CSI 445/660 – Part 9
(Introduction to Game Theory)

Ref: Chapters 6 and 8 of [EK] text.

Game Theory Pioneers

- John von Neumann (1903–1957)
- Ph.D. (Mathematics), Budapest, 1925
- Contributed to many fields including Mathematics, Economics, Physics and Computer Science.
- Taught at the Institute for Advanced Study in Princeton.
- A key participant in the Manhattan Project.

Note: The book "Theory of Games and Economic Behavior" by von Neumann and Morgenstern (which marks the beginning of Game Theory) was first published in 1944.

Game Theory Pioneers

- Oskar Morgenstern (1902–1977)
- Ph.D. (Political Science), University of Vienna, 1925.
- Taught at Princeton University and the Institute for Advanced Study at Princeton.
- Many contributions to Economics and Mathematics.

- John Nash (1928–2015)
- Ph.D. (Mathematics), Princeton, 1950.
- Many deep contributions to Mathematics.
- Taught at MIT.
- Nobel Prize in Economics in 1994 and the Abel Prize in Mathematics in 2015.

Game Theory – Introduction

Game Theory: Useful in analyzing situations where outcomes depend on a person's decisions as well as the choices made by others interacting with the person.

Some Applications:

- Pricing a product (when other companies have a similar product).
- Auctions.
- Choosing routes in transportation networks.
- International relations.

An example of a 2-person game:

- Two students ("players") **A** and **B**.
- They have an exam and a **joint** presentation the next day.
- Each can only prepare for one and not both.

Game Example (continued)

- Score for the exam:
  - If the student studies, then score $= 92$.
  - If the student doesn't study, then score $= 80$.
- Score for the presentation:
  - If both **A** and **B** prepare, then score $= 100$.
  - If only one student prepares, then score $= 92$.
  - If neither **A** nor **B** prepares, then score $= 84$.
- **A** and **B** **cannot** contact each other; however, they must make a decision.

Analysis:

1. Both **A** and **B** prepare for the presentation.
   - Each gets 100 for the presentation.
   - Each gets 80 for the exam.
   - Average score for each $= 90$.

Game Example (continued)

Analysis: (continued)

2. Both **A** and **B** study for the exam.
   - Each gets 92 for the exam.
   - Each gets 84 for the presentation.
   - Average score for each $= 88$.
3. **A** studies for the exam and **B** prepares for the presentation.
   - **A** gets 92 for the exam and 92 for the presentation. So, average score for **A** $= 92$.
   - **B** gets 80 for the exam and 92 for the presentation. So, average score for **B** $= 86$.
4. **A** prepares for the presentation and **B** studies for the exam.
   - **A** gets 80 for the exam and 92 for the presentation. So, average score for **A** $= 86$.
   - **B** gets 92 for the exam and 92 for the presentation. So, average score for **B** $= 92$.

Game Example (continued)

Summary of the Analysis – Payoff matrix:

		B	
		P	E
A	P	(90,90)	(86,92)
	E	(92,86)	(88,88)

- Table shows the actions for **A** and **B**.
- The payoff value (x, y) means that **A**'s (average) score is x and **B**'s (average) score is y.
- **Note:** **A**'s payoff depends on **B**'s actions as well.

Basic ingredients of a game:

- A set of players (**Focus:** 2-person games).
- A set of options (**strategies**) for each player.
- A **payoff matrix** that specifies the payoff values for the players for each combination of strategies.

Note: The game is completely captured by the payoff matrix.

Standard Assumptions

- **One-shot** games: Each player chooses an action (strategy) **without** knowing what the other player will choose.
- Everything players care about is specified in the payoff matrix.
- Each player knows all the possible strategies and the full payoff matrix. (If not, we have games of **incomplete information**.)
- Players behave **rationally**.
  - Each player wants to maximize his/her payoff.
  - Each player succeeds in selecting an optimal strategy.

Illustration – Reasoning in the Exam-Presentation Game:

- Consider the reasoning from **A**'s point of view. (**B**'s point of view is similar because of symmetry.)

Reasoning in the Exam-Presentation Game

		B	
		P	E
A	P	(90,90)	(86,92)
	E	(92,86)	(88,88)

Case 1: Suppose **B** chooses E.

- If **A** chooses P, payoff $= 86$.
- If **A** chooses E, payoff $= 88$.
- Due to rationality, **A** must choose E in this case.

Case 2: Suppose **B** chooses P.

- If **A** chooses P, payoff $= 90$.
- If **A** chooses E, payoff $= 92$.
- Due to rationality, **A** must choose E in this case also.

Conclusion: No matter what **B** does, **A** must choose E to get maximum payoff.

Exam-Presentation Game (continued)

		B	
		P	E
A	P	(90,90)	(86,92)
	E	(92,86)	(88,88)

- Here, **A** has a strategy (namely, E) that is **strictly better** than all of **A**'s other strategies, no matter what **B** chooses.
- This is an example of a **dominant strategy**.
- By symmetry, **B** also has the same dominant strategy.

Consequence: Both players choose E and each gets a payoff of 88. (Rationality dictates this outcome.)

Exam-Presentation Game (continued)

B

		P	E
A	P	(90,90)	(86,92)
	E	(92,86)	(88,88)

- Rational play (i.e., both players choose E) leads to a payoff of 88 for each.
- If they both choose P, note that each of them can get a better payoff (namely, 90).
- Based on the rationality assumption, that choice cannot happen. (If **A** agrees to choose P, **B** will choose E to get a better payoff of 92.)

Prisoner's Dilemma:

- Idea developed by Merrill Flood and Melvin Dresher in 1950; formalized by Albert Tucker.
- Two prisoners P1 and P2, interrogated in two separate rooms.
- Actions for each: Confess (C) or Not Confess (NC).

Prisoner’s Dilemma (continued)

Payoff Matrix for Prisoner’s Dilemma

		P2	
		C	NC
P1	C	(–4,–4)	(0,–10)
	NC	(–10,0)	(–1,–1)

- Payoff value “-4” means a 4 year jail term.
- Maximizing payoff implies less jail time.

Analysis by Prisoner P1:

Case 1: Suppose **P2** chooses C.

- If **P1** chooses C, then payoff $= -4$.
- If **P1** chooses NC, then payoff $= -10$.
- So, the rational choice is C.

Prisoner’s Dilemma (continued)

Analysis by Prisoner P1 (continued):

		P2	
		C	NC
P1	C	(–4,–4)	(0,–10)
	NC	(–10,0)	(–1,–1)

Case 2: Suppose **P2** chooses NC.

- If **P1** chooses C, then payoff $= 0$.
- If **P1** chooses NC, then payoff $= -1$.
- So, the rational choice is again C.

Consequences:

- So, the dominant strategy for both is C.
- Each gets a payoff of -4.
- Even though there is a better alternative (namely, the action NC for both), it can’t be achieved through rational play.

Prisoner's Dilemma (continued)

- Canonical example of situations where cooperation is difficult to establish because of individual self-interest.
- Has been used as a framework to study many real-world situations (generally referred to as **arms races**).

Example: Use of performance enhancing drugs in professional sports.

		A2	
		DU	U
A1	DU	(3,3)	(1,4)
	U	(4,1)	(2,2)

- **Strategies:** Use drugs (U) and Don't use drugs (DU).
- Dominant strategy for both players is U with $(2, 2)$ as the payoff.
- The alternative with better payoff (namely, $(3, 3)$) won't be reached.

Prisoner's Dilemma (continued)

- For situations like Prisoner's Dilemma to arise, payoffs must be chosen in a certain way.
- Even small changes to the payoff matrix can change the situation significantly.

Example: A modified payoff table for the Exam-Presentation game.

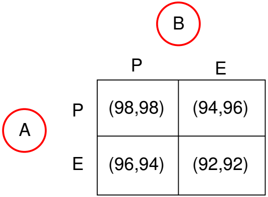
		B	
		P	E
A	P	(98,98)	(94,96)
	E	(96,94)	(92,92)

- Now, the dominant strategy for both players is P.
- The corresponding payoff is $(98, 98)$.

Some Formal Definitions

Best Response:

- Represents the best choice for a player, given the other player's choice.



		B	
		P	E
A	P	(98,98)	(94,96)
	E	(96,94)	(92,92)

- If **B** chooses E, **A**'s best response is P.

Notation:

- $P_1(x, y)$: Represents payoff to Player 1 when Player 1 uses strategy x and Player 2 uses strategy y.
- $P_2(x, y)$: Similar but represents payoff to Player 2.

Some Formal Definitions (continued)

Definition: A strategy s for Player P1 is a **best response** to strategy t for Player 2 if $P_1(s, t) \ \geq \ P_1(s', t)$ for **all** other strategies s' of P1.

Note: Best response strategy for P2 is defined similarly.

Additional Definitions:

- In general, there may be more than one best response.
- If there is a **unique** best response, it is a **strict best response**.
- A strategy s for P1 is a **strict best response** for strategy t by P2 if $P_1(s, t) \ > \ P_1(s', t)$ for **all** other strategies s' of P1.

Some Formal Definitions (continued)

Additional Definitions (continued):

- A **dominant strategy** for P1 is a strategy that is a **best response** to **every** strategy of P2.
- A **strictly dominant strategy** for P1 is a strategy that is a **strict best response** to **every** strategy of P2.

Example:

		B	
		P	E
A	P	(98,98)	(94,96)
	E	(96,94)	(92,92)

- Here, P is a **strictly dominant strategy** for both players.

Note: When a player has a strictly dominant strategy, the player should be expected to use it (due to rationality).

Strict Dominant Strategies

- **So far:** Games in which both players had strict dominant strategies.
- **Now:** Games in which **only one** player has a strictly dominant strategy.

The setting: (Manufacturing/Marketing)

- There are two versions, namely low cost (L) and upscale (U), of a product X. (Strategies: L and U.)
- There are two firms F1 and F2 (the players).
- **Market segment:** 60% of the population will buy L and 40% will buy U.
- F1 and F2 capture 80% and 20% of the market respectively.
- If only one firm manufactures L (or U), it will capture 100% of the corresponding market.

Market/Manufacturing Game (continued)

Computing Payoff Matrix:

- Both F1 and F2 manufacture L.
  - Market segment is 60%.
  - F1 captures 80% of the market (i.e., 48% overall) and F2 captures 12%.
  - So, the payoff for this case is $(48, 12)$.
- Other combinations can be computed similarly.

Resulting Payoff Matrix:

		F2	
		L	U
F1	L	(48,12)	(60,40)
	U	(40,60)	(32,8)

Market/Manufacturing Game (continued)

Analysis by F1:

		F2	
		L	U
F1	L	(48,12)	(60,40)
	U	(40,60)	(32,8)

- **Case 1:** F2 chooses L. Here, F1's strict best response is L.
- **Case 2:** F2 chooses U. Again, F1's strict best response is L.

Conclusion: L is **the** strictly dominant strategy for F1.

Analysis by F2:

- **Case 1:** F1 chooses L. F2's strict best response is U.
- **Case 2:** F1 chooses U. F2's strict best response is L.

Conclusion: F2 does **not** have a strictly dominant strategy.

Market/Manufacturing Game (continued)

What is the outcome of the game?

		F2	
		L	U
F1	L	(48,12)	(60,40)
	U	(40,60)	(32,8)

Reasoning used by F2:

- Due to rationality, F1 will choose L, its strictly dominant strategy.
- So, F2's best response is U and the resulting payoff is $(60, 40)$.

Note: F2's reasoning relies on **common knowledge**:

- Both players know the complete payoff matrix.
- Both players know that each player knows all the rules and will act rationally.

The Concept of Equilibrium

Motivation:

- Suppose we have a game where neither player has a strictly dominant strategy.
- John Nash proposed the concept of **equilibrium** to predict the outcomes of such games.

Example: Consider the following game.

F2

		A	B	C
F1	A	(4,4)	(0,2)	(0,2)
	B	(0,0)	(1,1)	(0,2)
	C	(0,0)	(0,2)	(1,1)

- In this game, no player has a strictly dominant strategy.
- **Reason:** If F2 chooses A, F1's best response is A; however, if F2 chooses B, F1's best response is B.

The Concept of Equilibrium (continued)

Definition: A pair of strategies (x, y) is a **pure Nash equilibrium** (pure NE) if x is a best response to y and vice versa.

Example:

F2

		A	B	C
	A	(4,4)	(0,2)	(0,2)
F1	B	(0,0)	(1,1)	(0,2)
	C	(0,0)	(0,2)	(1,1)

- Consider the strategy pair (A, A).
- The payoff is $(4, 4)$.

- If F1 plays A, F2's best response is A and vice versa.
- So, (A, A) is a **pure NE** for this game.
- Once the players choose (A, A), there is **no incentive** for either player to switch to another strategy **unilaterally**.

The Concept of Equilibrium (continued)

Example (continued)

		F2		
		A	B	C
F1	A	(4,4)	(0,2)	(0,2)
	B	(0,0)	(1,1)	(0,2)
	C	(0,0)	(0,2)	(1,1)

- Consider the strategy pair (B, B).
- The payoff is $(1, 1)$.
- If F1 plays B, F2's best response is C (with payoff $= 2$).
- So, F2 has an incentive to switch and (B, B) is **not** a pure NE.

Notes:

- Similarly, (B, C) is **not** a pure NE. (F1 has an incentive to switch to C.)
- In fact, the **only** pure NE for the game is (A, A).

Remarks on the Equilibrium Concept

- At an equilibrium, there is no force pushing it to a different outcome. (It is bad for a player to switch **unilaterally** to a different strategy.)
- If a pair of strategies (x, y) is **not** a pure NE, players cannot believe that this pair would actually be used (since one of the players has an incentive to switch).
- The equilibrium concept is not based on rationality alone.
- It is based on beliefs. (If each player believes that the other player will use a strategy which is part of an NE, then the other player has an incentive to use his/her part of the NE.)

Coordination Games

Example:

- Players A and B are preparing slides for a presentation.
- They can use Power Point (PP) or Keynote (KN).

Payoff matrix:

		P2	
		PP	KN
P1	PP	(1,1)	(0,0)
	KN	(0,0)	(2,2)

- This is a “coordination game” since the goal is to choose a common strategy by both players.

- For this game, both (PP, PP) and (KN, KN) are pure NEs.
- An **unbalanced** coordination game – payoffs for the two pure NEs are different.

Coordination Games (continued)

Contexts for coordination games – Some examples:

- Manufacturing companies work together to decide the unit of measurement (English or Metric) for their machinery.
- Units of an army must decide on a strategy to attack the enemy.
- People trying to meet each other in a shopping mall must decide where to meet.

Which Nash Equilibrium?

- A coordination game may have several pure NEs.
- Which will the players choose?
- Thomas Schelling introduced the idea of a **focal point** to study this.
- **Basic idea:** There may be natural reasons (possibly external to the payoff matrix) that allow people to choose an appropriate NE.

Coordination Games – Focal Point

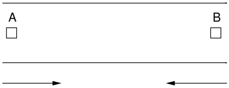
Example 1: Power Point vs Keynote game.

P2

		PP	KN
P1	PP	(1,1)	(0,0)
	KN	(0,0)	(2,2)

- The payoff is higher for the (KN, KN) equilibrium.
- So, if the focal point is “higher payoff”, players will prefer (KN, KN).

Example 2: Cars on a (dark) undivided road.



Strategies: L or R.

Coordination Games – Focal Point (continued)

Example 2 (continued):

		B	
		L	R
A	L	(Inf, Inf)	(–Inf, –Inf)
	R	(–Inf, –Inf)	(Inf, Inf)

- **Note:** "Inf" denotes ∞.
- Value $-\infty$ denotes "disaster".
- Value ∞ denotes "ok" (nobody gets hurt).

- Both (L, L) and (R, R) are pure NEs.
- The choice is based on **social convention**.
  - In USA, each driver uses R.
  - In UK, each driver uses L.

Coordination Games – Focal Point (continued)

Example 3 (Battle of the Sexes):

- Two people want to watch a movie together.
- **Strategies:** Action movie (A) or Romantic comedy (R).
- They want to coordinate on their choice.

		P2	
		R	A
P1	R	(1, 2)	(0, 0)
	A	(0, 0)	(2, 1)

- (R, R) and (A, A) are both pure NE.
- (R, R) is better for P2 while (A, A) is better for P1.

Consequence: Additional information (e.g. a convention that exists between the players) is needed to predict which equilibrium will be chosen.

Anti-Coordination Games

Hawk-Dove Game:

- Dividing a piece of food (weight: 6 lbs) among two animals (players).
- **Strategies:** Hawk (aggressive behavior) or Dove (passive behavior).

		P2	
		D	H
P1	D	(3, 3)	(1, 5)
	H	(5, 1)	(0, 0)

- If both choose H, they "destroy" each other and nobody gets anything.
- (H, D) and (D, H) are both pure NE; these correspond to "anti-coordination".
- We can't predict which of these equilibria will be chosen without additional information about the players.

Anti-Coordination Games (continued)

A context for the Hawk-Dove game:

- Two neighboring countries (the players).
- Hawk and Dove represent strategies with respect to foreign policy.
- If both countries are aggressive, they may go to war (which may be disastrous to both).
- If both are passive, then each country has an incentive to switch.
- **Equilibrium:** One country is aggressive and the other is passive.

Games Without Pure Nash Equilibria

- When games have one or more pure NE, we have some information about the outcome (i.e., the players are likely to choose the strategies corresponding to one of the equilibria).
- There are games where is there is **no** pure NE.
  (**Example:** Matching Pennies game – to be discussed next.)
- The notion of equilibrium for such games is based on **randomized** strategies (**mixed strategies**).

A Game Without any Pure Nash Equilibrium

Matching Pennies:

- Two players (P1 and P2), each holding a penny.
- **Strategies:** Head (H) or Tail (T).
- If coins match, P1 loses the penny to P2.
- Otherwise, P2 loses the penny to P1.

		P2	
		H	T
P1	H	(−1, +1)	(+1, −1)
	T	(+1, −1)	(−1, +1)

- An example of a **zero sum** game.
- In every outcome, what one player wins is exactly what the other player loses.

A Game Without any Pure Nash Equilibrium

Matching Pennies (continued):

P2

		H	T
P1	H	(–1, +1)	(+1, –1)
	T	(+1, –1)	(–1, +1)

- There is no dominant strategy for either player.
- There is no pure NE in this game.

Reason:

- For each pair of strategies, there is a player with a payoff of -1.
- That player has an incentive to switch.

What should the players do?

- If P1 knows what P2 is going to do, then P1 can always get a payoff of $+1$.
- So, P2 should make it difficult for P1 to **guess** what P2 will do; that is, employ **randomization**.

Mixed Strategies & Expected Payoff

Basic Ideas:

- Each player chooses a **probability** for playing H.
- So, each strategy is a real number in [0, 1].
- If probability of H is p, then probability of T $= 1 - p$.
- Players are "mixing" the options H and T (**mixed strategies**).
- When $p = 0$ or $p = 1$, we get the corresponding **pure** strategy.
- **Expected payoffs** must be considered.
- **Rationality:** Players want to maximize their expected payoffs.

Mixed Strategies & Expected Payoff (continued)

Notation:

- P1 and P2 play H with probabilities p and q respectively.
- Each mixed strategy is a probability value (i.e., the probability of playing H).

Definition: If P1's mixed strategy is p, then the **best response** of P2 is a probability value q that maximizes P2's expected payoff.

Definition: A **mixed Nash equilibrium** (mixed NE) is a pair (p, q) of probability values for P1 and P2 such that p is the best response for q and vice versa.

Note: In a mixed equilibrium, no player has an incentive to change his/her mixed strategy (i.e.,probability value) unilaterally.

A Mixed Nash Equilibrium for Matching Pennies

Lemma 1: No pure strategy can be part of a mixed NE for the Matching Pennies game.

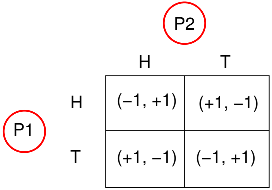
Proof sketch:

- We already know that there is no pure NE for the game; that is, both P1 and P2 **cannot** use pure strategies in an equilibrium.
- Suppose P1 uses **pure** strategy H while P2 uses mixed strategy q, where $0 < q < 1$.
- Now, P2 has the incentive to change the strategy to $q = 1$ (i.e., play H all the time) to ensure a win every time.
- Other cases are handled similarly.

Consequence: In any mixed NE for the Matching Pennies game, the probability values can't be either 0 or 1.

Mixed Strategies & Expected Payoff (continued)

Computing expected payoff (P2's Analysis):



P2

		H	T
P1	H	(−1, +1)	(+1, −1)
	T	(+1, −1)	(−1, +1)

- P2 plays H with probability q (and T with probability $1 - q$).

Case 1: Suppose P1 plays the pure strategy H.

- P1 loses 1 cent each time P2 plays H, that is, with probability q.
- P1 gains 1 cent each time P2 plays T, that is, with probability $1 - q$.
- So, expected payoff for P1 $= -q + (1 - q) = 1 - 2q$.

Mixed Strategies & Expected Payoff (continued)

Computing expected payoff (continued):

P2

		H	T
P1	H	(–1, +1)	(+1, –1)
	T	(+1, –1)	(–1, +1)

- P2 plays H with probability q (and T with probability $1 - q$).

Case 2: Suppose P1 plays the pure strategy T.

- P1 gains 1 cent each time P2 plays H, that is, with probability q.
- P1 loses 1 cent each time P2 plays T, that is, with probability $1 - q$.
- So, expected payoff for P1 $= q - (1 - q) = 2q - 1$.

Summary:

- P1's expected payoff when using pure strategy H $= 1 - 2q$.
- P1's expected payoff when using pure strategy T $= 2q - 1$.

Mixed Strategies & Expected Payoff (continued)

Lemma 2 (Generalization): Suppose P1 and P2 use strategies p and q respectively. Then

- The expected payoff for P1 $= (2p-1)(1-2q)$.
- The expected payoff for P2 $= (1-2p)(1-2q)$.

Lemma 3: If $1-2q \neq 2q-1$, then a pure strategy maximizes P1's expected payoff.

Proof sketch: Suppose $1-2q \neq 2q-1$. Then either $1-2q \; > \; 2q-1$ or $1-2q \; < \; 2q-1$.

Case 1: $1-2q \; > \; 2q-1$.

- Here, $1-2q \; > \; 0$.
- In this case, the expected payoff for P1 $= (2p-1)(1-2q)$.
- This function increases as p increases; it is maximized when $p = 1$.
- Thus, using pure strategy H maximizes P1's expected payoff.

Mixed Strategies & Expected Payoff (continued)

Proof sketch for Lemma 3 (continued)

Case 2: $1 - 2q < 2q - 1$.

- Pure strategy T maximizes P1's expected payoff. (The argument is similar to that of Case 1.)

Lemma 4: If $1 - 2q \neq 2q - 1$, then there is no mixed NE for the game.

Reason:

- When $1 - 2q \neq 2q - 1$, Lemma 3 shows that P1's best response is a pure strategy.
- However, Lemma 1 points out that no pure strategy can be part of a mixed NE for the game.

Mixed Strategies & Expected Payoff (continued)

Consequences of Lemma 4:

- P2 must choose q so that $1 - 2q = 2q - 1$, that is, $q = 1/2$ to get a mixed NE.
- Similarly, P1 must choose $p = 1/2$ for a mixed NE.
- Thus, the only mixed NE for the game is $(1/2, 1/2)$.

Additional Remarks:

- If P2 chooses $q < 1/2$ (i.e., plays T more often than H), then P1 will use the pure strategy H to gain advantage.
- If P2 chooses $q > 1/2$ (i.e., plays H more often than T), then P1 will use the pure strategy T to gain advantage.

Mixed Strategies & Expected Payoff (continued)

Additional Remarks (continued)

- When P2 chooses $q = 1/2$, both the pure strategies (H and T) give the same expected payoff to P1.
- The choice $q = 1/2$ by ensures that neither of the pure strategies offers any advantage to P1 (i.e., makes P1 **indifferent** between choosing H or T).

Theorem: [Nash 1950]

Every game with a finite number of players has at least one mixed equilibrium.

Mixed Strategies & Expected Payoff (continued)

Another example for Mixed NE Computation: Consider the following game.

P2

		A	B
P1	A	(90, 10)	(20, 80)
	B	(30, 70)	(60, 40)

- **Exercise:** Does this game have one or more pure NE?

P2's Analysis: Suppose P2 plays A with probability q (and B with probability $1 - q$).

Case 1: P1 chooses pure strategy A.

Outcome	**Probability**	**Payoff to P1**
(A,A)	q	90
(A,B)	$1 - q$	20

P1's expected payoff in Case 1 $= 90 \times q + 20 \times (1 - q) = 70q + 20.$

Mixed Strategies & Expected Payoff (continued)

Example for Mixed NE Computation (continued):

P2

		A	B
P1	A	(90, 10)	(20, 80)
	B	(30, 70)	(60, 40)

- **Case 2:** P1 chooses pure strategy B.

Outcome	Probability	Payoff to P1
(B,A)	q	30
(B,B)	$1-q$	60

P1's expected payoff in Case 2 $= 30 \times q + 60 \times (1-q) = -30q + 60.$

To make P1 indifferent with respect to pure strategy, we must have

$$70q + 20 \;=\; -30q + 60 \quad \text{or} \quad q = 0.4.$$

Mixed Strategies & Expected Payoff (continued)

Example for Mixed NE Computation (continued):

		P2	
		A	B
P1	A	(90, 10)	(20, 80)
	B	(30, 70)	(60, 40)

- A similar calculation shows that P1 must choose $p = 0.3$.
- So $(0.3, 0.4)$ is a mixed NE for this game.

Power Point vs Keynote coordination game:

		P2	
		PP	KN
P1	PP	(1,1)	(0,0)
	KN	(0,0)	(2,2)

- This game has two pure Nash equilibria, namely (PP, PP) and (KN, KN).
- It also has a **mixed** NE.

Games with Pure and Mixed NE

P2's Analysis: Suppose P2 plays PP with probability q (and KN with probability $1-q$).

Case 1: P1 chooses the pure strategy PP.

Outcome	**Probability**	**Payoff to P1**
(PP,PP)	q	1
(PP,KN)	$1-q$	0

P1's expected payoff in Case 1 $= q$.

Case 2: P1 chooses the pure strategy KN. P1's expected payoff in this case $= 2(1-q)$.

To obtain a mixed NE, we have $q = 2(1-q)$ or $q = 2/3$.

By symmetry, $p = 2/3$. So, $(2/3,\ 2/3)$ is a mixed NE for this game.

Complexity of Finding Nash Equilibria

- For the form of games we have considered (called **normal form**), determining whether a game has a pure NE is efficiently solvable.
- In general, with many players and more complex specifications of strategies, determining whether a game has a pure NE is **NP**-complete.
- Finding a mixed NE for a game is complete for another complexity class called **PPAD**.
- The class **PPAD** contains problems for which we know at least one solution exists but finding a solution is difficult ("needle in a haystack").
- It is believed that the class **PPAD** is different from the class **NP**.

Pareto and Social Optimality

Presentation-Exam Game (discussed earlier):

		B	
		P	E
A	P	(90,90)	(86,92)
	E	(92,86)	(88,88)

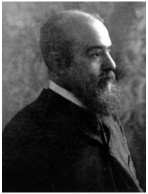
- E is a dominant strategy for both **A** and **B**.
- (E, E) is also a **pure NE**.
- The payoff for (E, E) is (88, 88).
- (P, P) is **not** a pure NE; **A** has an incentive to switch to E.

Additional Notes:

- Outcome (P, P) **can't** be reached under rational behavior (i.e., when players optimize **individually**).
- Other mechanisms are needed to allow such outcomes.

Exercise: Show that there is **no** mixed NE for the above game when the probability values are required to be **strictly between** 0 and 1.

Pareto Optimality



- Vilfredo Pareto (1848–1923)
- Ph.D. (Civil Engineering), University of Turin, Italy.
- Pareto Principle (or "80-20 rule") is named after him.
- Made many important contributions to Microeconomics.

Towards a definition of Pareto Optimality:

- The four payoff vectors in the Presentation-Exam game are:

  (90, 90), (86, 92), (92, 86), (88, 88)
- The vector $(90, 90)$ is **strictly better** than $(88, 88)$ (since it allows both players to do better).

Pareto Optimality (continued)

- Suppose we add one more vector $(88, 90)$ to the set to get:

  (90, 90), (86, 92), (92, 86), (88, 88), (88, 90)

- The vector $(88, 90)$ is **at least as good as** $(88, 88)$ since
  - no player is worse off choosing $(88, 90)$ over $(88, 88)$ and
  - at least one player's payoff is better off in $(88, 90)$ compared to that in $(88, 88)$.

- **Terminology:** Payoff vector $(88, 90)$ **dominates** the payoff vector $(88, 88)$. (Alternatively, $(88, 88)$ is **dominated by** $(88, 90)$.)

Pareto Optimality (continued)

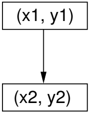
Definition: A payoff vector $(x_1,\ y_1)$ **dominates** another payoff vector $(x_2,\ y_2)$ if **all** the following conditions hold:

1. $x_1\ \geq\ x_2$,
2. $y_1\ \geq\ y_2$ and
3. **at least one** of these inequalities is **strict** (i.e., '$>$' instead of '$\geq$').

Examples:

- The vector $(88,\ 90)$ dominates $(88,\ 88)$.
- The vector $(86,\ 92)$ **does not** dominate $(88,\ 88)$.
- A vector $(x,\ y)$ **does not** dominate itself.

Representation:



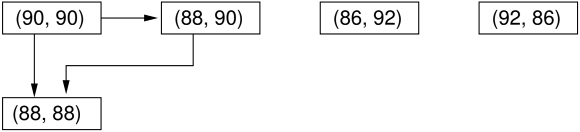
- $(x_1,\ y_1)$ dominates $(x_2,\ y_2)$.

Pareto Optimality (continued)

- Consider the following set X of vectors

$$X = \{(90, 90),\ (86, 92),\ (92, 86),\ (88, 88),\ (88, 90)\}.$$

- The domination relationship among these vectors is as follows:



- Vectors which don't have an incoming edge are "non-dominated".
- They represent **Pareto optimal** payoffs.

Definition: A pair of strategies is **Pareto optimal** if the payoff vector for the pair is **not** dominated by the payoff vector for any other pair of strategies.

Pareto Optimality (continued)

Example:

		B	
		P	E
A	P	(90,90)	(86,92)
	E	(92,86)	(88,88)

- Here, the Pareto optimal strategy pairs are (P, P), (P, E) and (E, P).
- The only pure Nash equilibrium (E, E) is **not** Pareto optimal. (Interestingly, that is the **only** strategy pair that is not Pareto optimal!)

How can players reach a Pareto optimal outcome?

- They must sign a binding contract before the game.
- If there is no such contract, some player may have an incentive to switch to another strategy (since a Pareto optimal strategy need not be a pure NE).

Social Optimality

- Some Pareto optimal strategies provide outcomes that are good for both players ("good for society").

  Example: In the Presentation-Exam game, the strategy pair $(P,\ P)$ (with payoff $= (90,\ 90)$) is better for both players than the strategy pair $(E,\ E)$ (with payoff $= (88,\ 88)$).

- There are other ways to define **social optimality**.

Definition: A pair of strategies (α, β) is a **social optimum** (or a **social welfare maximizer**) if it maximizes the **sum** of the payoffs to the two players.

Example: In the Presentation-Exam game, the strategy pair $(P,\ P)$ (with payoff $= (90,\ 90)$) is the **unique** social optimum with a total value of 180.

Pareto Optimality vs Social Optimality

Lemma: (1) Every social optimum is also Pareto optimal.
(2) A Pareto optimal solution need not be a social optimum.

Proof:

Part 1: Suppose a payoff vector $(x,\ y)$ is a social optimum but not Pareto optimal.

- Then, there must be another payoff vector $(x',\ y')$ which **dominates** $(x,\ y)$.
- Thus, $x' \ \geq \ x, \ \ y' \ \geq \ y,$ and at least one inequality is strict.
- Therefore, $x' + y' \ > \ x + y,$ and this contradicts the assumption that $(x,\ y)$ is a social optimum.

Part 2: In the Presentation-Exam game, $(86,\ 92)$ is Pareto optimal. However, it is not a social optimum (which is $(90,\ 90)$).

Nash Equilibrium vs Social Welfare Maximizer

Note: We consider pure Nash equilibria.

- A pure Nash Equilibrium need not be Pareto optimal.

  Example: In the Presentation-Exam game, (88, 88) is a pure NE but **not** Pareto optimal (it is dominated by (90, 90)).

- A pure Nash Equilibrium need not be a social optimum.

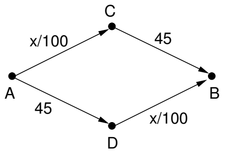
  Example: In the Presentation-Exam game, (88, 88) is a pure NE but **not** the social optimum (which is (90, 90)).

Note: We will consider two contexts where we can **quantify** how the total value of a pure NE compares with the social optimum.

- Traffic in transportation networks.
- Cost-sharing in computer networks.

Applying Game Theory to Network Problems

Example – Traffic in transportation networks:

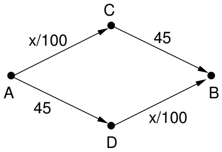


- Cars want to go from A to B.
- The value on each edge is the travel time.
- On the edges (A, C) and (D, B), travel time is a **linear** function of the number of cars x. (These edges are **sensitive to congestion**.)
- Number of cars = 4000.

- If all cars use the route A-C-B, travel time for each car = (4000/100) + 45 = 85.
- If all cars use the route A-D-B, travel time for each car is again 85.
- Suppose cars divide evenly between the two routes. Then travel time for each car = (2000/100) + 45 = 65.

Applying Game Theory ... (continued)

The underlying game:



- 4000 players (Drivers)
- **Strategies:** $\{$A-C-B, A-D-B$\}$
- Payoff for each player: Travel time

Notes:

- We will **minimize** payoffs.
- There is no dominant strategy for any player; the travel time for a route depends on the number of players using that route.
- There are **many** pure Nash equilibria for this game.

Applying Game Theory ... (continued)

Theorem:

1. Every combination of strategies that divides the 4000 cars **evenly** between the two routes is a pure NE.
2. In every pure NE, each route has the same number of cars.

Proof sketch for Part 1: Consider any combination of strategies that has 2000 cars along each route. (Travel time for each player $= 65$.)

Question: Does any single player have an incentive to switch to the other route?

- Suppose one player switches from A-C-B to A-D-B.
- After the switch, there will be 2001 cars along A-D-B.
- New travel time along A-D-B $= 45 + (2001/100) > 65$; that is, the payoff is worse.
- So, no player has an incentive to switch (unilaterally).

Applying Game Theory ... (continued)

Proof sketch for Part 2: Suppose there a pure NE with t cars on A-C-B and $4000 - t$ cars on A-D-B.

To prove: $t = 4000 - t$ (which implies that $t = 2000$).

Case 1: $t > 4000 - t$.

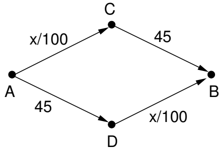
- Here, it is easy to verify that $4000 - t \leq t - 2$.
- Current travel time for player along A-C-B $= 45 + (t/100)$.
- Switch one player from A-C-B to A-D-B.
- New travel time for the player is

  $$45 + [(4000 - t) + 1]/100 \leq 45 + [(t - 2) + 1]/100 < 45 + (t/100)$$

- Thus, the player has an incentive to switch and we don't have a pure NE.

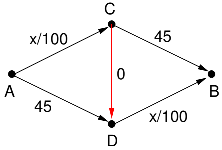
Case 2: $t < 4000 - t$: The proof is similar.

Braess's Paradox



- In any pure NE, each of the two routes is used by 2000 players.
- Travel time for each player = 65.

After adding the edge (C, D):



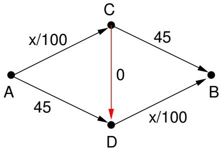
- **Strategies:** {A-C-B, A-C-D-B, A-D-B}.
- **Surprise:** There is a unique pure NE where **every** player uses the route A-C-D-B.
- Travel time for each player = 80.

Verifying that A-C-D-B a pure NE:

- Suppose a player wants to switch to A-D-B.
- New travel time $= 45 + (4000/100) = 85$.
- So, no player has an incentive to switch.

Braess's Paradox (continued)

Why A-C-D-B is a unique pure NE – A brief explanation:



- Consider the flow pattern with 2000 players using A-C-B and 2000 using A-D-B.
- Travel time for each player = 65.

- Suppose a player X switches from A-C-B to A-C-D-B.
- Travel time for X = $(2000/100) + (2001/100) = 40.01$.
- So, X has an incentive to switch.
- So, the above flow pattern is not a pure NE.

Note: A similar argument applies to other flow patterns.

Remark: Removing the red edge (C, D) creates a better pure NE.

Braess's Paradox (continued)

Braess's Paradox:

- Travel time in a pure NE increases even though resources were added to the system.
- Named after Dietrich Braess (1938–), a Mathematician from Germany.
- Result published in 1969.

Empirical observations supporting Braess's Paradox:

- In Seoul (South Korea), the destruction of a 6-lane highway (as part of a project called "Cheonggyecheon Restoration") actually reduced the commute time for many drivers.
- In Stuttgart (Germany), closing a major road actually decreased traffic congestion.
- In 1990, the closing of 42nd Street in New York City significantly reduced traffic congestion.

Braess's Paradox (continued)

Additional Remarks:

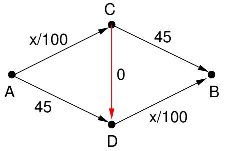
- Braess's paradox shows a situation where introducing a new choice (strategy) makes the payoff worse for everyone.
- Other such situations also exist.

Example: Prisoner's dilemma.

		P2	
		NC	C
P1	NC	(–1,–1)	(–10,0)
	C	(0,–10)	(–4,–4)

- If each player is given only one strategy, namely NC, things would be better for both.
- Adding a second strategy (C) introduces difficulties.
- For each player, C is the strictly dominant strategy. So, the outcome is (C, C), which is worse for both compared to (NC, NC).

Social Cost of Traffic at Equilibrium



- No. of players = 4000.
- **Pure NE:** All 4000 players use the route A-C-D-B.
- **Social optimum:** 2000 players use A-C-B and 2000 use A-D-B.

- For the pure NE, travel time for each player = 80.
- So, total time (cost) for this pure NE = $4000 \times 80 = 320,000$.
- For the social optimum, travel time for each player = 65.
- So, cost of social optimum = $4000 \times 65 = 260,000$.
- This example shows that cost of pure NE can be larger than that of social optimum.

A General Model for the Problem

Ref: **[Roughgarden & Tardos, 2002]**

- Road network represented by a directed graph with predefined origin and destination.
- For each edge e, a linear travel time function given by

  $$T_e(x) \ = \ a_e\, x + b_e$$

  where a_e and b_e are constants and x is the number of cars on the edge e.
- A **traffic pattern** specifies a path for each car. (Paths are assumed to be simple.)
- Social cost of a traffic pattern Z is the **sum** of the travel times for all the drivers.

Research Question 1: Under this model, is there always a (pure) Nash equilibrium?

Answer: Yes.

A General Model ... (continued)

Outline of algorithm for producing an equilibrium:

```
1  Start with any traffic pattern $Z$.
2  while ($Z$ is not an equilibrium) do
       Move one driver (chosen arbitrarily) to a better path.
       Let $Z$ denote the new traffic pattern.
```

Notes:

- The above algorithm **always** terminates.
- The traffic pattern produced when the algorithm terminates is a Nash equilibrium.

Proof idea: (**Potential Function Argument**)

- Define a suitable function (called a **potential function**).
- Argue that every time a driver is moved to a better path, the value of the function **decreases**.
- Also argue that the value of the function **cannot** decrease below a lower limit (at which point no switches can occur).

A General Model ... (continued)

Research Question 2: How does the total travel time at an equilibrium compare with the social optimum?

Theorem: **[Roughgarden & Tardos, 2002]**

There is always an equilibrium travel pattern Z such that the travel time of Z is **at most twice** the social optimum.

Further improvement: **[Anshlevich et al., 2004]**

There is always an equilibrium travel pattern W such that the travel time of W is **at most** $4/3$ times the social optimum.

Notes:

- For some non-linear travel cost functions, the cost of an equilibrium can be much larger than that of social optimum.
- For networks with more complicated travel cost functions, an equilibrium may not exist.

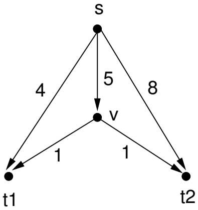
A Model for Multicast in Computer Networks

Ref: [Anshlevich et al. 2004]

- A directed graph with a cost $c(e) \geq 0$ for each edge, a designated **source** node s and k distinct **terminal** nodes t_1, t_2, …, t_k (one for each player).
- Each player P_i wants to set up a directed path from s to terminal t_i $(1 \leq i \leq k)$.
- Paths chosen by different players may **share** edges.
- If an edge e is shared by q players, then the cost for each player is $c(e)/q$. (So, there is an **incentive** to share edges.)
- Each player wants to **minimize** the cost of their path.
- Social cost of any solution is the **sum** of the costs of all the players.

A Model for Multicast ... (continued)

Example:

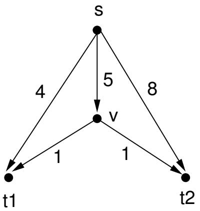


- Two players P_1 and P_2.
- Choices for P_1: $s \rightarrow t_1$ or $s \rightarrow v \rightarrow t_1$.
- Choices for P_2: $s \rightarrow t_2$ or $s \rightarrow v \rightarrow t_2$.
- **Initial choice:** P_1 uses the edge $s \rightarrow t_1$ and P_2 uses the edge $s \rightarrow t_1$.

Moves:

- P_1 notices that switching to $s \rightarrow v \rightarrow t_1$ does **not** decrease the cost.
- P_2 notices that switching to $s \rightarrow v \rightarrow t_2$ **does** decrease the cost (from 8 to 6), and does the switch.
- Now, P_1 notices that switching to $s \rightarrow v \rightarrow t_1$ **does** decrease the cost (4 to 3.5) because of the **shared** cost for $s \rightarrow v$.

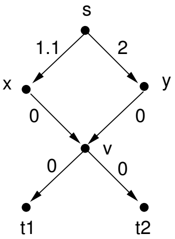
A Model for Multicast ... (continued)



Equilibrium:

- P_1 uses $s \to v \to t_1$ and
- P_2 uses $s \to v \to t_2$.
- Now, neither player has an incentive to switch.

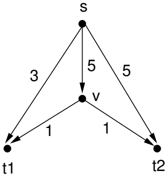
Example with multiple equilibria:



- **Equilibrium 1:** P_1 uses $s \to x \to v \to t_1$ and P_2 uses $s \to x \to v \to t_2$. (Cost for each player $= 1.1/2 = 0.55$.)
- **Equilibrium 2:** P_1 uses $s \to y \to v \to t_1$ and P_2 uses $s \to y \to v \to t_2$. (Cost for each player $= 2/2 = 1.0$.)

A Model for Multicast ... (continued)

Example – Social optimum need not be an equilibrium:



- **Social optimum:** P_1 uses $s \rightarrow v \rightarrow t_1$ and P_2 uses $s \rightarrow v \rightarrow t_2$.
- Total cost = 7. (Cost for each player = 3.5.)
- This is **not** an equilibrium.

Moves:

- P_1 has an incentive to switch to $s \rightarrow t_1$ (since the cost decreases from 3.5 to 3).
- Once P_1 switches, P_2 has an incentive to switch to $s \rightarrow t_2$ (since the cost decreases from 6 to 5).
- The situation where
  - P_1 uses $s \rightarrow t_1$ and
  - P_2 uses $s \rightarrow t_2$ is an equilibrium.
- Social cost at this equilibrium = 8.

A Model for Multicast ... (continued)

Ref: **[Anshlevich et al. 2004], [Kleinberg & Tardos, 2006]**

Research Question 1: Does every multicast problem have a Nash equilibrium?

Answer: Yes. (Proof uses the **potential function** technique.)

Research Question 2: How does the cost of a best equilibrium compare with the social optimum?

Answer: For any $k \geq 2$ players, the cost of a best equilibrium is **at most** H_k times the social optimum, where

$$H_k = 1 + (1/2) + (1/3) + \ldots + (1/k)$$

is the k^{th} **Harmonic Number**.

Note: $\ln k \; < \; H_k \; < \; \ln k + 1.$