leftarrow \cdots & & \\
 \downarrow & & \downarrow & & & & \\
 0 & & & & & &
 \end{array}$$

THEOREM: [MAYER-VIETORIS SPECTRAL SEQUENCE] ASSOCIATED TO \mathcal{U} , THERE IS A CONVERGENT SPECTRAL SEQUENCE $E_{p,q}^2 = H_p(C_*(K_U, F_q)) \Rightarrow H_{p+q}(X)$ [SINGULAR HOMOLOGY OF $X!$]

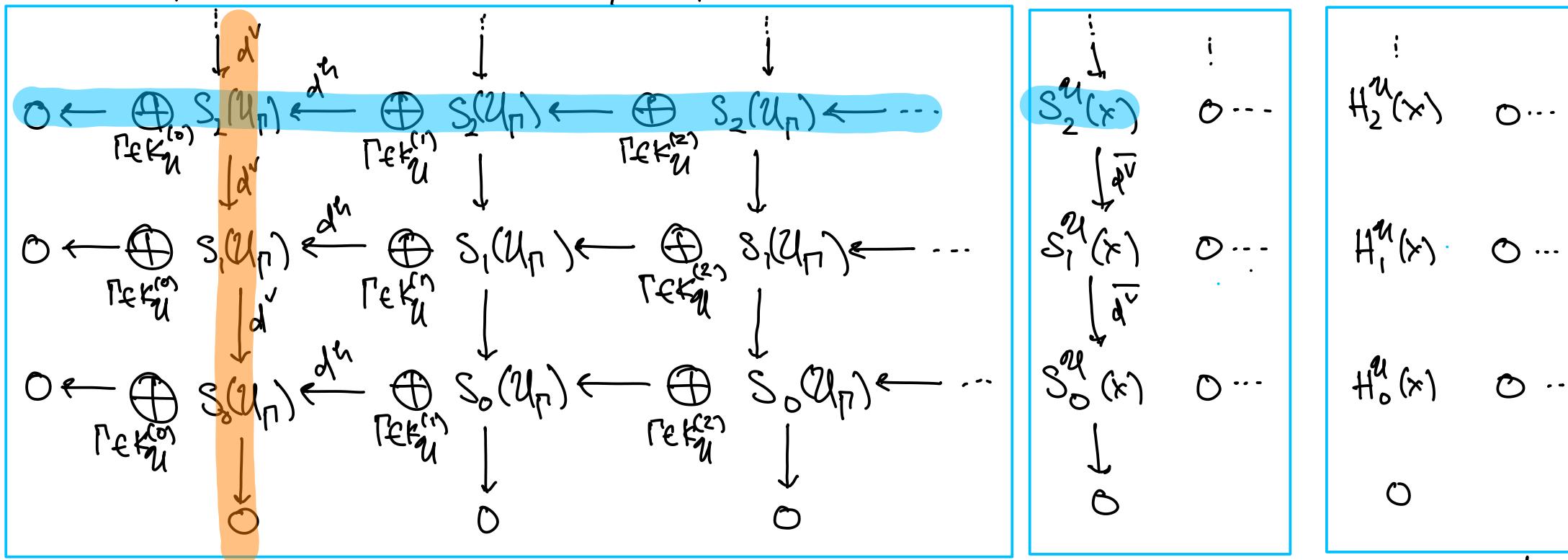
NOTE: FOR THE PURPOSES OF THIS TALK, WE DESCRIBE THE CONVERGENCE IN TERMS OF THE E^1 -PAGE:

$$E_{p,q}^1 = \bigoplus_{\Gamma \in K_U^{(p)}} H_q(U_\Gamma) = \bigoplus_{\gamma_1 < \dots < \gamma_p} H_q(U_{\gamma_1} \cap \dots \cap U_{\gamma_p}).$$

Topological Covers; Mayer-Vietoris (D)

(15)

PROOF: SINCE WE'VE SAID THE EXTENDED ČECH COMPLEX IS EXACT, THE HORIZONTAL SPECTRAL SEQUENCE ASSOCIATED TO THE MAYER-VIETORIS DOUBLE COMPLEX COLLAPSES TO THE y -AXIS BY THE FIRST PAGE & CONVERGES TO HOMOLOGY BY THE SECOND:



which tells us that ${}^n E_{p+2}^2 \Rightarrow H_{p+2}(\text{tot } X) = H_{p+2}^U(x) \cong H_{p+2}(X)$.

Converged!

THE RESULT IS PRECISELY THAT THE VERTICAL SPECTRAL SEQUENCE ALSO CONVERGES TO $H_{p+2}(X)$, WHICH IS NOTHING BUT THE STANDARD RESULT ABOUT SPECTRAL SEQUENCES: ${}^n E_{p+2}^2 \Rightarrow H_{p+2}(X)$, WHERE WE CAN DESCRIBE THE FIRST PAGE BY TAKING VERTICAL HOMOLOGY FIRST: JUST SINGULAR HOMOLOGY:

$${}^n E_{p+2}^1 = \bigoplus_{\Gamma \in K_{\bar{U}}^{(p)}} H_2(U_{\bar{U}})$$

Topological Covers ; Mayer-Vietoris (E)

(16)

Corollary: WE CAN RECOVER THE MAYER-VIETORIS LOCAL EXACT SEQUENCE FROM THE SPECTRAL SEQUENCE.

Show it:

$$\begin{array}{ccccc}
 & \downarrow & & \downarrow & \\
 S_2(U_0) \oplus S_2(U_1) & \xleftarrow{d_{1,2}^h} & S_2(U_0 \cap U_1) & \xleftarrow{\quad} & 0 \\
 & \downarrow & & \downarrow & \\
 S_1(U_0) \oplus S_1(U_1) & \xleftarrow{d_{1,1}^h} & S_1(U_0 \cap U_1) & \xleftarrow{\quad} & 0 \\
 & \downarrow & & \downarrow & \\
 S_0(U_0) \oplus S_0(U_1) & \xleftarrow{d_{1,0}^h} & S_0(U_0 \cap U_1) & \xleftarrow{\quad} & 0 \\
 & \downarrow & & \downarrow & \\
 0 & & 0 & &
 \end{array}$$

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 H_2(U_0) \oplus H_2(U_1) & \xleftarrow{\overline{d_{1,2}^h}} & H_2(U_0 \cap U_1) & \xleftarrow{\quad} & 0 \\
 H_1(U_0) \oplus H_1(U_1) & \xleftarrow{\overline{d_{1,1}^h}} & H_1(U_0 \cap U_1) & \xleftarrow{\quad} & 0 \\
 H_0(U_0) \oplus H_0(U_1) & \xleftarrow{\overline{d_{1,0}^h}} & H_0(U_0 \cap U_1) & \xleftarrow{\quad} & 0 \\
 & \circ & & \circ &
 \end{array}$$

THE FIRST PAGE OF THE VERTICAL SPECTRAL SEQUENCE GIVES US SHORT EXACT SEQUENCES: E¹-PAGE

$$0 \rightarrow \ker \overline{d_{1,2}^h} \rightarrow H_2(U_0 \cap U_1) \xrightarrow{\overline{d_{1,2}^h}} H_2(U_0) \oplus H_2(U_1) \rightarrow \text{coker } \overline{d_{1,2}^h} \rightarrow 0 \quad \star$$

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 0 & \xrightarrow{\text{coker } \overline{d_{1,2}^h}} & \xleftarrow{\ker \overline{d_{1,2}^h}} & 0 & \\
 0 & \xrightarrow{\text{coker } \overline{d_{1,1}^h}} & \xleftarrow{\ker \overline{d_{1,1}^h}} & 0 & \\
 0 & \xrightarrow{\text{coker } \overline{d_{1,0}^h}} & \xleftarrow{\ker \overline{d_{1,0}^h}} & 0 &
 \end{array}$$

E²-PAGE

AT THE SECOND PAGE, WE HAVE CONVERGED \Rightarrow . BY A STANDARD RESULT ABOUT SPECTRAL SEQUENCES, WE HAVE SHORT EXACT SEQUENCES

$$0 \rightarrow \text{coker } \overline{d_{1,2}^h} \rightarrow H_2(\text{DTM}) \xrightarrow{\overline{d_{1,2}^h}} \ker \overline{d_{1,2-1}^h} \rightarrow 0. \quad \star$$

$$H_2(X)$$

BUT THE CONVERGENCE HYPOTHESIS TELLS US $H_2(\text{DTM}) \cong H_2(X)$
THE RESULT FOLLOWS FROM SPLICING \star ; \star \star .

Topological Covers; Mayer-Vietoris (\mathbb{F})

(17)

[A PATHOLOGICAL PROOF OF THE INCLUSION-EXCLUSION PRINCIPLE]

PROPOSITION. IF U_0, \dots, U_n ARE FINITE SETS $\Rightarrow \Lambda_n = [n] = \{0, \dots, n\}$, THEN $\left| \bigcup_{i=0}^n U_i \right| = \sum_{\emptyset \neq \Gamma \subseteq \Lambda_n}^{|\Gamma|-1} (-1)^{|\Gamma|} \left| \bigcap_{y \in \Gamma} U_y \right|$.

PROOF: DEFINE A SPACE $X := \bigcup_{i=0}^n U_i$ WITH THE DISCRETE TOPOLOGY; CONSIDER THE OPEN COVER $\mathcal{U} := \{U_i\}_{i=0}^n$.

THE MAYER-VIETORIS SPECTRAL SEQUENCE GIVES ${}^V E_{p,q}^1 = \bigoplus_{\emptyset \neq \Gamma \subseteq \Lambda_n} H_q(\bigcap_{y \in \Gamma} U_y) \Rightarrow H_{p+q}(X) = \begin{cases} \mathbb{Z}[x] & \text{IF } p+q=0 \\ 0 & \text{OTHERWISE} \end{cases}$

SINCE SINGULAR HOMOLOGY IS AN ORDINARY, ADDITIVE HOMOLOGY THEORY, WE HAVE NONZERO HOMOLOGY VANISHING ON POINTS

\therefore THE ${}^V E^1$ -PAGE HAS COLLAPSED TO THE X-AXIS WITH

$${}^V E_{p,q}^1 = \bigoplus_{\emptyset \neq \Gamma \subseteq \Lambda_n} H_q(\bigcap_{y \in \Gamma} U_y)$$

$|\Gamma| = p+1$

EXPLICITLY, (AFTER EXTENDING EXACTNESS TO THE COKERNEL)

$$0 \leftarrow H_0(X) \leftarrow \bigoplus_i H_0(U_i) \leftarrow \bigoplus_{i < j} H_0(U_i \cap U_j) \leftarrow \dots \leftarrow H_0(U_0 \cap \dots \cap U_n) \leftarrow 0$$

LEMMA: IF $0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ IS AN EXACT SEQUENCE OF FINITELY GENERATED FREE ABELIAN GROUPS, THEN $\sum_{i=0}^n (-1)^i \text{RANK}(F_i) = 0$.

$\therefore \text{RANK}(H_0(X)) = \sum_{i=0}^n (-1)^i \text{RANK} \left(\bigoplus_{\substack{\emptyset \neq \Gamma \subseteq \Lambda_n \\ |\Gamma| = p+1}} H_0 \left(\bigcap_{y \in \Gamma} U_y \right) \right)$, WHICH IS EQUIVALENT TO

$$\left| \bigcup_{i=0}^n U_i \right| = \sum_{\emptyset \neq \Gamma \subseteq \Lambda_n} {}^{|\Gamma|-1} \left| \bigcap_{y \in \Gamma} U_y \right|, \text{ AS DESIRED!}$$

Topological Covers; Mayer-Vietoris (G)

EXAMPLE: COVER $X = S^1$ w/ $n+1$ OPEN BALLS: NO THREE INTERSECT: $u_{n-1}, u_n, \dots, u_0, u_1, u_2, \dots, u_{n-3}$

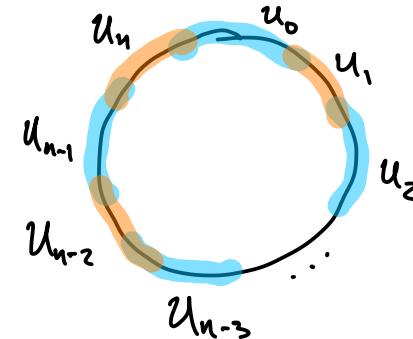
(18)

THE MAYER-VIETORIS SPECTRAL SEQUENCE CONVERGES

BY THE SECOND PAGE TO $H_n(S^1)$

THE FIRST PAGE IS

$$\begin{array}{ccccccc} 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 \\ & & \downarrow d_{1,0}^{n+1} & & & & \\ 0 & \leftarrow & \mathbb{Z}^{n+1} & \leftarrow & \mathbb{Z}^{n+1} & \leftarrow & 0 \\ & & & & & & \\ 0 & & 0 & & 0 & & 0 \end{array}$$

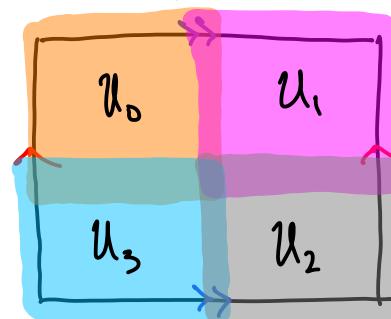


THE SECOND PAGE COINCIDES WITH THE SINGULAR HOMOLOGY OF S^1 , WHICH A PRIORI, WE DON'T KNOW, BUT IS ESSENTIALLY CALCULATED VIA THE KERNEL / COKERNEL OF $d_{1,0}^{n+1}$. Luckily, THESE MATRICES ARE SIMPLE; ONLY HAVE ZERO'S; ONES WHICH MAKE SIGHTLY BARE MORE SIMPLE.

WE OBTAIN $H_k(S^1) = \begin{cases} \mathbb{Z} & k=0,1 \\ 0 & \text{otherwise} \end{cases}$, AS EXPECTED



EXAMPLE: COVER THE TORUS $T^2 = S^1 \times S^1$ WITH FOUR OPEN SETS:



THE ASSOCIATED MAYER-VIETORIS SPECTRAL SEQUENCE CONVERGES

BY THE SECOND PAGE TO $H_n(T^2)$. THE FIRST PAGE HAS ONLY ONE NONZERO ROW:

$$0 \leftarrow \mathbb{Z}^4 \xleftarrow{d_1} \mathbb{Z}^{16} \xleftarrow{d_2} \mathbb{Z}^{16} \xleftarrow{d_3} \mathbb{Z}^4 \leftarrow 0$$

CALCULATING THE HOMOLOGY OF THIS REQUIRES COMPUTER AID. EVENTUALLY WE GET: $H_n(T^2) = \begin{cases} \mathbb{Z}, & n=0,2 \\ \mathbb{Z}^2, & n=1 \\ 0 & \text{else} \end{cases}$

Moral: IN SOME CASES, COMPLICATES THE SITUATION, IN OTHERS (CONFIGURATION SPACES, SHEAF HOMOLOGY) IMPROVES!

$$\begin{cases} \mathbb{Z}, & n=0,2 \\ \mathbb{Z}^2, & n=1 \\ 0 & \text{else} \end{cases}$$

HOMOTOPY COLIMIT SPECTRAL SEQUENCE (19)

DEFN: THE **ORDINAL CATEGORY**, DENOTED BY Δ HAS

$$\cdots \leftarrow \overset{\circ}{\leftarrow} \leftarrow \cdots \leftarrow^{\circ}$$

$\text{OBJ}(\Delta) = \{[n] : n \in \mathbb{Z}_{\geq 0}\}$ WHERE $[n] = \{0, \dots, n\}$ VIEWED AS A POSET w/ THE RELATION \leq .

$\text{MOR}_{\Delta}([m], [n]) = \{ \text{ORDER PRESERVING MORPHISMS } \theta : [m] \rightarrow [n], \text{ i.e., } i \leq j \Rightarrow \theta(i) \leq \theta(j)\}$.

DEFN: GIVEN A CATEGORY \mathcal{C} , A SIMPLICIAL OBJECT IN \mathcal{C} IS A FUNCTOR $X: \Delta^{\text{op}} \rightarrow \mathcal{C}$.

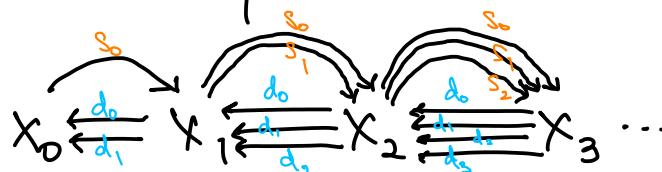
WE WRITE $\text{SC} = \text{Fun}(\Delta^{\text{op}}, \mathcal{C}) = \mathcal{C}^{\Delta^{\text{op}}}$ FOR THE CATEGORY OF SIMPLICIAL OBJECTS IN \mathcal{C} WITH MORPHISMS GIVEN BY NATURAL TRANSFORMATIONS $\tau: X \Rightarrow Y$.

THE BEHAVIOR ON MORPHISMS ON $X: \Delta^{\text{op}} \rightarrow \mathcal{C}$ CAN BE SHOWN TO BE "GENERATED" BY MORPHISMS

$d_i: X_n \rightarrow X_{n-1}$ FACE

$s_i: X_n \rightarrow X_{n+1}$ DEGENERACY

THIS ALLOWS US TO CONCRETELY ILLUSTRATE THE SIMPLICIAL STRUCTURE:



WE ARE PARTICULARLY INTERESTED IN sSet , sTop ; sAb .

PROP: THERE IS AN EQUIVALENCE OF CATEGORIES BETWEEN SIMPLICIAL ABELIAN GROUPS $\overset{\text{(POSITIVE)}}{\cong}$ CHAIN COMPLEXES
NAMELY GIVEN $X: \Delta^{\text{op}} \rightarrow \text{Ab}$, WE OBTAIN A CHAIN COMPLEX, CALLED THE ALTERNATING FREE MAP

COMPLEX OBTAINED BY SETTING $C_n = X_n$; $\partial_n: C_n \rightarrow C_{n-1}$: $\partial_n = \sum_{i=0}^n (-1)^i d_i$.

[THIS EQUIVALENCE IS CALLED THE DODD-KAN CORRESPONDENCE]

$$\text{sAb} \cong \underset{\mathbb{Z}_{\geq 0}}{\text{CH}}(\text{Ab})$$

HOMOTOPY COLIMIT SPECTRAL SEQUENCE (B)

THE EXTENDED ČECH COMPLEX USED IN CONSTRUCTING THE MAJER-VICTORIS SPECTRAL SEQUENCE NATURALLY ARISES FROM A SIMPLICIAL SPACE, CALLED THE SIMPLICIAL ČECH NERVE OF $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$.

$$\check{C}_*(\mathcal{U}) : \Delta^{\text{op}} \rightarrow \text{Top}$$

$$[n] \mapsto \check{C}_n(\mathcal{U}) = \bigsqcup_{\Gamma = (\gamma_0, \dots, \gamma_n) : \gamma_i \in A} U_\Gamma$$

$$\Gamma = (\gamma_0, \dots, \gamma_n) : \gamma_i \in A \quad ; \quad U_\Gamma \neq \emptyset$$

NOTICE THAT IN THE ČECH NERVE, WE ALLOW REPETITIONS OF INDICES. THIS IS CALLED THE UNORDERED ČECH NERVE, AS OPPOSED TO THE ALTERNATING OR ORDERED ČECH NERVES, THE LATTER ONE WHICH IS THE ONE WE USED. ALL THREE CAN BE SHOWN TO BE CHAIN HOMOTOPY EQUIVALENT THROUGH, WHICH I SHOWED IN MY THESIS, GENERALIZING A RESULT OF SERRE.

INTERESTINGLY, FOR A FUNCTOR $F : \text{Top} \rightarrow \text{AB}$, e.g., $H_2 : \text{Top} \rightarrow \text{AB}$, WE CAN PRECOMPOSE THE ČECH NERVE TO GET A SIMPLICIAL ABELIAN GROUP:

$$\Delta^{\text{op}} \xrightarrow{\check{C}_*(\mathcal{U})} \text{Top} \xrightarrow{H_2(\square)} \text{AB}$$

DEFN: GIVEN A FUNCTOR $F : \mathcal{C} \rightarrow \text{AB}$, ITS SIMPLICIAL REPLACEMENT, WRITTEN $\text{SREP}_* F$ IS A SIMPLICIAL ABELIAN GROUP w/ n -SIMPLICES $\text{SREP}_n F := \bigoplus_{C_0 \rightarrow \dots \rightarrow C_n} F(C_0)$; ON MORPHISMS $\theta : [m] \rightarrow [n]$

$\theta^* := \text{SREP}_*(F)(\theta) : \text{SREP}_n(F) \rightarrow \text{SREP}_m(F)$ IS DEFINED ON COMPONENTS BY:

$$f(C_0) \mapsto f(C_{0(0)})$$

$$C_0 \rightarrow \dots \rightarrow C_n$$

$$C_{0(0)} \rightarrow \dots \rightarrow C_{n(0)}$$

DEFN: THERE IS A FUNCTOR $|-| : \text{Set} \rightarrow \text{Top}$, CALLED THE CORDERIC REALIZATION WHICH IS A LEFT ADJOINT:

\therefore PRESERVES COHOMOLOGY. [EXPLICITLY ARISES AS A LEFT Kan EXTENSION OF $A \rightarrow \text{Top}$ ALONG THE YONEDA EMBEDDING, $y : A \hookrightarrow \text{Set}$]

HOMOTOPY COLIMIT SPECTRAL SEQUENCE (c)

$\Gamma_{\text{PERF}(C)}$ IS CONTRACTIBLE. $\text{IN } \text{PERF}(?)$

(2)

DEFN: FOR $F: C \rightarrow D$ IS A DIAGRAM (A C -INDEXED FUNCTOR VALUED IN D), WE DEFINE THE COLIMIT OF F AS THE INITIAL OBJECT IN COCOONE(F), WHERE A COCOONE OF F IS AN OBJECT d IN D TOGETHER WITH MORPHISMS $f(c) \rightarrow d$: FOR EVERY $x \xrightarrow{f} y$ IN C , THE FOLLOWING DIAGRAM COMMUTES: $\begin{array}{ccc} f(x) & \xrightarrow{f(f)} & f(y) \\ i_x \searrow & \curvearrowright & \swarrow i_y \\ & d & \end{array}$. IN PARTICULAR $\text{COLIM } F$ IS AN OBJECT IN D : FOR ANY $d \in D$ i MAP $f_x: f(x) \rightarrow d$: $f_y f(f) = f_x \circ x \xrightarrow{f} y$. $\exists! \beta: \text{COLIM } F \rightarrow d$

$$\begin{array}{ccc} f(x) & \xrightarrow{i_x} & \text{COLIM } F \\ f(f) \downarrow & \nearrow i_y & \xrightarrow{\beta} d \\ f(y) & \xrightarrow{i_y} & \end{array}$$

EXAMPLES: SAY $D = \text{TOP}$.

1) IF $C = \{x, y\}$, $f_x = X$, $f_y = Y$ IN TOP , THEN $\text{COLIM } F = X \sqcup Y$

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & X \sqcup Y & \xrightarrow{\quad} & Z \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ Y & \xrightarrow{\quad} & Z & \xrightarrow{\quad} & \end{array}$$

2) IF $C = \{0^0 \rightarrow 1^1 \rightarrow 2^2 \rightarrow \dots\} \stackrel{\approx N \text{ poset viewed as a category}}{\sim}$; $F(n) \subseteq \mathbb{R}^{n+1}$, THEN $\text{COLIM } F = \bigcup_{n=0}^{\infty} F(n)$

$\approx N$ poset viewed as a category

3) IF $C = \{ \begin{smallmatrix} 0^0 & \rightarrow & 1^1 \\ \downarrow & & \downarrow \\ x & \rightarrow & y \end{smallmatrix} \} \xrightarrow{F} \begin{array}{ccc} 2 & \xrightarrow{f} & X \\ g \downarrow & & \downarrow \\ y & & \end{array}$ THEN $\text{COLIM } F = \text{PUSHOUT} \left[\begin{array}{c} 2 \rightarrow X \\ \downarrow \\ y \end{array} \right] = X \sqcup Y / f(2) \sim g(2)$

$$\begin{array}{ccc} \text{e.g. } & & \\ \text{circle} & \xrightarrow{\text{CONST}_T} & \text{point} \\ \text{O}^2 & \xrightarrow{\quad} & \text{S}^1 \end{array}$$



HOMOTOPY COLIMIT SPECTRAL SEQUENCE (D) e.g., FUNDAMENTAL GROUP

(22)

A CURIOUS SITUATION ARISES WHEN TAKING HOMOTOPY OF PUSHOUTS. FOR EXAMPLE

$$\begin{array}{ccc} S^0 & \xrightarrow{\quad} & \{pt\} \\ \text{inc} \downarrow \doteq & & \downarrow \\ D' & \xrightarrow{\quad} & S^1 \end{array}$$

$$\pi_1 S^1 = \mathbb{Z}$$

OR, MORE GENERALLY:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad} & \{pt\} \\ \text{inc} \downarrow \doteq & & \downarrow \\ D^n & \xrightarrow{\quad} & S^n \end{array}$$

$$\pi_n S^n = \mathbb{Z} \quad [\text{USE FREUDENHAI SUSPENSION ARGUMENT}]$$

THE PROBLEM ARISES WHEN WE REPLACE OBJECTS w/ SOMETHING THEY ARE HOMOTOPY EQUIVALENT TO, $D^n \approx \{pt\}$

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad} & pt \\ \downarrow & & \downarrow \\ pt & \xrightarrow{\quad} & pt. \end{array}$$

NOT ISOMORPHIC!

$$\pi_n \{pt\} = 0$$

WE AMELIORATE THIS SITUATION BY REPLACING COLIMIT $\hat{+}$ WITH AN OBJECT THAT BEHAVES NICELY WITH RESPECT TO THIS SWAPPING: HOCOLIMF.

HOMOTOPY COLIMIT SPECTRAL SEQUENCE (E)

(23)

* Prop: If $f: \mathcal{C} \rightarrow \text{Top}$ is a diagram w/ \mathcal{C} a small category, then there exists a spectral sequence of the form $E_{p,q}^2 = H_p(\text{SRep}, H_q F) \Rightarrow H_{p+q}(\text{Hocolim } F)$

This is an application of the Bousfield-Kan spectral sequence applied to the two sided simplicial bar construction $B_*(\mathcal{S}\mathcal{P}\mathcal{T}_*, \mathcal{C}, F)$.

With some work we can show $E_{p,q}^2 = H_p(\text{SRep}, H_q F) \cong \text{COLIM}_p H_q F$, a left derived functor, which offers an alternative perspective when attempting calculations.

Let X be a space $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ an open cover.

Recall that we can associate to \mathcal{U} an abstract simplicial complex w/

vertices $k_{\mathcal{U}} = \{U_\alpha\}_{\alpha \in A}$; simplices $S_{k_{\mathcal{U}}} = \{\bar{\Gamma} : \{U_\gamma\}_{\gamma \in \Gamma} : \emptyset \neq \Gamma \subseteq A \text{ finite}; U_\gamma \neq \emptyset\}$

We can actually form a category also written $k_{\mathcal{U}}$ (w/ abuse of notation, whatever more my handwriting allows)

w/ $\text{obj}(k_{\mathcal{U}}) = S_k$; $\text{mor}_{k_{\mathcal{U}}}(\bar{\Gamma}', \bar{\Gamma}) := \{\text{inc}_{\bar{\Gamma}', \bar{\Gamma}}\} \text{ if } \bar{\Gamma}' \subseteq \bar{\Gamma}$

e.g., $\mathcal{U} = \{U_0, U_1\}$; $k_{\mathcal{U}}$ as ASC is $\{\{U_0\}, \{U_1\}, \{U_0, U_1\}\}$, $\{U_0\} \subseteq \{U_0, U_1\}$. $\{U_0\} \rightarrow \{U_0, U_1\}$

Proposition: [This is a result of BULGER/ISAKSEN extending one of Segal]. Define $F: k_{\mathcal{U}} \rightarrow \text{Top}$ to be the "inclusion": $\bar{\Gamma} = \{U_\gamma\}_{\gamma \in \Gamma} \mapsto U_\Gamma = \bigcap_{\gamma \in \Gamma} U_\gamma$. Then $\text{Hocolim } F \rightarrow X$ is a weak homotopy eqn. In particular $H_2(\text{Hocolim } F) \cong H_2(X)$

* This gives us a spectral sequence $E_{p,q}^2 = \text{COLIM}_p H_q F \Rightarrow H_{p+q}(X)$.

This can be shown to coincide w/ the M.G.S. for certain simple covers, i.e. in particular, one can recover the M.G.S. open question if they coincide entirely.

FURTHER INVESTIGATIONS

IN SHOWING $\text{COLIM}_n F \cong H_n(\text{REP. } F)$, I HAD TO INVESTIGATE THE "FUNCTOR TENSOR PRODUCT"
 $W \otimes_F F$ OF TWO AB -VALUED FUNCTORS. IN PARTICULAR I SHOWED PROPER ^{PP} OBJECTS WERE FLAT,
 GIVING PROPER CHARACTERIZATIONS OF WHAT THAT MEANS IN $\text{FUN}(C^P, AB) = AB^P$. IN PARTICULAR, ONCE
 I HAD THE TENSOR PRODUCT IN THIS AB -ENRICHED CATEGORY, I WAS ABLE TO DEFINE AND USE THE
 SAME SPECTRAL SEQUENCE ARGUMENT TO SHOW THAT IT WAS BALANCED.

A PERSISTENCE MODULE IS A FUNCTOR $R \rightarrow \text{Vect}$ IN vect^R . IT MAY BE POSSIBLE NOW TO DEFINE
 THE TENSOR PRODUCT ; TOR OF TWO PERSISTENCE MODULES, AS WELL AS USING THEM IN SPECTRAL SEQUENCES.
 I'M CURRENTLY INVESTIGATING WHETHER, IF ANY, THIS HAS.

ALSO, I HAVE FORGOTTEN THE PRECISE IMPORTANCE OF BEING IN AB^P AS OPPOSED TO AB^{PP} ; MOST INVESTIGATION IF
 ANY OF MY ARGUMENTS ; CONSTRUCTIONS FAIL.