CSI 445/660 – Network Science – Fall 2015

Solutions to Homework V

Problem 1: Let G be a connected undirected graph with 100 nodes such that the degree of each node in G is at least 50. Find the *largest* possible value for the farness centrality of a node of G. Be sure to explain how you arrived at your answer.

Solution to Problem 1: To compute the required value, we use the following lemma.

Lemma 1: If $G(V, E)$ is a connected graph with 100 nodes and the degree of each node is at least 50, then the diameter of the graph is at most 2.

Proof of Lemma 1: Consider any pair of nodes u and v. We will show that there is a path of length at most 2 between u and v. Let $N(u)$ denote the set of neighbors of u. ($N(u)$ does *not* include u.) Thus, $|N(u)| \geq 50$. Let $Q = V - N(u) - \{u\}$. Note that $|Q| = 100 - |N(u)| - 1 \leq 49$, since $|N(u)| \geq 50$. If $v \in N(u)$, then the distance between u and v is 1, and the lemma holds. So, let $v \in Q$. Note that the degree of v is at least 50. However, $|Q| < 50$ and v is not a neighbor of u. Therefore, v must be adjacent to at least one node, say w, in $N(u)$. Thus, in this case, there is a path $\langle u, w, v \rangle$ of length 2 between u and v. This completes the proof. ■

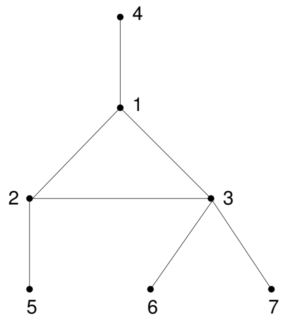
Calculating the largest possible farness centrality: Consider any node u of G and let d denote the degree of u. Thus, u has a path of length 1 to d nodes. By Lemma 1, the distance between u and the remaining $100 - d - 1 = 99 - d$ nodes (excluding u) is 2. Therefore, the farness centrality of u is $d + 2(99 - d) = 198 - d$. Since $d \geq 50$, the largest value of farness centrality is $198 - 50 = 148$.

Problem 1(b) (optional – for extra credit): Suppose the answer you arrived at for Problem 1 is α. Find a graph G which has 100 nodes and in which each node has a degree of at least 50 such that the farness centrality of *every* node in G is exactly α.

Your answer for Problem 1(b) must include a clear description of the graph (and *not* a drawing of the graph) along with an explanation of why the farness centrality of each node is α.

Solution to Problem 1(b): Let $G(V_1, V_2, E)$ be the complete bipartite graph with 50 nodes on each side of the bipartition. Consider any node u of G. In G, there are exactly 50 nodes which are at a distance of 1 from u and exactly 49 nodes which are at a distance of 2 from u. Thus, the farness centrality of u is $50 + 2 \times 49 = 148$, which matches the value derived above.

Problem 2: The underlying graph of a deterministic synchronous dynamical system (SyDS), where each node has a state value from {`0`, `1`}, is shown below. Assume that the system is *progressive*; that is, once a node reaches the state 1, it remains in that state forever.



The local function associated with each node is the 2-threshold function. Recall that a configuration specifies a state value for each node. This problem has two parts.

(a) Suppose the system starts at time 0 in the configuration where nodes 1, 6 and 7 are in state `1` while the other nodes are in state `0`. Show the successive configurations of the system until the system reaches a fixed point.

(b) Find an initial configuration with the *smallest number* of nodes in state `1` such that the system reaches the fixed point where every node is in state `1`. Be sure to indicate how you arrived at your solution.

Solution: Part (a): The sequence of configurations is shown below.

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Time t = 0:  (1, 0, 0, 0, 0, 1, 1)
Time t = 1:  (1, 0, 1, 0, 0, 1, 1)
Time t = 2:  (1, 1, 1, 0, 0, 1, 1)
```

The configuration `(1, 1, 1, 0, 0, 1, 1)` reached at time $t = 2$ is a fixed point.

Part (b): Consider node 4 whose degree is 1. Thus, the local function at node 1 has only two inputs. If the state of 4 is `0` in the initial configuration, node 4 will never change to state `1`, since the local function at node 4 is the 2-threshold function. Thus, in order to reach the final configuration where every node is in state `1`, the initial state of node 4 must be `1`. The same argument applies to nodes 5, 6 and 7. Thus, in any initial configuration from which the system can reach the fixed point `(1, 1, 1, 1, 1, 1, 1)` must have at least four of the nodes (namely, nodes 4, 5, 6 and 7) in state `1`.

Now, suppose the configuration of the system at time 0 is `(0, 0, 0, 1, 1, 1, 1)`. At time 1, node 3 will change to 1 and the resulting configuration is `(0, 0, 1, 1, 1, 1, 1)`. At time 2, nodes 1 and 2 will change to 1 and the resulting configuration is `(1, 1, 1, 1, 1, 1, 1)`, which is a fixed point.

Thus, the initial configuration with the smallest number of nodes in state `1`, which allows the system to reach the fixed point in which all nodes are in state `1` is `(0, 0, 0, 1, 1, 1, 1)`.

Problem 3: Let $G(V, E)$ be the projected network of an affiliation network G_A. Suppose G is connected and there is an independent set of size α in G. (In other words, G contains a set V' with α nodes such that there is no edge between any pair of nodes in V'.) **Prove or disprove:** The number of focal points in G_A is at least α.

Solution: The statement is **true**.

Proof: The proof is by contradiction. Suppose there is an affiliation network G_A with *less than* α focal points such that G is the projected network of G_A. Consider an independent set V' of size α in G. Since G is connected, at least one edge is incident on each node of V'. Thus, in G_A, each node of V' must have at least one edge to a focal point of G_A. However, since G_A has less than α focal points, by pigeonhole principle, two of the nodes in V', say u and v, must be adjacent to the same focal point, say f, of G_A. Then, the projected network must contain the edge $\{u, v\}$, contradicting the assumption that V' is an independent set. Hence, G_A must have at least α focal points. ∎
