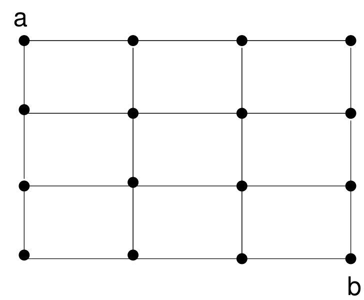
CSI 445/660 – Network Science – Fall 2015

Solutions to Homework I

Problem 1: Construct an undirected graph *without* edge weights that satisfies *all* of the following conditions: (i) the number of nodes in the graph is 16, (ii) each node has degree of at most 4 and (iii) the diameter of the graph is equal to 6.

Generalize the idea behind your construction for the above problem so that for any integer $n \geq 4$ such that n is a perfect square (i.e., $n = k^2$ for some integer k), your construction can be used to produce an undirected graph with n nodes so that the degree of each node is at most 4 and the diameter of the graph is close to $2\sqrt{n}$.

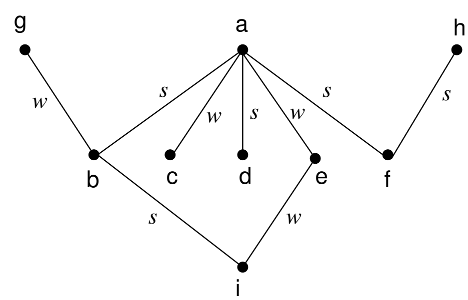
Solution: Consider the following graph, which contains 16 nodes arranged as a 4×4 grid.



In this graph, the degree of each node is at most 4. Further, its diameter is 6, which is the length of any shortest path between the two nodes marked as a and b. (Any path between a and b must traverse at least 3 horizontal edges and 3 vertical edges.)

To generalize the above construction for any integer $n = k^2$, we construct a $k \times k$ grid graph. (The above figure is the special case with $k = 4$.) The diameter of the graph is determined by a pair of opposite corner vertices; any path between such a pair of vertices must traverse at least $k-1$ horizontal edges and $k-1$ vertical edges. Thus, the diameter of the graph is $2k - 2 = 2\sqrt{n} - 2$, since $k = \sqrt{n}$.

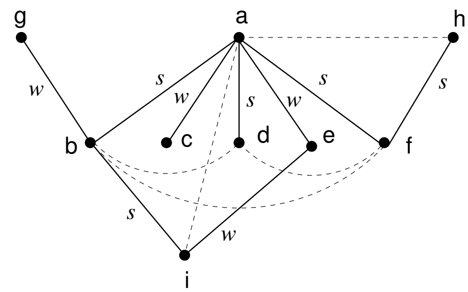
Problem 2: Consider the following graph in which edges are labeled s or w to indicate whether they represent strong or weak ties respectively.



Add a *minimum* number of undirected edges to the graph so that each node satisfies the *strong triadic closure* property.

Your answer must show the graph that results after you add all the necessary edges. You must also indicate the reason for the addition of each new edge.

Solution: The new edges needed to ensure that each node satisfies the strong triadic closure property are shown below as dashed lines.



The reasons for the new edges are as follows.

1. Edge $\{b, d\}$ was added because of the strong ties $\{a, b\}$ and $\{a, d\}$.
2. Edge $\{b, f\}$ was added because of the strong ties $\{a, b\}$ and $\{a, f\}$.
3. Edge $\{d, f\}$ was added because of the strong ties $\{a, d\}$ and $\{a, f\}$.
4. Edge $\{a, h\}$ was added because of the strong ties $\{a, f\}$ and $\{f, h\}$.
5. Edge $\{a, i\}$ was added because of the strong ties $\{b, a\}$ and $\{b, i\}$.

Problem 3: This problem has two parts. For each part, the graph example that you present must satisfy *both* of the following conditions: (i) it must be *connected* and (ii) it must have *at least* 5 nodes.

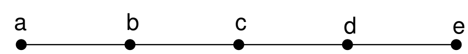
Recall that a **bridge** in a graph is an edge whose removal disconnects the graph. Further, an edge $\{x, y\}$ of a graph is a **local bridge** if x and y don't have any common neighbors.

(a) Show an example of an undirected graph G_1 such that each edge of G_1 is a bridge.

(b) Show an example of an undirected graph G_2 such that each edge of G_2 is a local bridge *but not* a bridge.

Solution:

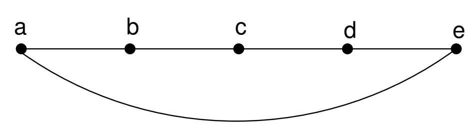
Part (a): Consider the following graph, which is a simple path with 5 nodes.



In the above graph, each edge is a bridge since deleting any edge splits the graph into two components.

Note: You can verify that if a graph is connected and each of its edges is a bridge, the graph must be a tree.

Part (b): Consider the following graph, which is a simple cycle with 5 nodes.



In the above graph, none of the edges is a bridge since deleting any edge *doesn't* split the graph into two or more components. However, each edge is a local bridge since for any edge $\{x, y\}$, the two end points x and y don't have any common neighbor.
