CSI 445/660 – Part 4
(Positive and Negative Relationships)

Ref: Chapter 5 of [Easley & Kleinberg].

Positive and Negative Relationships

- So far: Edges in a network represent **friendship** information (positive relationships).
- We also need to consider **conflicts** (negative relationships).
- The combination leads to the notion of **structural balance**.
- Provides another illustration of how **local** structure (i.e., a property involving a few nodes at a time) may have a **global** effect.

Model:

- The underlying graph is a **clique**; that is, each person has a positive or negative relationship with every other person. (General graphs will be considered later.)
- Each edge has a **label**: '+' (indicating a positive relationship) or '-' (indicating a negative relationship).
- A common model for studying international conflicts.

Positive and Negative ... (continued)

Model (continued):

- Ideas developed (in the sociological context) by Fritz Heider.

- Fritz Heider (1896–1988)
- Austrian Sociologist.
- Taught at the University of Kansas for many years.

- The mathematical development is due to Dorwin Cartwright and Frank Harary.

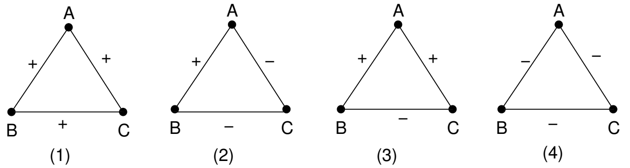
Positive and Negative ... (continued)

- Dorwin Cartwright (1915–2008)
- Areas: Psychology and Mathematics.
- One of the founders of Group Dynamics.
- University of Michigan, Ann Arbor, MI.

- Frank Harary (1921–2005)
- Mathematician who specialized in Graph Theory and its Applications.
- University of Michigan, Ann Arbor, MI and later New Mexico State University, Las Cruces, NM.

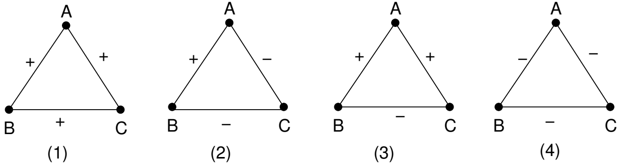
Structural Balance

Possible Edge Labelings for Three People:



- Labelings (1) and (2) have an **odd** number of '+' labels.
- Labelings (3) and (4) have an **even** number of '+' labels.
- Labeling (1): Three mutual friends; causes no problem.
- Labeling (2): Two friends and they both dislike the third; causes no problem.
- So, Labelings (1) and (2) have **structural balance**.

Structural Balance (continued)



- Labeling (3): A has two friends who don't like each other. This may be a source of "stress" for A. (It may cause A to lose the friendship with B or C.)

  Note: Recall (from the slides for Part 2) the study by Bearman & Moody [2004] involving the health records of teenage girls.

- Labeling (4): Here, two of the people may "team up" against the third person (i.e., there may be forces to change the label of one of the edges to '$+$').

- So, Labelings (3) and (4) have **structural imbalance**.

Structural Balance (continued)

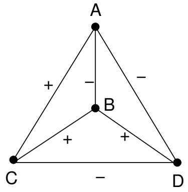
Balance condition for Three People:

- A labeled triangle is **balanced** if and only if the number of '+' labels is **odd**.

Extension – Structural Balance for all Cliques:

- A labeled clique is **balanced** if and only if each of its triangles is balanced (i.e., in each triangle, the number of '+' labels is **odd**).

Example:



- 4-clique.
- Not balanced.
- Triangle BCD has two edges labeled '+' (and so does triangle ABC).

Structural Balance (continued)

Testing the Structural Balance – An Easy Algorithm:

Input: A clique G with n nodes where each edge has a '+' or '-' label.

Output: "Yes" if G is balanced and "No" otherwise.

Outline of the Algorithm:

1. **for** each triple of nodes x, y and z **do**
   if (triangle $\{x, y, z\}$ is **not** balanced)
   Output "No" and **stop**.
2. Output "Yes".

Running time: $O(n^3)$ (since there are $\binom{n}{3} = O(n^3)$ triangles in a clique with n nodes).

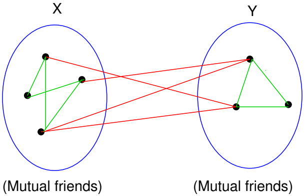
Characterizing Structural Balance

Note: The following trivial cases are ignored in the discussion.

- All edges of G are labeled '$+$': G is balanced.
- All edges of G are labeled '-': G is unbalanced.

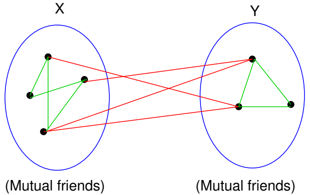
Idea of Battling Factions:

- Suppose we can partition the nodes of G into two sets X and Y such that the following conditions hold:
  - All edges inside X or inside Y are labeled '$+$' and
  - all edges that join a node in X to a node in Y are labeled '-'.



- Not all edges are shown.
- Each **green** edge has the label '$+$' and each **red** edge has the label '-'.

Characterizing Structural Balance (continued)



- Not all edges are shown.

- X and Y are called **battling factions**.
- In this structure, every triangle is balanced
(to be explained in class).
- **Key idea:** In any balanced clique, such a structure exists.

Terminology:

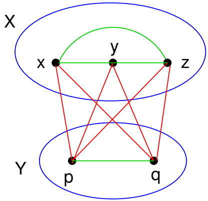
- **Internal edge:** An edge that joins two nodes in X or
two nodes in Y.
- **External edge:** An edge that joins a node in X to a node in Y.

Characterizing Structural Balance (continued)

Theorem: [Cartwright & Harary]
If a labeled complete graph G is balanced, then

- either all the edge labels in G are '$+$' or
- the nodes of G can be partitioned into two sets X and Y such that
  1. each **internal** edge is labeled '$+$' and
  2. each **external** edge is labeled '-'.

Example:



- This 5-clique is balanced.
- Partition: $X = \{x, y, z\}$ and $Y = \{p, q\}$.

Proof Sketch for the Cartwright-Harary Theorem

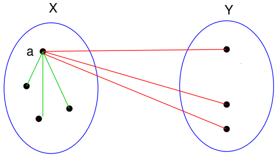
Notes:

- Ignore the (trivial) case where all edge labels are '+'.
- So, assume that at least one edge has the label '-'.
- The proof actually constructs the **battling factions** partition.

Construction:

- Choose any node a of G.
- Let the set X consist of a and all the nodes which are **friends** of a.
- Let Y be the remaining set of nodes (i.e., the **enemies** of a).

An Illustration:

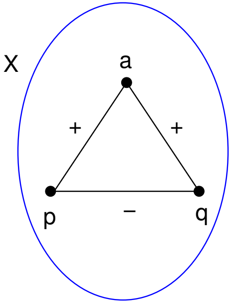


- Not all nodes/edges are shown.

Proof Sketch ... (continued)

Part 1: We must show that each internal edge in X has the label '$+$'.

- Consider any two nodes p and q in X.
- If one of p and q is the node a, the conclusion follows since all nodes in X are friends of a.
- So, assume that p and q are different from a.
- If p and q are enemies, we get the following **unbalanced** triangle in G:

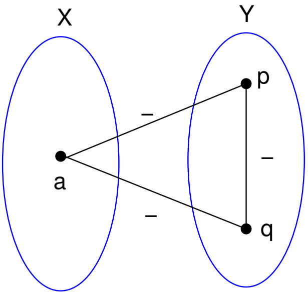


- This contradicts the assumption that G is balanced.

Proof Sketch ... (continued)

Part 2: We must show that each internal edge in Y has the label '$+$'.

- Consider any two nodes p and q in Y.
- If p and q are enemies, we get the following **unbalanced** triangle in G:

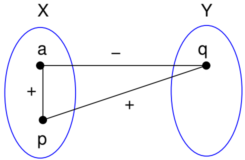


- A contradiction.

Proof Sketch ... (continued)

Part 3: We must show that each external edge has the label '-'.

- Consider any two nodes $p \in X$ and $q \in Y$.
- If p and q are friends, we get the following **unbalanced** triangle in G:



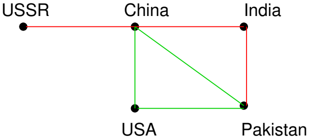
- A contradiction. (This completes the proof.)

Notes:

- The Cartwright-Harary Theorem leads to an $O(n^2)$ algorithm for the problem. (See Handout 4.1.)
- The running time is **linear** in the **size** of the input.

An Application – International Relations

Ref: **[Moore 1978]** (Reference [308] in the text).



- Relationships in 1972.
- There was a war between India and Pakistan.

- USA was trying to improve its relationship with China.
- The perception was that China and Pakistan were friends (since India was their common ‘enemy’).
- The structural balance theory suggests that USA should support Pakistan.

Other Related Topics (Brief Discussion)

- Some online networks allow people to express positive/negative sentiments.

Examples:

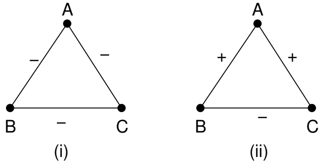
- Slashdot (`http://slashdot.org`): Allows people to designate each other as ‘friend’ or ’foe’.
- Epinions (`http://www.epinions.com`): A consumer review website where people could ‘trust’ or ‘distrust’ reviews. (These features were removed in 2014.)

Other Related Topics ... (continued)

- Evolving models of signed graphs (e.g. **[Antal et al. 2006]** – Ref [20] in the text).
  1. Start with a random labeling.
  2. Look for an unbalanced triangle and flip one of the labels to make it balanced.
  3. Repeat Step 2 until all triangles are balanced (or until the number of repetitions exceeds a set limit).
- Capture situations where people update their likes/dislikes as they strive for structural balance.

A Weaker Form of Structural Balance

- Two forms of structural **imbalance**:



- Some sociologists (e.g. James Davis, University of Chicago) have argued that (ii) is a stronger form of imbalance than (i).

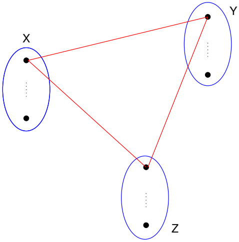
Definition: [Weaker form of Imbalance]

A clique with signed edges is **weakly balanced** if and only if there is no triangle with **exactly two** edges labeled '$+$'.

Note: One should expect a larger collection of possible structures that are weakly balanced.

A Weaker Form of Structural Balance (continued)

Example:



- This structure **allows** triangles with three edges labeled '-'.
- However, triangles with only one edge labeled '-' are **not allowed**.

Characterization of Weakly Balanced Cliques:

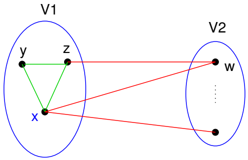
Theorem: (Also due to Cartwright & Harary)

Let G be a weakly balanced clique. Then the nodes of G **can be partitioned into groups** such that for any pair of nodes x and y

1. if x and y are in the same group, then x and y are friends and
2. if x and y are in different groups, then x and y are enemies.

Weak Balance for Cliques: Proof Idea

- Let G be the weakly balanced clique.
- Choose any node x of G and construct set $V1$ consisting of x and all the friends of X.
- Let $V2$ denote the set of remaining nodes.



- Each pair of nodes in $V1$ must be friends. (Otherwise, will have a triangle in $V1$ with exactly one edge labeled '-', which is **not** weakly balanced.)
- Also, for any node $z \in V1$ and any node $w \in V2$, z and w are enemies.

- Think of $V1$ as the first group.
- The complete graph on $V2$ is also **weakly balanced**. So, one can continue the process with $V2$, leading to several groups.

Strong Structural Balance for General Graphs

- **So far:** Balance conditions for cliques.
- **Now:** Strong structural balance for graphs which are not necessarily cliques.
- There are two possible definitions.

Definition 1: Let *G* a graph with each edge labeled '+' or '-'. *G* is **balanced** if signs can be assigned to the **missing edges** so that the resulting clique is (strongly) balanced.

Example:

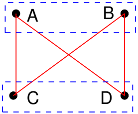
- The graph on the right assigns the '+' label to each missing edge.
- So, the graph on the left is **balanced**.

Balance for General Graphs (continued)

Definition 2: Let G a graph with each edge labeled '+' or '-'. G is **balanced** if the nodes of G can be partitioned into two sets $V1$ and $V2$ such that

1. Each edge inside $V1$ or $V2$ has the '+' label and
2. each edge that joins a node in $V1$ to a node in $V2$ has the '-' label.

Example:



Note: There need not be any internal edges.

Fact: Definitions 1 and 2 are **equivalent**; that is, a graph G is balanced according to Definition 1 and if and only if it is balanced according to Definition 2.

Balance for General Graphs (continued)

Reason for the Equivalence of Definitions:

- If it is possible to assign labels to missing edges so that the graph becomes balanced (by Definition 1), then we can obtain a "battling factions" partition that satisfies Definition 2.
- If the graph satisfies Definition 2, then all internal edges can be labeled '+' and all external edges can be labeled '-' to satisfy Definition 1.

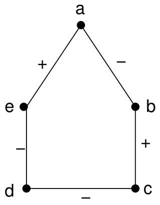
Note: Unfortunately, these definitions don't directly lead to an efficient algorithm for checking the balance condition for general graphs.

Theorem: **[Harary]**

A signed graph is balanced if and only if it does **not** contain any cycle with an **odd** number of edges with label '-'.

Explanation for Harary's Theorem

Example: The following graph has a cycle with an **odd** number of edges labeled '-'.



- In any "battling factions" decomposition, nodes a and e must be on the same side.
- Likewise, nodes b and c must be on the same side, but **different** from the side that contains a and e.

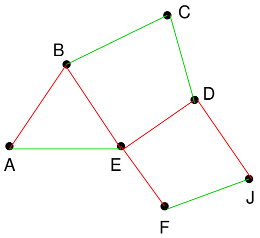
- Now, we can't add node *d* to either side.
- So, the above graph is **not** balanced.

Note: Harary's theorem leads to an efficient algorithm for testing the strong balance condition for general graphs.

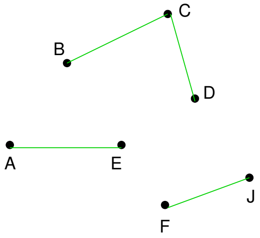
Algorithm Description: See Handout 4.2.

An Illustration for the Algorithm

Given signed graph G:

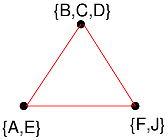


Graph G^+ **after Step 2:**



Note: G does not contain any edge labeled '-' joining two nodes in the same connected component.

Graph H **after Step 4:**



- H is **not** bipartite; it contains a cycle with 3 nodes.
- So, G is not balanced.

Notion of Approximate Balance (Brief Discussion)

- **So far:** "Perfect balance" (i.e., **all** triangles are balanced).
- Suppose we allow 0.1% of "unbalanced" triangles; that is, in the given signed clique G, 99.9% of the triangles are balanced. Then, the following result holds.

Theorem: Suppose G is a signed clique such that 99.9% of the triangles in G satisfy the strong balance condition. Then **at least one** of the following conditions hold:

- There is a subset V' with at least 90% of the nodes of G such that at least 90% of the edges inside $|V'|$ are labeled '$+$'.
- The nodes of G can be partitioned into two sets V_1 and V_2 such that
  1. at least 90% of the internal edges are labeled '$+$' and
  2. at least 90% of the external edges are labeled '-'.

Note: A proof of the above result is given in the text.