Strong Transitivity and Weyl Transitivity of Group Actions on Affine Buildings

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A.B., Kenyon College, 2007

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
May 2011

Abstract

Let H be a group acting on a building Δ. We analyze three transitivity properties that this action could have, namely strong, Weyl and weak transitivity. We present and analyze a collection of groups H and buildings Δ for which the action is not weakly transitive but may nonetheless be Weyl transitive. In these examples, the failure to be weakly transitive is in some sense precisely determined, and in some cases is shown to be extremely severe.

The first situation we consider is Chevalley groups. Let K be a local field and $G = \mathfrak{g}(K)$ a Chevalley group. Let (B, N) be the standard spherical BN-pair and $W = N/B \cap N$ the Weyl group. We precisely characterize which elements w of W admit only finite-order representatives in N. In particular for such a w of order m, all representatives of w in N have the same order, and that order is either m or $2m$. Using this we can find a variety of subgroups H of G, in particular if H is dense and torsionfree, such that H acts Weyl transitively but not weakly transitively on the affine building arising from G.

Next we consider the case of division algebras, where the failure to be weakly transitive can be more precisely characterized and shown to be very extreme. Let D be a finite-dimensional F-division algebra of degree $d > 2$, and let H be either $D^\times$ or $\mathrm{SL}_1(D)$. For any splitting field K, H admits an action on the buildings associated to $G = \mathrm{GL}_d(K)$ or $G = \mathrm{SL}_d(K)$. It is easy to show that this action is not weakly transitive, and in the present context we can show that it even fails "dramatically" to be weakly transitive. If F is a global field we can construct examples where the action of H on the affine building of G is nonetheless Weyl transitive. In the global case, for "most" D we can even show that $\mathrm{SL}_1(D)$ acts on the fundamental affine apartment only by translations - the most extreme possible situation.

Contents

Acknowledgments

First and foremost, credit is due to my advisor Peter Abramenko for introducing me to the world of buildings and for guiding me throughout my thesis research. It is an understatement to say that none of this would have been possible without his encouragement and support. I also thank Andrei Rapinchuk for many helpful conversations, related to my thesis work and beyond. Additionally I thank my fellow graduate students, in particular Tim Emerick, for listening to my building-related ramblings and letting me know when I ceased to make sense. I also want to thank the math and physics departments at Kenyon College, in particular Judy Holdener and Carol Schumacher, for infecting me with their boundless enthusiasm for mathematics. Lastly, many thanks to my family and my wife Sam for supporting my mathematical adventures over the years.

Chapter 0

Introduction

The overarching theme of this thesis is an analysis of certain transitivity properties of group actions on buildings. Buildings were introduced in the 1950's-60's by Jacques Tits as a way of geometrically studying semisimple algebraic groups over an arbitrary field. The original definition of a building Δ was a simplicial complex made up of a choice of *apartment system* $\mathcal{A}$ with certain properties. This non-canonical aspect, i.e., the need to keep track of a choice of apartment system, was alleviated in the 1980's with a more combinatorial, though equivalent definition of a building. Under the combinatorial approach, there is no need to refer to apartments or to any simplices other than the chambers. In fact all the structure of the building is encoded just in its set of chambers $\mathcal{C}$ and its *Weyl distance function* $\delta : \mathcal{C} \times \mathcal{C} \to W$, a function assigning to each pair of chambers a "distance," i.e., an element in the Weyl group W of the building. See Chapter 1 for details.

In the context of a group G acting on a building Δ, there are some standard types of transitivity. In the simplicial approach, we have a natural notion of *strong transitivity*, meaning that G acts transitively on the set of pairs (C, Σ) where C is a chamber in the apartment Σ. This of course depends on the choice of apartment system $\mathcal{A}$; we more specifically refer to the action as *strongly transitive with respect to* $\mathcal{A}$. In the combinatorial approach we want to avoid reference to apartments, and so we instead use the notion of *Weyl transitivity*, where for each $w \in W$, G acts transitively on the set of pairs (C, D) with $\delta(C, D) = w$. It is clear that strong transitivity implies

Weyl transitivity, and it turns out that they are equivalent in the important case of *spherical* buildings. However, outside the spherical case it is not immediately clear that Weyl transitivity is really weaker. This is the main motivation for this thesis, namely to exhibit and analyze many large classes of examples of Weyl transitive but not strongly transitive group actions on affine buildings. Tits provides a suggestion in [T] of a direction to look, and a class of explicit examples is worked out in [AB1] in the special case when the building is a tree. In the present work we find examples for affine buildings of arbitrary type and also achieve some very strong transitivity results in certain cases, even finding classes of examples for which the Weyl transitive action is "as far as possible" from being strongly transitive.

The transitivity properties of interest here are closely related to certain purely group-theoretic properties, namely Bruhat decompositions and BN-pairs. A BN*-pair* consists of a pair of subgroups B and N of G satisfying certain properties involving the Weyl group W. In particular, $N/(B\cap N)\cong W$. Also, we say (G,B) admits a *Bruhat decomposition* if there is a bijection $C : W \to B\backslash G/B$ satisfying certain properties. In case B arises from a BN-pair, we have the natural Bruhat decomposition $C(w) = BwB := B\tilde{w}B$ where $\tilde{w}\in N$ is such that $\tilde{w}(B\cap N)\mapsto w$ under the identification $N/(B\cap N)\cong W$. It turns out that Weyl transitive actions are essentially equivalent to Bruhat decompositions, and strongly transitive actions are equivalent to BN-pairs. Thus a Weyl transitive action that is not strongly transitive is equivalent to a Bruhat decomposition that does not arise from a BN-pair. Details can be found in Sections 1.3 and 2.1.

We also introduce a new notion, that of *weak transitivity* (Definition 1.3.1). A group G acts weakly transitively on a building Δ if there exists an apartment Σ such that $\mathrm{Stab}_G(\Sigma)$ acts transitively on the chambers of Σ. Clearly if the action of G is weakly transitive and transitive on $\mathcal{A}$, then it is strongly transitive with respect to $\mathcal{A}$. It turns out that a Weyl transitive action is strongly transitive if and only if it is weakly transitive. For our purposes then, we are interested in finding when weak transitivity fails. As seen in Lemma 2.4.2, it is not hard for a subgroup of a topological group acting Weyl transitivity to itself act Weyl transitively. The main thing to show is denseness. Thus the real work is in demonstrating the failure to be weakly transitive.

The key setup is the following observation, which is Lemma 2.4.3: Let G be a group acting strongly transitively on a building Δ with respect to the complete apartment system $\overline{\mathcal{A}}$. Choose an apartment Σ_0 and set $N = \mathrm{Stab}_G(\Sigma_0)$, $T = \mathrm{Fix}_G(\Sigma_0)$. Note that N/T can be identified with the Weyl group W of the building. Let H be any subgroup of G. Then H acts weakly transitively on Δ if and only if there exists $g \in G$ such that $(gHg^{-1} \cap N)T = N$, or equivalently, for all $n \in N$ it holds that $gHg^{-1} \cap nT \neq \emptyset$. Thus, showing that H does *not* act weakly transitively amounts to showing that any conjugate of H "misses" at least one coset of N/T. In some situations we can show that H even misses a large percentage of the cosets in N/T, as seen in Remark 5.4.11. Also, for certain conjugates of H we can sometimes obtain a *precise* description of which cosets are missed, as seen in Section 5.5.

The first batch of explicit examples arises in the context of Chevalley groups. Let G be any Chevalley group $\mathfrak{g}(K)$ over a field K, with root system Φ, as described in Section 3.1. Thinking of the standard (spherical) BN-data for G constructed in Section 3.3, we show that there exists an element w of the Weyl group $W = N/T$ such that all representatives of w in N have the same finite order. If we consider the natural action of the Weyl group on the Euclidean space $E = \langle \Phi \rangle_{\mathbb{R}}$, we show in Theorem 4.1.2 that for $w \in W$ the following are equivalent:

1) Acting on E, w has no eigenvalue 1.

2) All representatives of w in N have (the same) finite order, namely $|w|$ or $2|w|$.

Condition 1 simply states that w is a *generalized Coxeter element* in the language of [DW], in particular any Coxeter element will work.

Next we consider the case when K is complete with respect to a discrete valuation. Then by Proposition 3.3.4 G admits a *VRGD system*, or valuated root group data system (this name mirrors that of an *RGD system*), and so as shown in Theorem 2.3.4, we achieve the well-known result that there is a canonical affine building Δ_a on which G acts strongly transitively. In fact the action is strongly transitive with respect to $\overline{\mathcal{A}}$ as proved in Proposition 3.3.8. (This proposition is an explicit special case of the more general and abstract Theorem 17.9 in [W].) A subgroup of G that happens to be *torsionfree* thus cannot act weakly transitively on Δ, since it cannot represent any generalized Coxeter element in W. In fact it need only be m-torsionfree where $m = |w|$ for some generalized Coxeter element w. Here W is still the spherical Weyl group, which is a subgroup of the affine Weyl group. As before, if H is dense in G then the action of H is still Weyl transitive. Thus our goal in this context amounts to finding dense torsionfree subgroups, which we do in Section 4.2.

If K has characteristic 0 we can find many dense torsionfree congruence subgroups, as shown in Theorem 4.2.3. The canonical example is when $G = \mathrm{SL}_d(\mathbb{Q}_p)$ and H is the congruence subgroup $H = \{g \in \mathrm{SL}_d(\mathbb{Z}[\frac{1}{p}]) \mid g \equiv_q 1\}$, where $q > 2$ is relatively prime to p; see Theorem 4.2.2. If K has positive characteristic p, we can't necessarily hope for congruence subgroups to be torsionfree, but we can construct them to be m-torsionfree for $m = |w|$, assuming p does not divide m. This is also done in Theorem 4.2.3. Thus for any field K complete with respect to a discrete valuation, and any Chevalley group $G = \mathfrak{g}(K)$ with spherical Weyl group W such that the characteristic of K does not divide $2|W|$, there exist "many" subgroups H of G whose action on the affine building associated to G is Weyl transitive but not strongly transitive with respect to any apartment system. Group-theoretically this provides many examples of groups with Bruhat decompositions that do not arise from BN-pairs, and examples can be found for any affine type. These results are joint work with P. Abramenko and can be found also in [AZ].

There is a second way to define Chevalley groups, namely as groups of K-rational points, so that they are in fact linear algebraic groups; see Section 3.4. As seen in Proposition 3.4.3 the two constructions do not differ by much. It turns out that Theorem 4.1.2 still holds in this context, and all the results are essentially unchanged. We also inspect the specific cases of the classical groups, and use Theorem 4.1.2 to achieve a nice description of the generalized Coxeter elements using cycle decompositions; see Section 4.3.

The examples using congruence subgroups of Chevalley groups work fine but are rather *ad hoc*. They also are not really in the spirit of Tits' description in [T] of a rubric for producing examples of Weyl transitive but not strongly transitive group actions. In that paper Tits suggests looking instead at anisotropic groups over global fields. However, not until 2007 was an explicit example constructed, by Abramenko and Brown in [AB1]. In that paper, certain subgroups G of $\mathrm{SL}_2(\mathbb{Q}_p)$ are constructed that act Weyl transitively on a certain building but not strongly transitively with respect to any apartment system. More specifically, G is the norm-1 group of a quaternion division algebra $D = \left(\frac{\alpha,\beta}{\mathbb{Q}}\right)$ that splits over $\mathbb{Q}_p$, and Δ_p is the affine building (tree) associated to $\mathrm{SL}_2(\mathbb{Q}_p)$. G acts Weyl transitively on Δ_p, by virtue of G being dense in $\mathrm{SL}_2(\mathbb{Q}_p)$, but if $-1 \notin D^2$ then the action of G on Δ_p is not strongly transitive with respect to any apartment system.

Chapter 5 is devoted to working through this more natural situation in full generality, with the added bonus that weak transitivity is found to not only fail, but to fail "dramatically." Actions are constructed that may very well be Weyl transitive, but are "not even close" to being weakly transitive. We start with a (finite dimensional) division algebra D with center F, and embed D over F into $M_d(K)$ for any splitting field $K|F$ of D. Here d is the degree of D. As explained in Definition 5.3.4, we can think of $D^\times$ as a subgroup of $\mathrm{GL}_d(K)$, and can think of the reduced norm-1 group $\mathrm{SL}_1(D)$ as a subgroup of $\mathrm{SL}_d(K)$. If $H = \mathrm{SL}_1(D)$ or $D^\times$ we have a natural action of H on certain buildings associated to $G = \mathrm{SL}_d(K)$ or $\mathrm{GL}_d(K)$. There is always a spherical BN-pair, and if K has discrete valuation then there is an affine BN-pair, yielding a spherical building Δ and an affine building Δ_a on which G acts strongly transitively. In the latter case, if K is complete then G acts strongly transitively with respect to $\overline{\mathcal{A}}$ on the associated affine building Δ_a. As seen in Theorem 5.4.1 it is actually quite easy to exhibit at least one class of cosets in $W = N/T$ that no conjugate of H can intersect, immediately yielding Corollary 5.4.2 that H does not act weakly transitively.

It should be noted that there is a very nice, elementary argument due to A. Rapinchuk [R1] providing examples of such actions in the *full* generality that Tits outlined in [T]. In fact any non-split algebraic group that does not split over a quadratic extension will provide an example, in particular most anisotropic groups work. Corollary 5.4.2 is therefore in itself not very surprising. The methods we use here are nonetheless interesting for their ability to demonstrate and measure the *severe* failure of the actions to be weakly transitive. In particular we can exhibit a wide array of cosets that any conjugate of H will "miss." For example, thinking of the spherical Weyl group as the symmetric group S_d, if the cycle decomposition of $\sigma \in S_d$ features a k-cycle for $d/2 < k < d$, then H cannot "hit" σ; see Theorem 5.4.6. Alternately, if the cycle decomposition of $\sigma \in S_d$ features a unique cycle of smallest length (where any fixed point of σ in $\{1, \ldots, d\}$ is called a 1-cycle) and if this cycle is not a d-cycle, then again H cannot "hit" σ; see Theorem 5.4.4. These situations can be simultaneously generalized by a construction we call "lonely cycles," and Theorem 5.4.7 shows that indeed there is a huge class of cosets that no conjugate of H can intersect. A quick combinatorial argument given in Remark 5.4.11 shows that Theorem 5.4.6 alone implies that H "misses" at least 70% of S_d for large enough d. In the affine situation, since the affine Weyl group contains S_d as a subgroup, by similar arguments the action of H on Δ_a is very far from weakly transitive. However, if F is dense in K, e.g., if F is global and K is some completion of F, then H still acts Weyl transitively on Δ_a.

Another class of elements of interest in $W = S_d$ is the d-cycles, or Coxeter elements. The above arguments do not always discount the possibility that $H = \mathrm{SL}_1(D)$ could hit the d-cycles, but we use an additional argument to cover this case for a large class of situations. It is easy to see that any representative x of a d-cycle must satisfy $x^d = (-1)^{d-1}$ primitively, and have irreducible minimal polynomial. Thus if d is not a power of 2, as seen in Theorem 5.4.8 we can add the d-cycles to the list of cosets that don't intersect any conjugate of H. If d is a power of 2, we can still sometimes achieve this result for certain F. If $x \in H = \mathrm{SL}_1(D)$ represents a d-cycle, then D contains either a non-central primitive d_{th} or $2d_{th}$ root of unity, and so admits the corresponding cyclotomic subfield $F \subsetneq L \subseteq D$. Using considerations from the theory of global fields and the Brauer group, we show that this fails for many choices of F, in particular the natural case of $F = \mathbb{Q}$, as seen in Theorem 5.4.9.

In all these results we have been analyzing the failure to act strongly transitively with respect to *any* apartment system. We lastly turn our attention to situations where we can obtain even more precise results regarding the *fundamental* apartment. The most striking situation is $\mathrm{SL}_1(D)$ for D a division algebra over a global field F, acting on the appropriate building. In this context, we show in Section 5.5 that in "most" cases, $\mathrm{SL}_1(D)$ fails to represent *any* non-trivial element of the spherical Weyl group, and represents precisely the subgroup of translations in the affine Weyl group. We also show that that for arbitrary F the determination of a similar result would be at least as difficult as the well-known problem of relating the exponent and index of D.

There are a number of possible directions for future work. One goal would be to try and achieve a precise list of Weyl group elements represented by $D^\times$ and $\mathrm{SL}_1(D)$ for *arbitrary* apartments, not just the fundamental apartment. In particular M. Kassabov has suggested that in general $D^\times$ should only be able to represent permutations all of whose cycles have equal length, and the results for the fundamental apartment support this idea. Another direction for future work has been suggested by B. Mühlherr, namely to inspect the relationship between strong and Weyl transitivity in the non-spherical, non-affine case. It would be especially interesting to see if, for example, strong and Weyl transitivity are even equivalent for hyperbolic buildings.

Chapter 1

Buildings

A great deal of information about buildings can be found in the recent text by Abramenko and Brown [AB2], and this whole chapter draws primarily from that reference. We will be concerned with two different ways of thinking about buildings, and thus will present two definitions. The starting point of either definition is the notion of a *Coxeter group*.

Definition 1.0.1. A *Coxeter system* (W, S) is a group W together with a finite set of generators S such that W has a presentation

$$W = \langle S \mid (st)^{m_{s,t}} = 1 \text{ for all } s, t \in S \rangle$$

where each $m_{s,t}$ is in $\mathbb{N} \cup \{\infty\}$, $m_{s,t} = \infty$ is understood to mean that st has infinite order, and $m_{s,s} = 1$ for all $s \in S$. Note that this implies each $s \in S$ has order 2, and this in turn implies that $m_{s,t} = m_{t,s}$ for each $s, t \in S$.

We call the group W a *Coxeter group* and the generators s will occasionally be referred to as *involutions* or *transpositions*.

This is actually sufficient data to give the first definition of a building; see [AB2, Definition 5.1].

1.1 Combinatorial buildings

Definition 1.1.1. Let (W, S) be a Coxeter system. A *building of type* (W, S), or in case of ambiguity a *combinatorial building of type* (W, S), is a nonempty set $\mathcal{C}$ together with a map $\delta : \mathcal{C} \times \mathcal{C} \to W$ called the "Weyl distance function" such that the following axioms are satisfied for all $C, D \in \mathcal{C}$:

(CB1): $\delta(C, D) = 1$ if and only if $C = D$

(CB2): If $\delta(C, D) = w$ and $C' \in \mathcal{C}$ is such that $\delta(C', C) = s \in S$, then $\delta(C', D) = sw$ or w. If $\ell(sw) = \ell(w) + 1$ then $\delta(C', D) = sw$.

Here, $\ell : W \to \mathbb{Z}_{\geq 0}$ is the length function on W relative to the generating set S.

(CB3): If $\delta(C, D) = w$ and $s \in S$, there exists $C' \in \mathcal{C}$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$.

There is of course a huge amount of structure that is not immediately apparent from the axioms. To begin to unravel some of this structure, we need some more definitions.

Definition 1.1.2. The elements C of $\mathcal{C}$ are called *chambers*. Given a chamber C and an element s of S, we define the *s-panel* containing C to be

$$\mathcal{P} = \mathcal{P}(C, s) = \{D \in \mathcal{C} | \delta(C, D) \in \{1, s\}\}.$$

The s-panel containing C can be thought of as the set consisting of C itself, along with all chambers that are *s-adjacent* to C, i.e., are at a distance s from C. Once we establish the simplicial approach to buildings, the notion of adjacency will become more natural, as will the choice of the words panel and chamber. For now, let us look at a few standard examples of buildings.

Example 1.1.3. Define $\delta_W : W \times W \to W$ via $\delta_W(w_1, w_2) = w_1^{-1}w_2$. We claim that (W, δ_W) is a building. Indeed, (CB1) holds trivially, and (CB2) and (CB3) hold simply because $(w_1^{-1}w_2)(w_2^{-1}w_3) = w_1^{-1}w_3$. In fact, we have the stronger condition that

$$\delta_W(w_1, w_2)\delta_W(w_2, w_3) = \delta_W(w_1, w_3)$$

for all $w_1, w_2, w_3 \in W$. We call (W, δ_W) the *standard thin building of type* (W, S). Note that by construction W acts transitively on the chambers of (W, δ_W).

In general a building $(\mathcal{C}, \delta)$ of type (W, S) is called *thin* if for every chamber C and every element $s \in S$, there exists exactly one chamber at a distance s from C. In different contexts, thin buildings are called *Coxeter complexes* or *apartments*.

Of course there exist buildings that are not thin. The following is a standard example of a *thick building*, that is, one for which every panel contains at least three chambers. We will not give many details here; see [AB2, Section 4.3] for a complete account.

Example 1.1.4. Let K be a field and V an n-dimensional vector space over K. Let W be the symmetric group S_n, with generating set $S = \{(1\ 2), (2\ 3), \ldots, (n-1\ n)\}$. Let $\mathcal{C}$ be the set of *maximal flags* in V, i.e., chains of subspaces $V_1 < \cdots < V_{n-1}$ of V such that $\dim V_i = i$. We adopt the notation that $V_0 = 0$ and $V_n = V$. Let C and D denote the maximal flags $V_1 < \cdots < V_{n-1}$ and $W_1 < \cdots < W_{n-1}$ respectively. We want to define a Weyl distance $\delta(C, D)$ in S_n. For each $i \in \{1, \ldots, n\}$ it turns out we can choose $j_i \in \{1, \ldots, n\}$ such that

$$W_{i-1} + (W_i \cap V_k) = \begin{cases} W_{i-1} & \text{for } k < j_i \\ W_i & \text{for } k \geq j_i \end{cases}.$$

We now can define $\delta(C, D)$ to be the permutation taking each i to j_i. Note that if $C = D$ then $j_i = i$ for each i, so $\delta(C, D) = 1$. As promised we will give few details here, but it turns out that $(\mathcal{C}, \delta)$ satisfies (CB1)-(CB3) and is thick; see [AB2, Section 4.3].

In the next section we discuss the other definition of a building, as a simplicial complex. This approach has the advantage of being more concrete, and the disadvantage that one must keep track of a choice of *apartment system*.

1.2 Simplicial buildings

Let (W, S) be a Coxeter system and (W, δ_W) the standard thin building of type (W, S). We want to construct a simplicial complex Σ that encodes all the properties of (W, δ_W). For each subset J of S, define the *standard subgroup* $W_J := \langle J \rangle \leq W$. In particular, $W_S = W$ and $W_\emptyset = \{1\}$. Also define a *standard coset in* W to be any coset of the form wW_J for some $w \in W$, $J \subseteq S$. Note that the chambers of (W, δ_W), that is the elements of W, are the standard cosets corresponding to $J = \emptyset$. This fact motivates the following definition.

Definition 1.2.1. Let $\Sigma = \Sigma(W, S)$ be the poset of standard cosets in W, with the ordering $\leq$ given by $w_1 W_{J_1} \leq w_2 W_{J_2}$ if and only if $w_1 W_{J_1} \supseteq w_2 W_{J_2}$ as sets. Define the *Coxeter complex* associated to (W, S) to be the set Σ. We say that $w_1 W_{J_1}$ is a *face* of $w_2 W_{J_2}$ provided that $w_1 W_{J_1} \leq w_2 W_{J_2}$.

One immediate observation is that this ordering by *reverse* inclusion ensures that the maximal elements of Σ are really the minimal cosets, namely the singleton sets $wW_\emptyset = \{w\}$. Also note that W itself is the unique minimal element of Σ, and is a face of every other element. This face relation and the choice of the word "complex" should be hints that Σ can be realized as a simplicial complex; see also [AB2, Theorem 3.5].

Proposition 1.2.2. *Let (W, S) be any Coxeter system. Then the poset $\Sigma = \Sigma(W, S)$ is simplicial.*

Proof. We must show that any two elements of Σ have a greatest lower bound. Since by construction Σ is closed under taking faces, this will suffice. See [AB2, Appendix A] for a more detailed explanation of this definition of simplicial complex. Let wW_J,

$w'W_{J'}$ be arbitrary elements of Σ. If we can show that $(w')^{-1}wW_J$ and $W_{J'}$ have a greatest lower bound $w''W_{J''}$, then $w'w''W_{J''}$ will be the greatest lower bound of wW_J and $w'W_{J'}$, and so we may assume without loss of generality that $w' = 1$. Also note that the face relation is given by reverse inclusion, so W is a global lower bound.

Consider any lower bound of wW_J and $W_{J'}$. It must contain both sets, and so in particular it contains the identity and is a standard subgroup, not just a standard coset. Call it $W_{J''}$. Since $w \in wW_J \subseteq W_{J''}$, and $W_{J''}$ is a group, also $W_J \subseteq W_{J''}$. Thus, any such $W_{J''}$ contains w, W_J, and $W_{J'}$, and so the greatest lower bound should be $W_{J''} = \langle w, W_J, W_{J'} \rangle$. It is not immediately obvious that there really exists $J'' \subseteq S$ such that $W_{J''} = \langle w, W_J, W_{J'} \rangle$, but this can be easily shown using the theory of Coxeter groups, and we reference [AB2, Proposition 2.16] for this result. □

As stated earlier, the maximal simplices of Σ are the singletons $\{w\}$ for $w \in W$, so thinking of the combinatorial approach it seems reasonable to refer to the maximal simplices as *chambers*. It is also true that Σ is *pure*, i.e., all maximal simplices have the same dimension. To see this, we must decode how to determine the dimension of a given simplex.

Lemma 1.2.3. *For $J \subsetneq S$, the dimension of the simplex $A = wW_J$ in Σ is $|S| - |J| - 1$. In particular all maximal simplices have dimension $|S| - 1$.*

Proof. The faces of A are precisely the simplices $wW_{J'}$ for $J \subseteq J' \subseteq S$. Thus, there is a 0-dimensional face for every J' containing J with $|S \backslash J'| = 1$. But there are precisely $|S| - |J|$ such J', and so A has $|S| - |J|$ vertices and $\dim A = |S| - |J| - 1$. □

It is often more natural to refer to the *rank* of a simplex, defined to be $\mathrm{rk}(A) = \dim A + 1$. Thus, $\mathrm{rk}(wW_J) = |S| - |J|$, and maximal simplices have rank $|S|$. We also say that the Coxeter complex $\Sigma(W, S)$ has rank $|S|$. This of course syncs up with the notion of rank in a Coxeter system, where the rank of (W, S) is $|S|$.

We are now equipped to give the second, simplicial definition of a building, and eventually demonstrate its equivalence to the combinatorial definition.

Definition 1.2.4. Let (W, S) be a Coxeter system. A (simplicial) *building of type* (W, S) is a simplicial complex Δ such that Δ is a union of subcomplexes Σ, called *apartments*, and the following axioms are satisfied:

(SB1): Each Σ is a Coxeter complex of type (W, S).

(SB2): For any two simplices $A, B \in \Delta$, there exists Σ containing A and B.

(SB3): If Σ and Σ' both contain A and B, then there is an isomorphism from Σ to Σ' fixing A and B pointwise.

We immediately note that if $A = B = \emptyset$ and Σ, Σ' are any two apartments, then by (SB3) $\Sigma \cong \Sigma'$. Thus, the specification in (SB1) that each Σ have the same type is redundant. Either way, it is important that Δ indeed has a well-defined type (W, S).

At this point we can discuss the equivalence of the two approaches. We will only give an overview; see the proof of [AB2, Corollary 5.93] for details. Let Δ be a simplicial building of type (W, S). Let $\mathcal{C} = \mathcal{C}(\Delta)$ be the set of maximal simplices in Δ. For each apartment Σ, let $\mathcal{C}(\Sigma)$ denote the set of chambers in $\mathcal{C}$ contained in Σ. By (SB1) and Section 3.5 of [AB2] each apartment comes equipped with a Weyl distance function $\delta_\Sigma : \mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \to W$. In fact we can define a Weyl distance δ on all of Δ; for $C, D \in \mathcal{C}$ set $\delta(C, D) := \delta_\Sigma(C, D)$ where Σ is any apartment containing C and D, the existence of which is guaranteed by (SB2). This is independent of the choice of Σ, and in fact $(\mathcal{C}(\Delta), \delta)$ satisfies (CB1)-(CB3) and is of type (W, S); see [AB2, Propositions 4.81,4.84].

This shows that simplicial buildings are also combinatorial. Now let $(\mathcal{C}, \delta)$ be a combinatorial building of type (W, S). Producing a simplicial complex $\Delta(\mathcal{C})$ satisfying (SB1)-(SB3) is an involved process, the details of which take up Sections 5.3 through 5.6 in [AB2]. We mention a few of the steps here. As we have seen, the maximal simplices correspond to chambers and the codimension-1 simplices correspond to panels, i.e., are determined by s-adjacency. Similarly the lower-dimensional simplices are determined by J*-adjacency* for $J \subseteq S$. Two chambers C and D are J-adjacent if $\delta(C, D) \in W_J$. Apartments are determined by a sort of converse of (SB1); any subcomplex of $\Delta(\mathcal{C})$ that is "isometric" to $\Sigma(W, S)$ is declared to be an apartment. Finally, Theorem 5.91 in [AB2] shows that $\Delta(\mathcal{C})$ satisfies (SB1)-(SB3) and has type (W, S).

This shows that combinatorial buildings are also simplicial. What's more, the two approaches really are equivalent. We state this fact as a Proposition here; see [AB2, Corollary 5.93] for the proof.

Proposition 1.2.5. *Let Δ be a simplicial building of type (W, S), with Weyl distance function δ. Let $(\mathcal{C}(\Delta), \delta)$ be the combinatorial building associated to Δ. Then the simplicial complex $\Delta(\mathcal{C}(\Delta))$ is canonically isomorphic to Δ.*

Now let $(\mathcal{C}, \delta)$ be a combinatorial building of type (W, S). Let $\Delta(\mathcal{C})$ be the simplicial building arising from $(\mathcal{C}, \delta)$. Then $(\mathcal{C}(\Delta(\mathcal{C})), \delta) = (\mathcal{C}, \delta)$.

We can now analyze Example 1.1.4 in the language of simplicial buildings. As before we leave most of the details out; see [AB2, Section 4.3].

Example 1.2.6. Let K be a field and V an n-dimensional vector space over K as in Example 1.1.4. Let Δ be the flag complex of V, i.e., the simplicial complex whose simplices are flags in V. Here a *flag* is a chain of proper subspaces $V_1 < \cdots < V_k$ in V. The face relation is given by inclusion of flags, that is $V_1 < \cdots < V_k$ is a face of $W_1 < \cdots < W_\ell$ if for all i there exists j such that $V_i = W_j$. Thus the maximal simplices are flags $V_1 < \cdots < V_{n-1}$ with $\dim V_i = i$ for each i, which are the maximal flags from Example 1.1.4.

For each K-basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of V we have a subcomplex whose simplices are flags $V_1 < \cdots < V_k$ such that each V_i has as a basis some subset of $\mathcal{B}$. Denote this subcomplex by $\Sigma_{\mathcal{B}}$. It turns out that Δ is a building, with an apartment $\Sigma_{\mathcal{B}}$ for each choice of basis $\mathcal{B}$. (It is possible that different bases yield identical apartments, and so to avoid counting apartments more than once we really take the apartments to be $\Sigma_{[\mathcal{B}]}$, where $[\mathcal{B}]$ is the equivalence class of bases $\mathcal{B}'$ such that $\Sigma_{\mathcal{B}'} = \Sigma_{\mathcal{B}}$.) It is clear that Δ is the union of these apartments, and (SB1)-(SB3) follow by arguments in [AB2, Section 4.3]. As promised, the simplicial approach makes this example more concrete; it is easier to work with apartments than with the Weyl distance function.

In particular it is now easy to see that Δ is thick, i.e., every panel is contained in at least three chambers. For example, if $\mathcal{P}$ is the panel $V_1 < \cdots < V_{n-2}$ then any choice of $n-1$ dimensional subspace V_{n-1} containing V_{n-2} will produce a chamber $V_1 < \cdots < V_{n-1}$ containing $\mathcal{P}$. Clearly there are at least three distinct lines in V/V_{n-2}, even if K only has two elements, and so $\mathcal{P}$ is contained in at least three distinct chambers. A similar argument works for any panel, and Δ is indeed thick.

As mentioned earlier, one feature of the simplicial approach is that one must keep track of which subcomplexes have been designated as apartments. Obviously there might be more than way to decompose a given building into a union of apartments. To avoid potential imprecision we introduce the notion of an *apartment system* $\mathcal{A}$. This is a collection of subcomplexes Σ satisfying the above axioms. Many properties of buildings makes sense only "with respect to" a given apartment system, or may only hold for certain choices of $\mathcal{A}$. There is a specific apartment system that will often be of use, namely the *complete apartment system* $\overline{\mathcal{A}}$. This is the collection of all subcomplexes $\Sigma \subseteq \Delta$ such that $\Sigma \in \mathcal{A}$ for some $\mathcal{A}$. Since every building by definition admits at least one apartment system, this definition makes sense, and it turns out that $\overline{\mathcal{A}}$ really is an apartment system; see [AB2, Theorem 4.54].

In some cases, $\overline{\mathcal{A}}$ is actually the *only* apartment system. One such case is the *spherical* case, which is particularly important in its own right.

Definition 1.2.7. Let Δ be a building of type (W, S). If W is finite we say Δ is *spherical*.

Note that for a Coxeter system (W, S) with W finite there is a unique w_0 in W

with $\ell(w_0)$ maximal, called the *longest word*. Thus, in a Coxeter complex, given any chamber C there is a unique chamber $-C$, called the *opposite chamber*, such that $\delta(C, -C) = w_0$. We similarly say two chambers C, D in a building of spherical type (W, S) are *opposite* if $\delta(C, D) = w_0$. The presence of opposite chambers forces spherical buildings to be "tightly controlled;" in particular we get the following result. See [AB2, Section 4.7] for details.

Proposition 1.2.8. *Let Δ be spherical. Then there is a unique system of apartments, and the apartments are precisely the convex hulls in Δ of pairs of opposite chambers.*

Referring to "convex hulls" makes use of the simplicial approach to buildings and the notion of *galleries*. We will not need these notions any more, and so will just reference [AB2] once again and take this proposition for granted. The point is that in the spherical case, given a chamber $C \in \mathcal{C}$, every apartment containing C is determined uniquely by the opposite chamber it contains. For the sake of future brevity, we set some notation here:

Let $\mathcal{A}$ be an apartment system. For $C \in \mathcal{C}$ set $\mathcal{A}(C) := \{\Sigma \in \mathcal{A} \mid C \subseteq \mathcal{A}\}$. For $\Sigma \in \mathcal{A}$ set $\mathcal{C}(\Sigma) := \{C \in \mathcal{C} \mid C \subseteq \Sigma\}$ as before. Note that by the above discussion, in the spherical case $\mathcal{A} = \overline{\mathcal{A}}$ and $\mathcal{A}(C) \leftrightarrow \{D \mid \delta(C, D) = w_0\}$.

Before moving to the next section we should mention another type of building that will be of particular importance later, namely an *affine building*. This is simply any building of type (W, S) for W an *affine Coxeter group*. We will actually not make use of any particular properties of such groups, and so will simply define them to be Coxeter groups with a particular *Coxeter diagram*; see [W, Chapter 1]. As a sidenote, affine buildings are often called *Euclidean buildings* since the apartments can be geometrically realized as tilings of Euclidean space.

1.3 Strong and Weyl transitivity

The rest of this thesis will be concerned with groups acting on buildings, usually via type-preserving automorphisms. Here by "type" we do not mean (W, S) but rather a coloring of the vertices of the building; a type-preserving action on a building is thus the analog of an action on a graph without inversion. From here on, all group actions will be assumed to be type-preserving, unless otherwise stated.

Let Δ be a building with chamber set $\mathcal{C}$. We say the group G acts *chamber transitively* on Δ if the action is transitive on $\mathcal{C}$. As often happens in geometric group theory, we can learn a lot about the structure of G by analyzing stabilizers in G. Since G is acting on a building, the natural stabilizers to inspect are stabilizers of chambers and stabilizers of apartments. We define three types of transitivity that are similar but have some important differences.

Definition 1.3.1. We say the action of G on Δ is *weakly transitive* if there exists an apartment $\Sigma \in \overline{\mathcal{A}}$ such that $\mathrm{Stab}_G(\Sigma)$ acts transitively on $\mathcal{C}(\Sigma)$.

Definition 1.3.2. We say the action of G on Δ is *strongly transitive with respect to* $\mathcal{A}$ if it is transitive on $\mathcal{A}$ and there exists an apartment $\Sigma \in \overline{\mathcal{A}}$ such that $\mathrm{Stab}_G(\Sigma)$ acts transitively on $\mathcal{C}(\Sigma)$, or equivalently if it is chamber transitive and there exists $C \in \mathcal{C}$ such that $\mathrm{Stab}_G(C)$ acts transitively on $\mathcal{A}(C)$ [AB2, Section 6.1.1].

Definition 1.3.3. We say the action of G on Δ is *Weyl transitive* if it is chamber transitive and there exists $C \in \mathcal{C}$ such that $\mathrm{Stab}_G(C)$ acts transitively on the "w-sphere" $\{D \in \mathcal{C} \mid \delta(C, D) = w\}$ for all $w \in W$ [AB2, Section 6.1.3].

One advantage of Weyl transitivity is that it does not reference apartments, and so does not depend on a choice of $\mathcal{A}$. For this reason, and due to the presence of δ in the definition, Weyl transitivity is of particular interest in the combinatorial approach. It is less natural in this approach to talk about apartments, but in order to relate these types of transitivity it is important to establish the following lemma, which is Lemma 6.13 in [AB2].

Lemma 1.3.4. *If G acts Weyl transitively on Δ, then for any apartment Σ the orbit $G\Sigma = \{g\Sigma \mid g \in G\}$ is an apartment system.*

Proof. It suffices to check that any two chambers C, D are contained in an apartment. The other conditions follow since $G\Sigma$ is a subset of $\overline{\mathcal{A}}$. By chamber transitivity we may assume $C \in \mathcal{C}(\Sigma)$. Let $w = \delta(C, D)$. Choose $D' \in \mathcal{C}(\Sigma)$ such that $\delta(C, D') = w$. Since Σ is a Coxeter complex of type (W, S) we know such a D' exists. By Weyl transitivity there exists $g \in \mathrm{Stab}_G(C)$ such that $D = gD'$. Thus, $C, D \in g\Sigma$, and indeed $G\Sigma$ is an apartment system. □

Weyl and strong transitivity are in certain cases equivalent, and so while strong transitivity may be germane in the simplicial approach to buildings, one may switch to Weyl transitivity in the combinatorial approach to avoid reference to apartments, and not lose any generality or specificity. The following proposition, which is Proposition 6.14 in [AB2], relates strong, Weyl, and weak transitivity in the general case.

Proposition 1.3.5. *If G acts strongly transitively with respect to any apartment system then it acts Weyl transitively and weakly transitively. If G acts Weyl transitively and weakly transitively then G acts strongly transitively with respect to some apartment system.*

Proof. Suppose G acts strongly transitively with respect to $\mathcal{A}$. By definition then it acts weakly transitively. Let $C, D, D' \in \mathcal{C}$ with $\delta(C, D) = \delta(C, D') = w$. Choose $\Sigma, \Sigma' \in \mathcal{A}$ such that $C, D \in \mathcal{C}(\Sigma)$ and $C, D' \in \mathcal{C}(\Sigma')$. Since G acts strongly transitively, we can choose $g \in \mathrm{Stab}_G(C)$ such that $g\Sigma = \Sigma'$. Since g preserves δ, $\delta(C, gD) = w = \delta(C, D')$, and so $gD = D'$ since $\delta(C, -)$ is bijective on $\mathcal{C}(\Sigma')$.

Now suppose conversely that G acts Weyl transitively and weakly transitively. Let $\Sigma \in \overline{\mathcal{A}}$ be such that $\mathrm{Stab}_G(\Sigma)$ acts transitively on $\mathcal{C}(\Sigma)$. Since G acts transitively on $G\Sigma$, and this is an apartment system by Lemma 1.3.4, G acts strongly transitively with respect to $G\Sigma$. □

In light of this proposition, it is possible in theory to have a group action on a building that is Weyl transitive but not strongly transitive, even with respect to *any* apartment system. The key is to find a situation where Weyl transitivity holds, but for no apartment $\Sigma \in \overline{\mathcal{A}}$ does $\mathrm{Stab}_G(\Sigma)$ act transitively on $\mathcal{C}(\Sigma)$. Such a situation seems reasonable, but we show here that in the spherical case it is actually impossible; see also [AB2, Proposition 6.15].

Proposition 1.3.6. *Let G be a group acting chamber transitively on a spherical building Δ of type (W, S). Let w_0 be the longest word in W. The following are equivalent:*

1. *The action is strongly transitive with respect to $\overline{\mathcal{A}}$.*
2. *The action is Weyl transitive.*
3. *For $C \in \mathcal{C}$, $\mathrm{Stab}_G(C)$ acts transitively on $\{D \mid \delta(C, D) = w_0\}$.*

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are immediate. If $\mathrm{Stab}_G(C)$ acts transitively on $\{D \mid \delta(C, D) = w_0\}$ then it also acts transitively on $\overline{\mathcal{A}}(C)$, by Proposition 1.2.8, and so G acts strongly transitively with respect to $\overline{\mathcal{A}}$. □

Note that the presence of opposite chambers is crucial, at least in this proof, to showing the equivalence of strong and Weyl transitivity. Outside the spherical case, then, it is at least reasonable to look for examples of Weyl transitive actions that are not strongly transitive with respect to any apartment system.

1.4 Roots and root systems

In analyzing group actions on buildings we will be particularly interested in groups that have a structure informed by *root systems*, and so in this section we collect some facts about roots in Coxeter complexes. We also establish the notion of a root system as a subset of some Euclidean space, as that will become important in Chapter 3.

Let $\Sigma = \Sigma(W, S)$ be the standard thin building of type (W, S) defined in 1.1.3. Throughout this section we assume W is finite. As in [AB2, Section 5.5.4], for each $s \in S$ we set $\alpha_s := \{w \in W \mid \ell(sw) > \ell(w)\}$. We call these *simple roots*, and declare that any subset $\alpha \subseteq W$ isometric to some α_s is called a *root*. In fact, according to Proposition 5.81(3) of [AB2], we have that for each root α there exists $w \in W$ and $s \in S$ such that $\alpha = w\alpha_s$. For a root α we also define the *opposite root* $-\alpha$ to be simply $W \backslash \alpha$.

Let Φ denote the set of roots, and $\Pi \subseteq \Phi$ the set of simple roots. Note that for each $s \in S$ we have $1_W \in \alpha_s$. This motivates the designation of a set Φ_+, called the set of *positive roots*, given by $\Phi_+ := \{\alpha \in \Phi \mid 1_W \in \alpha\}$. We thus have $\Pi \subseteq \Phi_+ \subseteq \Phi$. Another notion that we will need is that of an *interval of roots*. Let $\alpha, \beta \in \Phi$ be distinct roots such that $\alpha \cap \beta \neq \emptyset$. We define the *closed interval* $[\alpha, \beta]$ to be $[\alpha, \beta] := \{\gamma \in \Phi \mid \alpha \cap \beta \subseteq \gamma\}$. Similarly define the *open interval* $(\alpha, \beta) := [\alpha, \beta] \backslash \{\alpha, \beta\}$. The intervals $[\alpha, \beta)$ and $(\alpha, \beta]$ are also defined in the natural way.

It is easier to picture roots using the simplicial approach to Coxeter complexes. A root in a simplicial Coxeter complex is essentially just half of the complex. Details and precise explanations can be found in [AB2, Section 3.4].

We can also think of the roots of a Coxeter complex as vectors in a Euclidean space. Set $E := \mathbb{R}^{|S|}$, and choose a basis (e_s) indexed by $s \in S$. For $s, t \in S$, set

$$\langle e_s, e_t \rangle := -\cos \frac{\pi}{m(s,t)}$$

where $m(s, t)$ is the order of st in W. Extend $\langle \cdot, \cdot \rangle$ to a symmetric bilinear form on E.

If, as claimed, we can view the root system as living in E, we need an action of W on E that corresponds to the action of W on Σ. For $x \in E$, $s \in S$, set

$$s(x) := x - \frac{2\langle x, e_s \rangle}{\langle e_s, e_s \rangle} e_s.$$

By arguments in [AB2, Section 2.5] this provides a well-defined, faithful action of W on E. Since every root is a translate of a simple root by an element of W, we now have a canonical way of viewing Φ as a set of vectors in E.

We can in fact give a second definition of a root system, which immediately encodes the Euclidean structure. In future chapters we will often speak of a fixed root system Φ interchangeably in the sense of half-spaces in a Coxeter complex or in the Euclidean sense. We will give very few details here regarding the proof of the equivalence of the two approaches; far more details can be found in [AB2, W].

Definition 1.4.1. Let $E = \mathbb{R}^n$ be n-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$. We say a finite set $\Phi \subseteq E \backslash \{0\}$ is a (reduced, crystallographic) *root system* provided that the following properties hold:

1. $E = \operatorname{span} \Phi$.
2. For $\epsilon \in \mathbb{R}$, if $\alpha, \epsilon\alpha \in \Phi$ then $\epsilon = \pm 1$.
3. Φ is closed under the reflections σ_α, where $\sigma_\alpha(v) := v - \dfrac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$ for $\alpha \in \Phi$, $v \in E$.
4. For $\alpha, \beta \in \Phi$, $\dfrac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. We set $\alpha^\vee := \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$, so this condition is equivalent to $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$.

We include conditions 2 and 4 in the definition; these specify that all our root systems are reduced and crystallographic [AB2, Section 1.1]. Also, in the future it will sometimes make notational sense to write s_α instead of σ_α.

Lemma 1.4.2. *Let $\alpha \in \Phi$, $v, w \in E$. Then $\langle \sigma_\alpha(v), \sigma_\alpha(w) \rangle = \langle v, w \rangle$.*

Proof.

$$\begin{aligned}
\langle \sigma_\alpha(v), \sigma_\alpha(w) \rangle &= \left\langle v - \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, w - \frac{2\langle w, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \right\rangle \\
&= \langle v, w \rangle - \left\langle \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, w \right\rangle - \left\langle v, \frac{2\langle w, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \right\rangle + \left\langle \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \frac{2\langle w, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \right\rangle \\
&= \langle v, w \rangle - 2\frac{\langle v, \alpha \rangle \langle \alpha, w \rangle}{\langle \alpha, \alpha \rangle} - 2\frac{\langle v, \alpha \rangle \langle w, \alpha \rangle}{\langle \alpha, \alpha \rangle} + 4\frac{\langle v, \alpha \rangle \langle w, \alpha \rangle \langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle^2} \\
&= \langle v, w \rangle
\end{aligned}$$

□

Definition 1.4.3. Keeping the notation of Definition 1.4.1, we let W be the group of isometries of E given by $W = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle$. We call W the *Weyl group* of the root system Φ. Note that by Lemma 1.4.2, W really is a group of isometries.

The notion of an interval of roots makes sense here as well. Let $\alpha \neq \pm\beta \in \Phi$. We set $[\alpha, \beta] := \{\gamma \in \Phi \mid \text{there exist } p_\gamma, q_\gamma \in \mathbb{R}_{\geq 0} \text{ such that } \gamma = p_\gamma\alpha + q_\gamma\beta\}$. By specifying whether p_γ and/or q_γ may be zero, we also have natural definitions for $(\alpha, \beta]$, $[\alpha, \beta)$, and (α, β).

Definition 1.4.4. Let Φ be a root system. We call a choice of subset $\Phi_+ \subseteq \Phi$ a set of *positive roots* provided that the following hold:

1. For each $\alpha \in \Phi$, we have $\alpha \in \Phi_+$ if and only if $-\alpha \notin \Phi_+$.
2. If $\alpha, \beta \in \Phi_+$, and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi_+$.

Given a choice of Φ_+ we have the corresponding set of *negative roots* $\Phi_- := \Phi \backslash \Phi_+$. We refer to Φ_+ and Φ_- as choices of *positive and negative root systems.*

Since Φ is finite, the following definition makes sense:

Definition 1.4.5. Given a choice positive root system Φ_+, we say $\alpha \in \Phi_+$ is *simple* if it cannot be written as a sum of two elements of Φ_+. Let Π denote the set of simple roots. We call Π a choice of *simple root system.*

We state without proof the following lemma, found as Theorem 5.5 in [C2].

Lemma 1.4.6. *Let* Φ *be a root system with positive root system* Φ_+ *and simple root system* Π*. Then* Π *is a basis of* E*.*

In particular $\ell := |\Pi|$ is independent of the choice of Π, so it makes sense to define the *rank* $\operatorname{rk}\Phi$ to be ℓ. Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. It turns out that every root is an integral combination of simple roots. Also, there exists a *highest root* $\tilde{\alpha} = \sum_{i=1}^{\ell} a_i\alpha_i$ such that for any other root $\beta = \sum_{i=1}^{\ell} b_i\alpha_i$, $b_i \leq a_i$. See [C1, Proposition 2.1.6; C2, Proposition 12.9]. This will be important in the setup of Theorem 2.3.4.

Note that the Weyl group preserves the conditions in Definitions 1.4.4 and 1.4.5. Thus for $w \in W$, Φ_+ a positive root system, and Π a simple root system, $w\Phi_+$ is another positive root system and $w\Pi$ is another simple root system.

Lemma 1.4.7. *Let* $w \in W$*,* Φ_+ *a positive root system, and* $\Pi \subseteq \Phi_+$ *a simple root system. If* $w\alpha \in \Phi_+$ *for all* $\alpha \in \Pi$ *then* $w = 1$*.*

Proof. By [C1, Proposition 2.1.3], Φ_+ contains one and only one simple root system. Thus since $w\Pi \subseteq \Phi_+$ and $w\Pi$ is a simple root system, in fact $w\Pi = \Pi$. By [C1, Theorem 2.2.4], W acts faithfully on the set of simple root systems, and so we conclude $w = 1$. □

This immediately tells us that if w acts trivially on Φ then $w = 1$, and so the action of W on Φ is faithful.

We state without proof one more property of W that will be important later. See [C1, Proposition 2.1.8(ii); C2, Theorem 5.13] for the proof.

Proposition 1.4.8. W *is generated by the set* $S := \{\sigma_\alpha \mid \alpha \in \Pi\}$*.*

Chapter 2

Groups acting on buildings

In this chapter we will describe some structural features that a group G could have, and how the group structure can both inform and be informed by the action of G on a building Δ.

2.1 BN-pairs and Bruhat decompositions

The first two group-theoretic definitions are that of a BN-pair and a Bruhat decomposition. They seem a bit esoteric on first glance, but will correspond very nicely to strong and Weyl transitivity. Our reference for this whole section is [AB2, Section 6.2].

Definition 2.1.1. Let G be a group, together with subgroups B and N that generate G. Let $T := B \cap N$. We say (B, N) is a *BN-pair* if $T \triangleleft N$ and $W := N/T$ admits a set of generators S with the following properties:

(BN1): For any $s \in S$, $w \in W$, we have $sBw \subseteq BswB \cup BwB$.

(BN2): For any $s \in S$, $sBs^{-1} \nleq B$.

We call W the *Weyl group* of (B, N). If the Weyl group is spherical (resp. affine) we will call the BN-pair spherical (resp. affine). Also, we often refer to the quadruple (G, B, N, S) as a *Tits system*.

We will take for granted that given a Tits system (G, B, N, S), S is uniquely determined and (W, S) is a Coxeter system. See [AB2, Theorem 6.56].

We now define Bruhat decomposition.

Definition 2.1.2. Let G be a group, B a subgroup of G, (W, S) a Coxeter system, and $C : W \to B\backslash G/B$ a bijection. We say that C provides a *Bruhat decomposition of type* (W, S) of (G, B) provided that the following condition is satisfied:
(BD): For all $s \in S$, $w \in W$, we have

$$C(sw) \subseteq C(s)C(w) \subseteq C(sw) \cup C(w),$$

and if $\ell(sw) = \ell(w) + 1$ then $C(s)C(w) = C(sw)$.

For a group G and subgroup B, if (G, B) admits a Bruhat decomposition we say B is a *Tits subgroup* of G.

Now let G be a group acting Weyl transitively on a building $\Delta = (\mathcal{C}, \delta)$ of type (W, S). Choose a fundamental chamber C, and set $B = \mathrm{Stab}_G(C)$. There is an obvious bijection between G/B and $\mathcal{C}$ via $gB \mapsto gC$. Thus also there is a bijection between $B\backslash G/B$ and the B-orbits in $\mathcal{C}$. For $b \in B$, $g \in G$, $\delta(C, bgC) = \delta(C, gC)$, so every B-orbit in $\mathcal{C}$ corresponds to a unique $w \in W$, and since δ is surjective by (CB3) we get a bijection between W and the B-orbits in $\mathcal{C}$.

We thus have a bijection $C : W \to B\backslash G/B$. We can realize C explicitly: $C(w) = BgB$ if and only if $\delta(C, gC) = w$. The map C in fact provides a Bruhat decomposition of (G, B), as seen in Theorem 6.21 of [AB2] proved below.

Theorem 2.1.3. *For $s \in S$ and $w \in W$, $C(sw) \subseteq C(s)C(w) \subseteq C(sw) \cup C(w)$. Also, if $\ell(sw) = \ell(w) + 1$ then $C(s)C(w) = C(sw)$.*

Proof. Let $h \in C(s)$ and $g \in C(w)$, so $C(s) = BhB$ and $C(w) = BgB$. By the explicit construction of the bijection C, we get that $\delta(C, hC) = s$ and $\delta(C, gC) = w$. By (CB2) we have $\delta(C, hgC) = \delta(h^{-1}C, gC) = sw$ or w. This implies that hg is an element of either $C(sw)$ or $C(w)$, and so $C(s)C(w) \subseteq C(sw) \cup C(w)$. Also, if $\ell(sw) = \ell(w) + 1$ then again by (CB2) $\delta(C, hgC) = sw$, so $C(s)C(w) \subseteq C(sw) = BhgB \subseteq BhBBgB = C(s)C(w)$ implying $C(s)C(w) = C(sw)$.

The only thing left to show is that $C(sw) \subseteq C(s)C(w)$. By (CB3) we can choose a specific $h \in C(s)$ ensuring that $\delta(C, hgC) = sw$, so $BhgB = C(sw)$. Then $C(sw) = BhgB \subseteq BhBBgB = C(s)C(w)$, and we are done. □

In this way, a group acting Weyl transitively on a building admits a canonical Bruhat decomposition. We also have a natural converse, namely that a group with a Bruhat decomposition acts Weyl transitively on a canonically associated building.

Theorem 2.1.4. *Let G be a group, B a subgroup of G, (W, S) a Coxeter system, and $C : W \to B\backslash G/B$ a bijection satisfying* (BD). *Let $\mathcal{C} := G/B$ and define $\delta : \mathcal{C}\times\mathcal{C} \to W$ via $\delta(gB, hB) = w$ if and only if $C(w) = Bg^{-1}hB$. Then $\Delta = (\mathcal{C}, \delta)$ is a building of type (W, S) and G acts Weyl transitively on Δ.*

The proof is informed by arguments following Definition 6.31 in [AB2].

Proof. Since the map $G/B \times G/B \to B\backslash G/B$ given by $(gB, hB) \mapsto Bg^{-1}hB$ is clearly well-defined we see that δ, being the composite of this with C^{-1}, is also well-defined. We now verify the axioms (CB1), (CB2), and (CB3). Clearly $\delta(gB, hB) = 1$ if and only if $C(1) = Bg^{-1}hB$. If $w = C^{-1}(B)$, we have that for any $s \in S$, $C(s) = C(s)C(w) \supseteq C(sw)$; thus $C(s) = C(sw)$ and $s = sw$ for all s, so $w = 1$. Thus $C(1) = Bg^{-1}hB$ if and only if $g^{-1}h \in B$. This verifies (CB1).

Now suppose $\delta(gB, hB) = w$ and $\delta(g'B, gB) = s$, so $C(w) = Bg^{-1}hB$ and $C(s) = Bg'^{-1}gB$. By (BD), $C(s)C(w) \subseteq C(sw)\cup C(w)$, so $Bg'^{-1}hB \subseteq C(s)C(w) \subseteq C(sw)\cup C(w)$, and so either $Bg'^{-1}hB = C(sw)$ or $Bg'^{-1}hB = C(w)$. We conclude that $\delta(g'B, hB) = sw$ or w. Additionally, if $\ell(sw) = \ell(w) + 1$ then by (BD) $Bg'^{-1}hB = C(sw)$, and $\delta(g'B, hB) = sw$. This verifies (CB2).

Lastly, suppose $\delta(gB, hB) = w$ and $s \in S$. Suppose that for all $g' \in G$ satisfying $g'^{-1}g \in C(s)$ we in fact have $g'^{-1}h \in C(w)$. Then $C(s)g^{-1}h \subseteq C(w)$, so $C(s)Bg^{-1}hB \subseteq C(w)$. But $Bg^{-1}hB = C(w)$ and (BD) tells us that $C(sw) \subseteq C(s)C(w)$, so this implies $C(sw) \subseteq C(w)$, which is impossible as seen earlier. Thus, in fact there exists g' such that $\delta(g'B, gB) = s$ and $\delta(g'B, hB) = sw$, verifying (CB3).

We now show that G acts Weyl transitively on Δ. Of course G acts on G/B by left translation, and thus on Δ. What's more, this action clearly preserves δ and is chamber transitive. Now let gB and $g'B$ be two chambers, both some fixed distance from B. Then $BgB = Bg'B$, so there exists $b \in B$ such that $g'B = bgB$. Thus B is transitive on w-spheres, and the action of G on Δ is Weyl transitive. □

Remark 2.1.5. If (G, B) admits a Bruhat decomposition and the resulting building is thick, we also call the Bruhat decomposition *thick*. Since every Bruhat decomposition has a type (W, S), it also makes sense to refer to a Bruhat decomposition as *spherical* or *affine* if (W, S) is. Also note that this is a true equivalence, namely if we construct a building Δ from a Bruhat decomposition of G and then take the Bruhat decomposition corresponding to the action of G on Δ, we recover the original Bruhat decomposition. Similarly we recover Δ from the Bruhat decomposition arising from Δ.

We can also realize the building arising from a Bruhat decomposition as a simplicial complex. We will not discuss many details here, but see Section 6.2.4 of [AB2] for a very detailed discussion of *standard parabolic cosets* and the simplicial building. We will state the following proposition as a fact, and take for granted that the simplicial building $\Delta(G, B)$ is well-defined and satisfies the building axioms.

Proposition 2.1.6. *Let G be a group with a subgroup B yielding a Bruhat decomposition of type (W, S). Let $\Delta = \Delta(G, B)$ be the simplicial complex of standard parabolic cosets of G, ordered by reverse inclusion. Then Δ is a building of type (W, S) on which G acts Weyl transitively.*

Having seen that Bruhat decompositions and Weyl transitivity are essentially equivalent, we now turn our attention to BN-pairs and strong transitivity, which will enjoy a similar correspondence. We already know that strong transitivity implies Weyl transitivity, so it will be prudent now to show that a BN-pair yields a Bruhat decomposition; see also [AB2, Theorem 6.52].

Proposition 2.1.7. *Let (G, B, N, S) be a Tits system with Weyl group W. Then (G, B) admits a Bruhat decomposition of type (W, S).*

Proof. Let $w \in W$ with $w = nT = Tn$. Since $T \leq B$, we may refer to constructions like wB and Bw without ambiguity. Define $C : W \to B\backslash G/B$ via $C(w) = BwB$. Since G is generated by B and N, C is surjective. We show C is injective by induction. If $C(w) = B$, i.e., $BwB = B$, then $w = 1$ since $T = B \cap N$. Let $r > 0$ and suppose that for any w with $\ell(w) < r$, $Bw'B = BwB$ implies $w' = w$. Now let w have length r and suppose $Bw'B = BwB$. Choose some $s \in S$ such that $\ell(sw) < \ell(w)$. Clearly $BsBw'B = BsBwB$. By (BN1) this product consists of either one or two double cosets. If one then specifically $Bsw'B = BsBw'B = BsBwB = BswB$ and by induction hypothesis $sw' = sw$. If two then $Bsw'B \cup Bw'B = BsBw'B = BsBwB = BswB \cup BwB$, and since $Bw'B = BwB$ and the union is disjoint, the previous case shows that $sw' = sw$. In either case $w' = w$ and C is injective.

We now show that (BD) holds. Let $w \in W$, $s \in S$. Then $C(sw) = BswB \subseteq BsBBwB = C(s)C(w)$, which is the first inclusion of (BD). Also, using (BN1), $C(s)C(w) = BsBwB \subseteq BswB \cup BwB = C(sw) \cup C(w)$.

Now suppose $w \in W$ and $s \in S$ with $\ell(sw) = \ell(w) + 1$. If w has length 0 then $C(s)C(w) = C(sw)$ holds trivially. Suppose $\ell(w) = r > 0$, and that for any w' with $\ell(w') < r$ and any $t \in S$, $\ell(tw) = \ell(w) + 1$, $C(t)C(w') = C(tw')$. Write $w = s_1 \cdots s_r$ as a reduced word in S; let $w' = s_1 \cdots s_{r-1}$ and $t = s_r$. By repeated use of the induction hypothesis, $C(w')C(t) = C(w)$. Now, $\ell(sw')$ must be $\ell(w') + 1 = r$, since $r + 1 = \ell(sw) = \ell(sw't) \leq \ell(sw') + 1 \leq r + 1$. Thus,

$$C(s)C(w) = C(s)C(w')C(t) = C(sw')C(t)$$

again by the induction hypothesis.

At this point we suppose that $C(s)C(w)$ is in fact a union of two double cosets. Then $C(s)C(w) = C(sw) \cup C(w)$. But we just found another representation of $C(s)C(w)$ as a product, namely $C(sw')C(t)$. This is the inverse of $C(t)C((w')^{-1}s)$, and since we're assuming it is a union of two double cosets we have

$$C(sw')C(t) = (C(t)C((w')^{-1}s))^{-1} = (C(t(w')^{-1}s))^{-1} \cup (C((w')^{-1}s))^{-1}$$
$$= C(sw't) \cup C(sw') = C(sw) \cup C(sw').$$

We conclude $C(sw) \cup C(w) = C(sw) \cup C(sw')$. We know $sw \neq w$ and $w \neq w'$ by length arguments, so by the injectivity of C we get that $w = sw'$. But then $sw = w'$, contradicting the assumption that $\ell(sw) = \ell(w) + 1$. □

We close this section wth the following theorem, which is Theorem 6.56 in [AB2].

Theorem 2.1.8. *Strong transitivity and BN-pairs are related in the following way:*

1. *Let (G, B, N, S) be a Tits system with Weyl group W. Then there exists a thick building $\Delta = \Delta(G, B)$ of type (W, S) with an action of G, strongly transitive with respect to some $\mathcal{A}$, such that $B = \mathrm{Stab}_G(C)$ and $N \leq \mathrm{Stab}_G(\Sigma_0)$ for some fundamental chamber C and apartment Σ_0, and N is transitive on $\mathcal{C}(\Sigma_0)$.*

2. *Suppose the group G acts on a thick building Δ, strongly transitively with respect to some $\mathcal{A}$, with fundamental chamber C and apartment Σ_0. Let $B = \mathrm{Stab}_G(C)$ and $N \leq \mathrm{Stab}_G(\Sigma_0)$ such that N acts transitively on $\mathcal{C}(\Sigma_0)$. Then (B, N) is a BN-pair and $\Delta = \Delta(G, B)$.*

Proof of forward implication. Since (B, N) is a BN-pair, we have a Bruhat decomposition and a building $\Delta = \Delta(G, B)$ of type (W, S). We know G acts Weyl transitively on Δ. Let C be the fundamental chamber corresponding to B in G/B, so $B = \mathrm{Stab}_G(C)$. We want to show that N stabilizes some apartment Σ_0 and acts transitively on $\mathcal{C}(\Sigma_0)$. This will in particular show that G acts weakly transitively on Δ and thus strongly transitively, by Proposition 1.3.5.

Let $\Sigma_0 = \{wB \mid w \in W\} \subseteq \mathcal{C}$. Let $\phi : W \to \Sigma_0$ be given by $\phi(w) = wB$. Then ϕ is clearly an isometry and so Σ_0 is an apartment. By construction, N stabilizes Σ_0 and acts transitively on $\mathcal{C}(\Sigma_0)$.

It remains to show that Δ is thick, or equivalently that for any $C \in \mathcal{C}$, $s \in S$, there exist at least two chambers a distance s from C. Since the action is chamber transitive it suffices to show just that the fundamental chamber B has this property. Let $s \in S$. By construction $\delta(B, sB) = s$. By (BN2) we know that $sBs \not\leq B$, so in particular $BsB \not\leq sB$. Choose $g \in BsB \backslash sB$, so $gB \neq sB$. However, $g \in C(s)$, so $\delta(B, gB) = s$. Thus Δ is indeed thick. $\square$

Proof of reverse implication. We now suppose G acts strongly transitively, and thus Weyl transitively, on some thick building Δ. Using the setup preceding Theorem 2.1.3, we know that (G, B) admits a Bruhat decomposition, given by a bijection C, and that $\Delta = \Delta(G, B)$. Considering the natural epimorphism $\pi : N \twoheadrightarrow W = \mathrm{Aut}_0(\Sigma_0)$ it is clear that $B \cap N =: T = \ker \pi$ and that we can identify W with N/T.

Now let $g \in G$. We can choose an apartment containing C and gC, and by strong transitivity it must be of the form $b\Sigma_0$ for some $b \in B$ since B acts transitively on $\mathcal{A}(C)$. Since $b^{-1}gC \in \Sigma_0$, there exists $n \in N$ such that $b^{-1}gC = nC$, again by strong transitivity. Thus $n^{-1}b^{-1}g \in B$, implying that $G = BNB$, so in particular $G = \langle B, N \rangle$.

We now need to verify the BN axioms. Since $G = BNB$, the Bruhat decomposition map $C : W \to B\backslash G/B$ can be realized explicitly as $C(w) = BwB$, without ambiguity. Then (BN1) is just a restatement of (BD). Now let $s \in S$. By thickness we can choose $g \in G$ such that $\delta(C, sC) = \delta(C, gC) = s$ but $sC \neq gC$. Just from Weyl transitivity we can choose $b \in B$ such that $gC = bsC$. Then $sbsC = sgC \neq C$, so $sbs \notin B$. Since $s = s^{-1}$, (BN2) follows. $\square$

2.2 RGD systems

In Chapter 3 we will focus our attention on Chevalley groups. These groups are generated by a family of subgroups, called *root groups*, and the relations between these subgroups are partially determined by the corresponding root system structure. The next two sections deal with certain constructions of Tits systems that lend themselves well to analyzing Chevalley groups. We refer to Section 1.4 for the relevant definitions and notation regarding root systems. The reference for this section is [AB2, Section 7.8].

Let Σ be a Coxeter complex of spherical type (W, S), and let Φ be a root system in Σ. Let Π be a set of simple roots in Φ, parameterized by $s \in S$. For $\alpha_s \in \Pi$ we will notationally identify s with α_s and $-s$ with $-\alpha_s$.

Let G be a group, with a family of subgroups U_α parameterized by $\alpha \in \Phi$. We say the pair $(G, (U_\alpha)_{\alpha\in\Phi})$ is an *RGD system of type* (W, S) if it satisfies the following axioms:

(RGD0): The U_α are all nontrivial.

(RGD1): For $\alpha \neq \pm\beta$, $[U_\alpha, U_\beta] \leq U_{(\alpha,\beta)}$

(RGD2): For each $s \in S$ there is a function $m : U_s^* \to G$ such that for $u \in U_s^*$ and $\beta \in \Phi$, $m(u) \in U_{-s}uU_{-s}$ and $m(u)U_\beta m(u)^{-1} = U_{s\beta}$.

(RGD3): For each s, $U_{-s} \not\leq U_+$.

(RGD4): $G = T\langle U_\alpha | \alpha \in \Phi\rangle$ where $T = \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$.

Here, $U_+ := \langle U_\alpha \mid \alpha \in \Phi_+\rangle$, $U_{(\alpha,\beta)} := \langle U_\gamma \mid \gamma \in (\alpha, \beta)\rangle$, and $H^* := H\backslash\{1\}$ for any group H. It should be mentioned that functions m in (RGD2) actually exist for every root, not just simple roots [W], and so we get the more general formula $m(u)U_\beta m(u)^{-1} = U_{s_\alpha\beta}$ for any α. Of course if we want to verify the RGD axioms it is less work to just check (RGD2) for simple roots. Also note that since $s_\alpha = s_{-\alpha}$, we could equivalently replace $m(u)U_\beta m(u)^{-1} = U_{s_\alpha\beta}$ with $m(u)^{-1}U_\beta m(u) = U_{s_\alpha\beta}$.

As implied, if $(G, (U_\alpha)_{\alpha\in\Phi})$ is an RGD system of type (W, S) then G has a canonical BN-pair of type (W, S). The proof of this fact takes up thirteen pages in [AB2, Section 7.8], and we will not attempt to recreate all the lemmas and details here. It is worth describing some key results and steps, however, before moving on to VRGD systems in the next section. We describe the setup here and provide a sketch of the proof, omitting details as necessary.

Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD system of type (W, S). Let $m : U_s^* \to G$ be the maps described in (RGD2). Set $B := TU_+$ and $N = \langle T, \{m(u) \mid u \in U_s^*, s \in S\}\rangle$. By [AB2, Section 7.8.3], there is a map $\pi : N \twoheadrightarrow W$ such that $\ker \pi = T$ and $\pi(m(u)) = s$ for $u \in U_s^*$. This shows immediately that $N/T \cong W$. We also already have a canonical generating set for N/T that syncs up with S, namely $\{\tilde{s}T \mid s \in S\}$ where each $\tilde{s}$ is $m(u)$ for some choice of $u \in U_s^*$.

Lemma 2.2.1. *$G = \langle B, N\rangle$.*

Proof. By (RGD4), $G = T\langle U_\alpha \mid \alpha \in \Phi\rangle$. For any $\alpha \in \Phi_+$ we have $U_\alpha \le U_+ \le B$, so all we have to show is that $U_\alpha \le \langle B, N\rangle$ for $\alpha \in \Phi_-$. As explained in Section 1.4, we can choose $w \in W$, $s \in S$ such that $\alpha = w\alpha_s$. Let $x \in N$ be such that $w = xT$. Then by (RGD2) $U_\alpha = xU_s x^{-1} \le \langle B, N\rangle$. □

Lemma 2.2.2. *$B \cap N = T$.*

Proof. Clearly $T \le B \cap N$. Thus, it suffices to show that $N \cap U_+ \le T = \ker \pi$. Let $x \in N \cap U_+$, with $w := \pi(x)$. By (RGD2), for $\alpha \in \Phi_+$ we have $U_{w\alpha} = xU_\alpha x^{-1} \le U_+$. Now, by (RGD3), $U_{-s} \not\le U_+$, so $w\alpha \ne -\alpha_s$ for any $s \in S$. But if $w \ne 1$ then by Lemma 1.4.7 there exist $\alpha \in \Phi_+$ and $s \in S$ such that $w\alpha = -\alpha_s$. Thus, $w = 1$ and in fact $x \in \ker \pi$. □

Corollary 2.2.3. *For any $s \in S$, $U_{-s} \not\le B$.*

Proof. Let $v \in U_{-s}^*$, and let $m(v) \in U_s v U_s$ be as in (RGD2), or more precisely as in the comments below the RGD axioms. Conjugation by $m(v)$ interchanges U_s with U_{-s}, so $m(v)$ does not normalize U_s and so is not in T. Of course $m(v)$ is in N, and since $B \cap N = T$ we know that $m(v) \notin B$. Then since $m(v) \in U_s v U_s \subseteq BvB$ we conclude that $v \notin B$. □

Theorem 2.2.4. *Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD system of type (W, S). Set $B := TU_+$ and $N = \langle T, \{m(u) \mid u \in U_s^*, s \in S\}\rangle$. Then (G, B, N, S) is a Tits system of type (W, S).*

Proof. By Lemmas 2.2.2 and 2.2.1, and the preceding paragraph, all the setup of Definition 2.1.1 holds and we just need to show that (BN1) and (BN2) are satisfied. As often happens, (BN1) is very computationally messy to verify. As such, we will simply reference the proof of Theorem 7.115 in [AB2] for the verification of (BN1).

We will, however, verify (BN2). Let $s \in S$. Then $sBs^{-1} \geq sU_s s^{-1} = U_{-s}$, and by Corollary 2.2.3 $U_{-s} \not\leq B$, so $sBs^{-1} \not\leq B$. □

Before moving on we prove part (a) of the exercise on page 36 of [S3], regarding normalizers of certain subgroups. The exercise is phrased in terms of Chevalley groups, which we will discuss in depth in Chapter 3, but the statement is true for general RGD systems too.

Lemma 2.2.5. *Let $B = TU_+$, $B_- = TU_-$, and N be as above. Then $N_G(B) = N_G(U_+) = B$, $N_G(B_-) = N_G(U_-) = B_-$.*

Proof. Clearly B normalizes U_+ and B. Let $g \in N_G(U_+)$. Thanks to the Bruhat decomposition we can choose $u, v \in U_+$, $x \in N$ such that $g = uxv$. Then x normalizes U_+. Let $\pi(x) = w \in W$, so $w\Phi_+ = \Phi_+$ (see (RGD2)). By Lemma 1.4.7 this implies $w = 1$, so $g \in B$. By a similar argument, coupled with Corollary 2.2.3, we see that B is also the full normalizer of itself. A parallel argument shows $N_G(B_-) = N_G(U_-) = B_-$. □

2.3 VRGD systems

Let G be a group with a family of subgroups U_α parameterized by $\alpha \in \Phi$, such that $(G, (U_\alpha)_{\alpha\in\Phi})$ is an RGD system of type (W, S). Here we think of Φ as a root system in the Euclidean sense. We will also assume as in [W] that G is generated by the subgroups U_α. For each $\alpha \in \Phi$ let $\phi_\alpha : U_\alpha^* \to \mathbb{Z}$ be a map. We say the triple $(G, (U_\alpha)_{\alpha\in\Phi}, (\phi_\alpha)_{\alpha\in\Phi})$ is a *VRGD system of type* (W, S) if it satisfies the following axioms:

(VRGD0): Each ϕ_α is surjective.

(VRGD1): For each $\alpha \in \Phi$ and each $k \in \mathbb{Z}$, $U_{\alpha,k} := \langle u \in U_\alpha | \phi_\alpha(u) \geq k \rangle$ is a subgroup of U_α, where $\phi_\alpha(1)$ is considered to be ∞.

(VRGD2): For all $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$, $[U_{\alpha,k}, U_{\beta,\ell}] \subseteq \prod_{\gamma \in (\alpha,\beta)} U_{\gamma, p_\gamma k + q_\gamma \ell}$, where p_γ and q_γ are as defined in Section 1.4. In particular $p_\gamma, q_\gamma > 0$ and $\gamma = p_\gamma \alpha + q_\gamma \beta$.

(VRGD3): For $\alpha, \beta \in \Phi$, $u \in U_\alpha^*$, $x \in U_\beta^*$, we have that $\phi_{s_\alpha(\beta)}(m(u)^{-1} x m(u)) - \phi_\beta(x)$ is independent of x, where $m : U_\alpha^* \to G$ is as defined in (RGD2).

(VRGD4): For $\alpha \in \Phi$, $u \in U_\alpha^*$, $x \in U_\alpha^*$, we have that $\phi_{-\alpha}(m(u)^{-1} x m(u)) - \phi_\alpha(x) = -2\phi_\alpha(u)$, independent of x.

The term *VRGD system* is new, but is natural in light of the term RGD system. Where RGD stands for root group data, VRGD stands for *valuated* root group data. Note that in (VRGD3) we refer to the maps m existing for any root α, as is done in [W]. This is fine, as explained in the comments below the RGD axioms. Also note that by our assumption that G is generated by the root groups, by [AB2, Corollary 125] we know that T is already contained in $\langle \{m(u) \mid u \in U_s^*, s \in S\} \rangle$ and so the latter group already equals N. Also note that since we have maps m for any root, we could write $N = \langle \{m(u) \mid u \in U_\alpha^*, \alpha \in \Phi\} \rangle$ when convenient.

If G admits a VRGD system, it then of course admits an RGD system and thus has a BN-pair. This BN-pair is spherical, and we get an action of G on a spherical building. Of course this will never provide examples of Weyl transitive actions that are not weakly transitive, but it turns out that a group admitting a VRGD system also has an *affine* BN-pair, and so acts canonically on an affine building.

The special case where G is a p-adic Chevalley group is covered in detail in [IM]. See also [BT] for a more general situation. We will define such groups and show they admit a VRGD system in Chapter 3. We are interested in the more general case, however, and present the connection between VRGD systems and affine BN-pairs. The rest of this section will be an overview of Chapter 14 of [W], in which the general case is covered. The theorem of interest is Theorem 14.38, though we need some setup before we can even state it here.

Let N be as above and let $U := \langle U_{\alpha,k_\alpha} \rangle$ where $k_\alpha = 0$ if $\alpha \in \Phi^+$ and $k_\alpha = 1$ if $\alpha \in \Phi^-$. Let $\Phi_a := \Phi \times \mathbb{Z}$, and as usual let E be the Euclidean space spanned by Φ. For $(\alpha, k) \in \Phi_a$ and $v \in E$, we define the *affine reflection*

$$\sigma_{\alpha,k}(v) := \sigma_\alpha(v) + \frac{2k}{\langle \alpha, \alpha \rangle}\alpha = \sigma_\alpha(v) + k\alpha^\vee$$

and the *affine Weyl group* W_a to be $W_a := \langle \sigma_{\alpha,k} \mid (\alpha, k) \in \Phi_a \rangle$. A quick calculation verifies that affine reflections have order 2. Lastly, for each $(\alpha, k) \in \Phi_a$ define the *affine half-space* $[\alpha, k]$ to be $[\alpha, k] := \{v \in E \mid \langle v, \alpha \rangle \geq -k\}$. (We will take for granted that W_a is really an affine Coxeter group; see [B3, Section 6.2.1; W, Chapter 2].)

By Proposition 14.4 of [W], there exists a surjective homomorphism $\pi : N \twoheadrightarrow W_a$ such that $\pi(m(u)) = \sigma_{\alpha,k}$ for $u \in U_\alpha$ and $k = -\phi_\alpha(u)$. Set $T_a := \ker \pi$ and $B_a := T_a U$. We will see in Lemma 2.3.1 that T_a normalizes each $U_{\alpha,k}$ so B_a is really a subgroup. Lastly, we recall the identification of S with a set of certain choices of $m(u_\alpha)$, where $u_\alpha \in U_\alpha^*$ and $\alpha \in \Pi$. Let $\tilde{\alpha}$ be the unique highest root in Φ and choose $u_{-\tilde{\alpha}} \in U_{-\tilde{\alpha}}^*$. Define $\tilde{s} := m(u_{-\tilde{\alpha}})$ and $S_a := S \cup \{\tilde{s}\}$. We can in fact choose the u_α to satisfy $\phi_\alpha(u_\alpha) = k_\alpha$ for all $\alpha \in \Pi \cup \{-\tilde{\alpha}\}$ by (VRGD0).

We claim that (G, B_a, N, S_a) is a Tits system. To see this we need a series of lemmas.

Lemma 2.3.1. *Let $(\alpha, k) \in \Phi_a$. Let $g \in N$. Then $\pi(g)[\alpha, k] = [\beta, \ell]$ for some β, ℓ, and $g^{-1}U_{\alpha,k}g = U_{\beta,\ell}$.*

This is Proposition 14.19 in [W]. Since the proof is constructive we present it here, though it uses Theorem 3.41 of [W], a refinement of (VRGD3), which we will simply assume.

Proof. We may assume $g = m(u)$ for some $u \in U_\gamma^*$. Let $j = \phi_\gamma(u)$. Then

$$\begin{aligned}\pi(g)([\alpha,k]) &= \sigma_{\gamma,-j}([\alpha,k]) = \sigma_{\gamma,-j}(\{v \in V | \langle v,\alpha\rangle \geq -k\}) \\ &= \{v | \langle \sigma_{\gamma,-j}(v),\alpha\rangle \geq -k\} = \{v | \langle(\sigma_\gamma(v) - j\gamma^\vee),\alpha\rangle \geq -k\} \\ &= \{v | \langle \sigma_\gamma(v),\alpha\rangle \geq -(k - j\langle\alpha,\gamma^\vee\rangle)\} = \{v | \langle v,\sigma_\gamma(\alpha)\rangle \geq -\ell\}\end{aligned}$$

where $\ell = k - j\langle\alpha,\gamma^\vee\rangle$.

So $\pi(g)([\alpha,k]) = [\beta,\ell]$ for $\beta = \sigma_\gamma(\alpha)$ and $\ell = k - j\langle\alpha,\gamma^\vee\rangle$. Now, by Theorem 3.41 of [W], for $z \in U_{\sigma_\gamma(\alpha)}$, $\phi_\alpha(gzg^{-1}) = \phi_{\sigma_\gamma(\alpha)}(z) + j\langle\alpha,\gamma^\vee\rangle$. Thus, $g^{-1}U_{\alpha,k}g = \{z \in U_{\sigma_\gamma(\alpha)} | \phi_\alpha(gzg^{-1}) \geq k\} = \{z \in U_{\sigma_\gamma(\alpha)} | \phi_{\sigma_\gamma(\alpha)}(z) \geq \ell\} = U_{\beta,\ell}$. □

As an immediate consequence, we see that T_a normalizes each $U_{\alpha,k}$, so B_a is really a subgroup of G. Another result we will need is the following lemma, which we will not prove here. See [W, Proposition 14.33(iii)].

Lemma 2.3.2. *For each $\alpha \in \Phi$, $U_\alpha \cap B_a = U_{\alpha,k_\alpha}$.*

Before we verify the BN axioms we need to check that the preliminary conditions are satisfied. We will take for granted that S_a generates W_a; see Chapter 2 and Proposition 14.36 of [W]. Specifically we can identify $s \in S$ with σ_{α_s} and $\tilde{s}$ with $\sigma_{-\tilde{\alpha},1}$.

Lemma 2.3.3. *$G = \langle B_a, N\rangle$ and $T_a = B_a \cap N$.*

Proof. To show $G = \langle B_a, N\rangle$ it suffices to show that $U_\alpha \leq \langle B_a, N\rangle$ for each $\alpha \in \Phi$. Let $\alpha \in \Phi$, $u \in U_\alpha^*$. Let $k = \phi_\alpha(u)$. If $k \geq k_\alpha$ then $u \in U_{\alpha,k_\alpha} \leq U \leq B_a$. Suppose $k < k_\alpha$, so $-k \geq -k_\alpha + 1$, which equals $k_{-\alpha}$ since $1 = k_\alpha + k_{-\alpha}$. Then $m(u)^{-1}um(u) \in U_{-\alpha,-k}$ by Lemma 2.3.1, and so $m(u)^{-1}um(u) \in U_{-\alpha,k_{-\alpha}} \leq B_a$.

Now we claim $T_a = B_a \cap N$. The argument in [W] uses the structure of the spherical Coxeter complex $\Sigma(W,S)$. We will give a more direct proof here, inspired by the proof of Lemma 2.2.2. This proof only works if the rank is at least 2, so for the proof of the rank 1 case we just reference [W] again. VRGD systems of rank 1 are of course interesting, and were the focus of [AB1] and [AB2], but since the present lemma has such a nice proof when the rank is at least 2 we present it now. We have one inclusion by construction, so it suffices to show $N \cap U \subseteq T_a$.

Let $x \in N \cap U$. Recall that $N = \langle \{m(u) \mid u \in U_\alpha^*, \alpha \in \Pi\} \rangle$. Say $x = m(u_r) \cdots m(u_1)$ for $u_i \in U_{\alpha_i}^*$, $\alpha_i \in \Pi$; by [W, Equation 3.9] $m(u^{-1}) = m(u)^{-1}$ so this is really an arbitrary element. Set $w_a = \pi(x) = \sigma_{\alpha_r,k_r} \cdots \sigma_{\alpha_1,k_1}$, where $k_i = -\phi_{\alpha_i}(u_i)$. Also let $w = \sigma_{\alpha_r} \cdots \sigma_{\alpha_1} \in W$. For $\alpha \in \Phi$, by Lemma 2.3.1 we have $U_{w\alpha,\ell} = x^{-1} U_{\alpha,k_\alpha} x$ where

$$\ell = k_\alpha + \left(\sum_{i=1}^{r} k_i \left\langle \sigma_{\alpha_{i-1}} \cdots \sigma_{\alpha_1} \alpha, \alpha_i^\vee \right\rangle \right).$$

Since $x \in U$, also $U_{w\alpha,\ell} \leq U$. Thus by Lemma 2.3.2 and (VRGD0) we have $\ell \geq k_{w\alpha}$, so

$$\begin{aligned} k_{w\alpha} &\leq k_\alpha + \sum_{i=1}^{r} k_i \left\langle \sigma_{\alpha_{i-1}} \cdots \sigma_{\alpha_1} \alpha, \alpha_i^\vee \right\rangle \\ &= k_\alpha + \left\langle \alpha, \sum_{i=1}^{r} k_i \sigma_{\alpha_1} \cdots \sigma_{\alpha_{i-1}} \alpha_i^\vee \right\rangle \\ &=: k_\alpha + \langle \alpha, v \rangle \end{aligned}$$

for all $\alpha \in \Phi$. Note that v does not depend on α so this definition is fine.

The only way $\langle \alpha, v \rangle$ can be negative is if $\alpha \in \Phi_-$ and $w\alpha \in \Phi_+$. Suppose $v \neq 0$. Then precisely half the roots must have negative inner product with v, and so *every* negative root α satisfies $\alpha \in \Phi_-$ and $w\alpha \in \Phi_+$. Let $\alpha \in \Phi_+$. Then $k_\alpha = k_{w(-\alpha)} = 0$, and $k_{-\alpha} = k_{w\alpha} = 1$. We conclude that $1 \leq \langle \alpha, v \rangle$ and $-1 \leq \langle -\alpha, v \rangle$ so in fact $1 = \langle \alpha, v \rangle$ for all $\alpha \in \Phi_+$. Since we are assuming Φ has rank greater than 1 this is a contradiction, since $\langle \alpha + \beta, v \rangle = \langle \alpha, v \rangle + \langle \beta, v \rangle$. Thus in fact $v = 0$ and $k_{w\alpha} \leq k_\alpha$ for all $\alpha \in \Phi$, so w acts trivially.

Since $w\alpha = \alpha$ and $v = 0$, we conclude that $w_a[\alpha, k] = [\alpha, k]$ for all $[\alpha, k]$, and so $w_a = 1$ and $x \in T_a$. $\square$

Theorem 2.3.4. *Let* $(G, (U_\alpha)_{\alpha\in\Phi}, (\phi_\alpha)_{\alpha\in\Phi})$ *be a VRGD system of type* (W, S)*. Then, using the notation as before,* (G, B_a, N, S_a) *is a Tits system of type* (W_a, S_a)*.*

Proof. All the setup of Definition 2.1.1 holds and we just need to show that (BN1) and (BN2) are satisfied. As in the RGD case, the verification of (BN1) is very long and would require a huge digression into the interaction between the root groups, the valuations, and the spherical building associated to the underlying RGD system. As such we will simply reference the proof of Theorem 14.38 in [W].

It is, however, a simple exercise to verify (BN2). We present here a different proof than that in [W]. We will show $s^{-1}B_a s \not\leq B_a$. Since $s^2 \in T_a \leq B_a$ it's fine to reverse the order of conjugation this way. Let $s \in S_a$. Say $s = m(u)$, where $u \in U_\alpha^*$ for the appropriate $\alpha \in \Pi \cup \{-\tilde{\alpha}\}$. As before, $\phi_\alpha(u) = k_\alpha$. By Lemma 2.3.1, $\pi(s)[\alpha, k_\alpha] = [\beta, \ell]$, where $\beta = s\alpha = -\alpha$ and $\ell = k_\alpha - \phi_\alpha(u)\langle\alpha, \alpha^\vee\rangle = -k_\alpha$, i.e., $\pi(s)[\alpha, k_\alpha] = [-\alpha, -k_\alpha]$. Thus, $s^{-1}U_{\alpha,k_\alpha}s = U_{-\alpha,-k_\alpha}$. We conclude that for any $s \in S_a$, $s^{-1}B_a s \not\leq B_a$, by Lemma 2.3.2. □

In these two sections we have seen that groups admitting RGD systems and VRGD systems have BN-pairs of spherical and affine type, respectively. Thus, if a group admits an RGD system it acts strongly transitively on some spherical building, and if a group admits a VRGD system it acts strongly transitively on some affine building, with respect to some apartment system. In the next section we collect some easy but crucial lemmas that can test whether a subgroup enjoys these transitivity properties.

2.4 Transitivity properties and subgroups

We first establish an easy lemma regarding group actions in general.

Lemma 2.4.1. *Let* G *be a group acting transitively on a set* X*, and let* $H \leq G$*. For* $x \in X$ *let* $G_x = \mathrm{Stab}_G(x)$*. Then the action of* H *on* X *is transitive, if and only if* $HG_x = G$ *for all* $x \in X$*, if and only if* $HG_x = G$ *for some* $x \in X$*.*

Proof. Suppose H acts transitively on X. Let $g \in G$, $x \in X$. Choose $h \in H$ such that $hx = gx$. Then $h^{-1}g \in G_x$. Now suppose $HG_x = G$ for some $x \in X$. Let $y \in X$ and $g \in G$ with $gx = y$. Choose $h \in H$ such that $h^{-1}g \in G_x$. Then $hx = h(h^{-1}g)x = gx = y$, so H acts transitively on X. □

Note that the condition $HG_x = G$ is equivalent to the condition that the map $H \to G/G_x$ given by $h \mapsto hG_x$ is surjective.

We now consider a group G acting either Weyl transitively or strongly transitively on a building Δ and determine criteria by which the action of a subgroup $H \leq G$ is also Weyl transitive or weakly transitive.

First suppose G acts Weyl transitively on $\Delta = (\mathcal{C}, \delta)$. Fix a fundamental chamber $C \in \mathcal{C}$ and set $B = \mathrm{Stab}_G(C)$. Set $X := \{D \in \mathcal{C} \mid \delta(C, D) = w\}$, so B acts transitively on X.

Lemma 2.4.2. *If G is a topological group and H is dense in G, and if B is an open subgroup of G, then the action of H on Δ is also Weyl transitive.*

Proof. First, since B is open and H is dense, H intersects all cosets of B. Thus $HB = G$, and by Lemma 2.4.1 H acts chamber transitively on Δ. Now fix some D_0 in X, and choose $g \in G$ such that $D_0 = gC$. Then since $H \cap B$ is dense in B and all cosets in $B/(B \cap gBg^{-1})$ are open in B, we know that each coset intersects $H \cap B$, and so $(H \cap B)(B \cap gBg^{-1}) = B$. But $B \cap gBg^{-1}$ is just $\mathrm{Stab}_B(D_0)$, and so by Lemma 2.4.1 again, $H \cap B$ acts transitively on X. Since C and w were arbitrary, this implies that H acts Weyl transitively on Δ. □

Now suppose G acts strongly transitively on Δ with respect to $\overline{\mathcal{A}}$, in particular G acts weakly transitively. Fix an apartment Σ_0, such that $N = \mathrm{Stab}_G(\Sigma_0)$ acts transitively on $X = \mathcal{C}(\Sigma_0)$. Let $T = \mathrm{Stab}_N(C)$.

Lemma 2.4.3. *The action of H on Δ is weakly transitive if and only if there exists $g \in G$ such that $(gHg^{-1} \cap N)T = N$.*

Proof. Suppose H acts weakly transitively, so there exists $\Sigma \in \overline{\mathcal{A}}$ such that $\mathrm{Stab}_H(\Sigma)$ acts transitively on $\mathcal{C}(\Sigma)$. Choose $g \in G$ such that $g\Sigma = \Sigma_0$, so $\mathrm{Stab}_{gHg^{-1}}(\Sigma_0) = gHg^{-1} \cap N$ acts transitively on $\mathcal{C}(\Sigma_0)$. By Lemma 2.4.1 then, $(gHg^{-1} \cap N)T = N$. Conversely, if $(gHg^{-1} \cap N)T = N$ then $gHg^{-1} \cap N$ acts transitively on $\mathcal{C}(\Sigma_0)$, and thus so does $\mathrm{Stab}_{gHg^{-1}}(\Sigma_0)$. But this means $\mathrm{Stab}_H(g^{-1}\Sigma_0)$ acts transitively on $\mathcal{C}(g^{-1}\Sigma_0)$, and so H acts weakly transitively. $\square$

Remark 2.4.4. Note that the hypothesis that G acts strongly transitively on Δ with respect to the *complete* apartment system $\overline{\mathcal{A}}$ is vital, since we can't control Σ. As Proposition 3.3.8 will later indicate, showing that a given G acts transitively on $\overline{\mathcal{A}}$ can be a rather involved process, even if the explicit structure of G is known. In the next chapter we describe one such explicit structure that a group admitting an RGD or VRGD system can have.

Chapter 3

Chevalley groups

Chevalley groups are certain groups whose structure is informed by three factors: a root system, a choice of representation of a certain Lie algebra, and a field. Throughout this chapter we follow [S3], though we will use notation from [AB2, Section 7.9.2]. Also see [C1] for a detailed account of *adjoint* Chevalley groups.

3.1 Definitions

Many details will be skipped in this section, as the constructions are well known. The construction of Chevalley groups is given in [C1, S3], and the relevant details from semisimple Lie algebra theory can be found in [C2, B3]. Let $\mathfrak{g}$ be a (finite dimensional) complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let Φ be the corresponding *root system*. A *root* here is a weight of the adjoint map ad, i.e., a linear functional $\alpha : \mathfrak{h} \to \mathbb{C}$ such that $\mathfrak{g}_\alpha := \{y \in \mathfrak{g} \mid \operatorname{ad} x(y) = \alpha(x)y \text{ for all } x \in \mathfrak{h}\}$ is nonzero. Since $\mathfrak{h}$ is abelian and self-normalizing we can think of $\mathfrak{h}$ as $\mathfrak{g}_0$, and we get a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

We know that Φ spans $\mathfrak{h}^*$ as a $\mathbb{C}$-vector space. Let $E \subseteq \mathfrak{h}^*$ be the $\mathbb{R}$-span of Φ. We can define a symmetric, non-degenerate bilinear form $\langle , \rangle$ on E thanks to the non-degeneracy of the *Killing form*; see [C2] for details and definitions. As in

Section 1.4 we also set $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ for a root $\alpha \in \Phi$, so that $2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle = \langle\alpha,\beta^\vee\rangle$. Quotients of this sort will appear very often, and this notation will prove very convenient.

It is a fact that Φ is a root system in the sense of 1.4. We will not prove this here, but will reference the standard texts [B3, C2, S3]. Because of this, we have a Weyl group $W = W(\Phi)$ with canonical generating set S, a choice of positive roots Φ_+, and simple roots $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. The structure of the root system informs the structure of $\mathfrak{g}$ to a large degree. In particular there is a nice canonical basis of $\mathfrak{g}$ called the *Chevalley basis* that is indexed by Φ and Π. The basis consists of an element $x_\alpha \in \mathfrak{g}_\alpha$ for each root α and an element $h_i \in \mathfrak{h}$ for each $1 \le i \le \ell$. These $|\Phi| + \ell$ elements interact nicely and allow for a well-understood set of relations for $\mathfrak{g}$ [C2, S3].

From the Lie algebra we next move to the *universal enveloping algebra* (UEA) of $\mathfrak{g}$, denoted $\mathcal{U} = \mathcal{U}(\mathfrak{g})$. This is a certain associative algebra with 1 containing $\mathfrak{g}$ as a subspace, see [C2, Section 9.1] for details. Given the basis of $\mathfrak{g}$ described above, a basis of $\mathcal{U}$ is given by the elements

$$\prod_{i=1}^{\ell} h_i^{e_i} \prod_{\alpha\in\Phi} x_\alpha^{f_\alpha},$$

where the e_i and f_α are nonnegative integers. Morally, moving from $\mathfrak{g}$ to $\mathcal{U}$ is a step closer to group theory as we now have the associative property.

One last preliminary step before constructing a Chevalley group is to construct the *Kostant $\mathbb{Z}$-form* $\mathcal{U}_\mathbb{Z}$. This is a certain $\mathbb{Z}$-algebra contained in $\mathcal{U}$, defined to be the $\mathbb{Z}$-subalgebra of $\mathcal{U}$ generated by all $x_\alpha^n/n!$, where $\alpha \in \Phi$ and $n \in \mathbb{N}$. Note that since we started with a Lie algebra $\mathfrak{g}$ specifically over $\mathbb{C}$, $x_\alpha^n/n!$ is a true element of $\mathcal{U}$. This is an important step since we want to construct a Chevalley group over an arbitrary field, and now that we are in the realm of $\mathbb{Z}$-algebras we can use tensor products.

Let K be an arbitrary field, and let V be an arbitrary (finite dimensional) faithful representation of $\mathfrak{g}$. According to [C2, Proposition 9.3], V is also a representation of $\mathcal{U}$. By Corollary 1 of Theorem 2 in [S3], there exists a lattice M of V that is invariant under the action of $\mathcal{U}_{\mathbb{Z}}$. In particular for each $\alpha \in \Phi$, $n \in \mathbb{N}$, we have that $\frac{x_\alpha^n}{n!}$ stabilizes M. For indeterminant y, we thus have that $y^n \frac{x_\alpha^n}{n!}$ stabilizes $M \otimes \mathbb{Z}[y]$, where this and all tensor products in this section are over $\mathbb{Z}$. Also, since any x_α^n acts as zero on M for large enough n (see [S3, Lemma 11]), we can refer to the infinite sum

$$\exp(yx_\alpha) := \sum_{n=0}^{\infty} \frac{y^n x_\alpha^n}{n!}$$

and this sum acts on $M \otimes \mathbb{Z}[y]$. Extend this to an action on $M \otimes \mathbb{Z}[y] \otimes K$, and lastly filter the action through the homomorphism $y \mapsto \lambda$ for some fixed $\lambda \in K$. We thus get a well-defined construction $x_\alpha(\lambda) := \exp(\lambda x_\alpha)$ acting on $M \otimes K =: V^K$.

Definition 3.1.1. The *Chevalley group* $\mathfrak{g}(K) = \mathfrak{g}(\Phi, \Lambda, K)$ is defined to be the subgroup of $\mathrm{Aut}(V^K)$ generated by the $x_\alpha(\lambda)$, where $\alpha \in \Phi$, $\lambda \in K$. Here, Λ is the weight lattice of the representation V, and encodes the fact that a different choice of V may yield a different Chevalley group.

Remark 3.1.2. In fact the possibilities for Λ are tightly controlled. If V is the adjoint representation, Λ is the *root lattice* Λ_r, generated by Φ. If V is the *universal representation*, i.e., the direct sum of irreducible representations with the fundamental weights as highest weights, then Λ is the *full weight lattice* Λ_w, generated by all weights of all representations; see [C2, Sections 10.1-10.3] for details and definitions. For any V with lattice Λ, it turns out that $\Lambda_r \leq \Lambda \leq \Lambda_w$ [S3, Lemma 27(c)], i.e., the adjoint and universal representations provide lower and upper bounds on the possible choices.

For each root $\alpha \in \Phi$ define the *root group* $U_\alpha := \langle x_\alpha(\lambda) \mid \lambda \in K \rangle$. Note that the usual properties of the *exp* map hold, and $x_\alpha(\lambda)x_\alpha(\mu) = x_\alpha(\lambda + \mu)$, so $U_\alpha \cong (K, +)$. (This shows that $\mathfrak{g}(K)$ is actually a subgroup of $\mathrm{Aut}(V^K)$; *a priori* it wasn't clear that the $x_\alpha(\lambda)$ were invertible.) Also define

$$m_\alpha(\lambda) := x_\alpha(\lambda)x_{-\alpha}(-\lambda^{-1})x_\alpha(\lambda) \text{ and } h_\alpha(\lambda) := m_\alpha(\lambda)m_\alpha(1)^{-1}.$$

Note that by construction, $m_\alpha(\lambda)^{-1} = m_\alpha(-\lambda)$. We reference here two crucial facts regarding the action of $h_\alpha(\lambda)$ and $x_\alpha(\lambda)$ on V^K, the proofs of which can be found in [S3, Lemma 19] and [S3, Lemma 11] respectively.

Lemma 3.1.3. *Let γ be a weight of V^K, $\alpha \in \Phi$, and $\lambda \in K$. Then $h_\alpha(\lambda)$ acts on the weight space V^K_γ via multiplication by $\lambda^{\langle \gamma, \alpha^\vee \rangle}$.*

Note that $\langle \gamma, \alpha^\vee \rangle$ really is an integer; see the corollary to Theorem 3 in [S3].

Lemma 3.1.4. *Let γ be a weight of V^K, $\alpha \in \Phi$. Then*

$$x_\alpha V^K_\gamma \subseteq V^K_{\gamma+\alpha},$$

where $V^K_{\gamma+\alpha} = 0$ if $\gamma + \alpha$ is not a weight.

As an immediate corollary to these lemmas, we get the following

Corollary 3.1.5. *Let $G = \mathfrak{g}(K)$ be a Chevalley group arising from V^K, so $G \leq \mathrm{GL}(V^K)$, and let Φ be the root system. Then there exists an ordering of a basis of V^K with respect to which the following hold:*

1. *For $\alpha \in \Phi_+$, $\lambda \in K$, $x_\alpha(\lambda)$ is upper triangular with 1's on the diagonal.*

2. *For $\alpha \in \Phi_-$, $\lambda \in K$, $x_\alpha(\lambda)$ is lower triangular with 1's on the diagonal.*

3. *For $\alpha \in \Phi$, $\lambda \in K$, $h_\alpha(\lambda)$ is diagonal.*

Proof. Define a partial ordering $>$ on the weights of V^K given by: $\gamma > \delta$ iff $\gamma = \delta + \alpha$ for some $\alpha \in \Phi_+$. Choose a basis of each weight space V^K_γ, and take the union of these bases over all weight spaces to obtain a basis $\mathcal{B}$ of V^K. Define a partial ordering on $\mathcal{B}$ where if $v \in \mathcal{B} \cap V^K_\gamma$ and $w \in \mathcal{B} \cap V^K_\delta$ with $\gamma > \delta$ then $v < w$. Extend this to a total ordering of $\mathcal{B}$. Then by the previous lemmas and the construction

$$x_\alpha(\lambda) = 1 + \lambda x_\alpha + \lambda^2 \frac{x_\alpha^2}{2!} + \lambda^3 \frac{x_\alpha^3}{3!} + \cdots$$

the results follow. □

If we let $d = |\mathcal{B}|$, we get $G \leq \mathrm{GL}_d(K)$. In fact, $G \leq \mathrm{SL}_d(K)$ since the $x_\alpha(\lambda)$ generate G and have determinant 1. What's more, in the construction of $\mathcal{B}$ we may as well choose $\mathcal{B}$ to be a $\mathbb{Z}$-basis of the lattice M. Since M is invariant under the action of $\mathcal{U}(\mathbb{Z})$, this ensures that the entries of any $x_\alpha^n/n!$ are integers.

Corollary 3.1.6. *Thinking of $G \leq \mathrm{SL}_d(K)$, the nondiagonal entries of $x_\alpha(\lambda)$ are all elements of $\mathbb{Z}[\lambda]$, the subring of K generated by λ, and are in particular divisible by λ.*

Proof. Since $x_\alpha(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n x_\alpha^n}{n!}$ and the entries of each $x_\alpha^n/n!$ are integers, this follows immediately. □

Set $N := \langle m_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in K^* \rangle$ and $T := \langle h_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in K^* \rangle$. By Lemma 3.1.3, T is abelian. Let W be the Weyl group of Φ. Clearly $T \leq N$. We claim that $N/T \cong W$. To see this we must first establish the *Chevalley relations*, a collection of relations in G that we will use here and in future sections. Note that these are not necessarily defining relations, but they are the only ones we will need.

We list the Chevalley relations here. Some we have already proved, and we will take the others for granted. See [S3, page 30] or [C1, Chapter 12]. Here, $\alpha, \beta \in \Phi$ and $\lambda, \mu \in K$, specifically $\lambda, \mu \in K^*$ when necessary.

(CR1): $x_\alpha(\lambda)x_\alpha(\mu) = x_\alpha(\lambda + \mu)$.

(CR2): If $\alpha \neq \pm\beta$, $[x_\alpha(\lambda), x_\beta(\mu)] = \prod_{\gamma \in (\alpha,\beta)} x_\gamma(c_\gamma \lambda^{p_\gamma} \mu^{q_\gamma})$ for some integers c_γ independent of λ, μ. Here, p_γ and q_γ are as in Section 1.4.

(CR3): $m_\alpha(\lambda)h_\beta(\mu)m_\alpha(\lambda)^{-1} = h_{s_\alpha\beta}(\mu)$.

(CR4): $m_\alpha(1)x_\beta(\lambda)m_\alpha(1)^{-1} = x_{s_\alpha(\beta)}(\pm\lambda)$ where the $\pm$ only depends on α and β.

(CR5): $h_\alpha(\lambda)x_\beta(\mu)h_\alpha(\lambda)^{-1} = x_\beta(\lambda^{\langle \beta, \alpha^\vee \rangle}\mu)$.

Since $h_\alpha(\lambda) = m_\alpha(\lambda)m_\alpha(1)^{-1}$, combining (CR4) and (CR5) we get:

(CR6): $m_\alpha(\lambda)x_\beta(\mu)m_\alpha(\lambda)^{-1} = x_{s_\alpha(\beta)}(\pm\lambda^{\langle s_\alpha(\beta), \alpha^\vee \rangle}\mu)$.

An immediate consequence of (CR3) is that $T \lhd N$. This is part (a) of Lemma 22 in [S3], and the next lemma covers parts (b) and (c):

Lemma 3.1.7. *There is an isomorphism $\phi : W \to N/T$.*

Proof. Construct a homomorphism $\phi : W \to N/T$ via $\phi(s_\alpha) = Tm_\alpha(1)$ (we use right cosets here for convenience). We must check that the relations in W are satisfied in N/T, so that ϕ will be well-defined. Note that $Tm_\alpha(1) = Th_\alpha(-1)m_\alpha(1) = Tm_\alpha(-1) = Tm_\alpha(1)^{-1}$, so for any α we have $s_\alpha^2 \mapsto Tm_\alpha(1)m_\alpha(1)^{-1} = T$.

Also, $s_\alpha s_\beta s_\alpha^{-1} s_{s_\alpha(\beta)}^{-1} \mapsto Tm_\alpha(1)m_\beta(1)m_\alpha(1)^{-1}m_{s_\alpha(\beta)}(1)^{-1}$. But this is just T since $m_\alpha(1)m_\beta(1)m_\alpha(1)^{-1} = m_{s_\alpha(\beta)}(c)$ by (CR4), where $c = \pm 1$. If $c = -1$ we must also again use the fact that $Tm_\alpha(-1) = Tm_\alpha(1)$. These relations define W, so ϕ is well-defined, and is clearly surjective.

Now suppose $w = s_{\alpha_1} \cdots s_{\alpha_k} \mapsto T$, so $m_{\alpha_1}(1) \cdots m_{\alpha_k}(1) =: t \in T$. Let $\alpha \in \Phi$. By (CR5) $tx_\alpha(1)t^{-1} = x_\alpha(\lambda)$ for some $\lambda \in K^*$ depending on α and the α_i. But by (CR4) $tx_\alpha(1)t^{-1} = x_{w\alpha}(\pm 1)$, so in fact $x_\alpha(\lambda) = x_{w\alpha}(\pm 1)$ for all α. Now, if $w \neq 1$, there exists $\alpha \in \Phi_+$ such that $w\alpha \in \Phi_-$. By Lemma 3.1.5 however, this implies that $x_\alpha(\lambda) = x_{w\alpha}(\pm 1)$ is impossible. Thus, $w = 1$, and ϕ is an isomorphism. □

In the following then, we will often identify W, the Weyl group of Φ, with N/T. Note that in constructing $\mathfrak{g}(K)$, the fact that K is a field is incidental. In fact K can be any commutative ring with 1. What's more, $\mathfrak{g}(\cdot)$ is functorial, i.e., if A and B are commutative rings with a ring homomorphism $\phi : A \to B$, we get an induced group homomorphism $\tilde{\phi} : \mathfrak{g}(A) \to \mathfrak{g}(B)$, simply via $x_\alpha(\lambda) \mapsto x_\alpha(\phi(\lambda))$. This is fine since the construction of $\mathfrak{g}(K)$ does not depend on any properties of K other than its being a commutative ring with 1 [S3, Section 6].

We close this section with an observation that will be important in Section 4.2.

Proposition 3.1.8. *Let A be a commutative topological ring with 1, and let B be a subring of A. If B is dense in A then $\mathfrak{g}(B)$ is dense in $\mathfrak{g}(A)$.*

Proof. Thinking of $\mathfrak{g}(B)$ and $\mathfrak{g}(A)$ as subgroups of $\mathrm{SL}_d(A)$, they are topological groups with topology induced by that of A. Thus it suffices to show that a generating set of

$\mathfrak{g}(A)$ is contained in the closure of $\mathfrak{g}(B)$. But clearly each $x_\alpha(\lambda)$ for $\lambda \in A$ is indeed in the closure of $\mathfrak{g}(B)$, and so $\mathfrak{g}(B)$ is dense in $\mathfrak{g}(A)$. □

3.2 Local and global fields

We are specifically interested in Chevalley groups over *local* and *global* fields. Such fields will also be important in Chapter 5. Good references for these topics include [L2, Chapters 23-25] and [O, Part 1].

Definition 3.2.1. A field K is called *local* if it is of one of the following two types:

1. K is *archimedean*: K is either $\mathbb{R}$ or $\mathbb{C}$.
2. K is *non-archimedean*: K is complete with respect to a discrete valuation and the residue field k is finite.

Recall that a *discrete valuation* is a group homomorphism $\nu : K^\times \to \mathbb{Z}$ such that $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$, with the *ad hoc* declaration that $\nu(0) = \infty$ so that ν is defined on K. For our purposes we assume ν is surjective. Set $R = \{x \in K \mid \nu(x) \geq 0\}$; this is clearly a subring of K, called the *valuation ring*. Note that $\nu(1) = \nu(1) + \nu(1)$ so $\nu(1) = 0$. Thus, the units in R are precisely $R^\times = \{x \in K \mid \nu(x) = 0\}$. Also, given $\pi \in R$ with $\nu(\pi) = 1$, it is clear that πR is a maximal ideal in R. Thus $k = R/\pi R$ is a field, called the *residue field*, which for our purposes is assumed to be finite.

Let $p = \operatorname{char} k$. There is a natural absolute value on K given by $|x| = p^{-\nu(x)}$. This is clearly positive definite and multiplicative, and also satisfies the *strong triangle inequality* $|x+y| \leq \max\{|x|, |y|\}$. We can recover all the data from the previous paragraph just using $|\cdot|$, namely $R = \{x \in K \mid |x| \leq 1\}$, $R^\times = \{x \in K \mid |x| = 1\}$, π has absolute value p^{-1}, and $k = R/\pi R$. Note that in the archimedean case when K is $\mathbb{R}$ or $\mathbb{C}$ we also have a natural absolute value, though it only satisfies the standard, weaker triangle inequality. Either way, every local field has a topology induced by the absolute value. Also, since $\nu(1) = 0$ we have $2\nu(-1) = 0$, and so $\nu(-1) = 0$. Thus, $\nu(x-y) = \nu(y-x)$ and the distance function $d(x,y) = |x-y|$ is symmetric. All the other axioms for a metric are immediate and so the topology on K is a metric topology. It thus makes sense to refer to K being *complete* with respect to ν.

One standard example of a non-archimedean valuation is the map $\nu_p : \mathbb{Q} \twoheadrightarrow \mathbb{Z}$ for prime p given by $\nu_p(p^e m/n) = e$, where m and n are integers not divisible by p. The completion of $\mathbb{Q}$ with respect to ν_p is the well-studied local field of *p-adic numbers*, $\mathbb{Q}_p$, with valuation ring $\mathbb{Z}_p$ and residue field $\mathbb{F}_p$; see [L2, Section 23.F6]. The other standard example of a local field is $K = k((t))$, the field of formal Laurent series over a finite field k. Here, $\nu(a_m t^m + \cdots + a_i t^i + \cdots) = m$, again with the *ad hoc* declaration that $\nu(0) = \infty$; see [L2, Section 24.F2]. It is clear that in this case k is also the residue field.

It will be important in Section 5.2 that a finite extension of a local field is local, with a natural relationship between the absolute values. The proof of the following fact is standard, see for instance [L2, Theorem 23.4].

Proposition 3.2.2. *Let K be a local field complete with respect to the absolute value $|\cdot|$. Let $L|K$ be a finite extension of degree n. Then $|\cdot|$ extends uniquely to an absolute value on L, with respect to which L is complete, and for $x \in L$ we have $|x| = |N_{L|K}(x)|^{1/n}$. Also, if $|\cdot|$ is (non-)archimedean on K then it is the same on L.*

Now let K be a non-archimedean local field with residue field k, and L a finite extension K. By the proposition L is also a non-archimedean local field, say with residue field ℓ. Since the absolute values on K and L agree, it is clear that we can think of ℓ as an extension of k. What's more, we always have $[\ell : k] \leq [L : K]$. In case $[\ell : k] = [L : K]$ we say that the extension $L|K$ is *unramified.*

We can now define global fields. Essentially a global field is a field all of whose completions are local [L2, Section 25.F2]. As a working definition, however, we can use the following characterization:

Definition 3.2.3. A *global field* is a field K of one of the following two types:

1. Any finite extension K of $\mathbb{Q}$.

2. Any finite extension of the field of rational functions $k(t)$ for finite field k.

The completions of $\mathbb{Q}$ are precisely $\mathbb{R}$, in the archimedean case, and $\mathbb{Q}_p$ for prime p in the non-archimedean case. For the field $\mathbb{F}_q(t)$ with valuation induced by the degree map on $\mathbb{F}_q[t]$, the completion is $\mathbb{F}_q((t))$. These are the standard examples, and will be used to motivate some general results in Section 4.2.

3.3 Chevalley groups and VRGD systems

We ultimately want Chevalley groups to act on buildings, and the key to this is RGD and VRGD systems.

We first show that Chevalley groups admit a canonical RGD system. Let $G = \mathfrak{g}(K)$ be a Chevalley group, with root system Φ and Weyl group W. For each root $\alpha \in \Phi$, let $U_\alpha = \langle x_\alpha(\lambda) \mid \lambda \in K \rangle$ be the standard root group.

Proposition 3.3.1. *The pair* $(G, (U_\alpha)_\alpha)$ *is an RGD system.*

Proof. First recall that although we think of Φ in the "Euclidean sense" when dealing with Chevalley groups, we can also treat it as the set of roots in the Coxeter complex $\Sigma(W, S)$; see Section 1.4.

Since each U_α is isomorphic to $(K, +)$, (RGD0) holds trivially.

Let $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$. Then $(\alpha, \beta) = \{\gamma \in \Phi \mid \text{there exist } p_\gamma, q_\gamma \in \mathbb{R}_+ \text{ such that } \gamma = p_\gamma\alpha + q_\gamma\beta\}$, and so (RGD1) follows from the Chevalley relation (CR2).

For each $\alpha \in \Phi$ and $u = x_\alpha(\lambda) \in U_\alpha^*$ set $m(u) := m_{-\alpha}(-\lambda^{-1})$ as defined in 3.1. Then by construction $m(u) \in U_{-\alpha}uU_{-\alpha}$, and by the Chevalley relation (CR6) we get that $m(u)U_\beta m(u)^{-1} = U_{s_\alpha\beta}$ for all $\beta \in \Phi$. This proves (even the stronger version of) (RGD2).

To see that (RGD3) holds, we think of G as a matrix group, as in Corollary 3.1.5. In this context U_{-s} is non-diagonal and lower triangular, whereas U_+ is upper triangular. Thus (RGD3) is an immediate consequence.

Lastly, (RGD4) holds trivially, since G is even generated by the U_α. □

In the proof of Proposition 3.3.1 we never needed the fact that $T = \langle h_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in K^*\rangle$ really equals $\bigcap_{\alpha\in\Phi} N_G(U_\alpha)$, but this usually is indeed the case. We will need this fact later, and we have been referring to both groups as T, so it would be wise to prove this now.

Proposition 3.3.2. *Let $G = \mathfrak{g}(K)$ be a Chevalley group with root groups U_α. Then*

$$\langle h_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in K^*\rangle = \bigcap_{\alpha\in\Phi} N_G(U_\alpha).$$

Proof. Denote $\bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ by T'. By (CR5), each $h_\alpha(\lambda)$ normalizes each U_α, so $T \le T'$. Now, the type of an RGD system only depends on Φ, so by Theorem 2.2.4 we know that $N/T' \cong W$. But we also know by Lemma 3.1.7 that $N/T \cong W$. Since W is finite and $T \le T'$ we conclude that $T = T'$. □

We now prove part (b) of the exercise on page 36 of [S3], along with a related result, which will be important later.

Lemma 3.3.3. *If $|K| > 3$ then $N = N_G(T)$. If $|K| > 5$ then additionally $T = C_G(T)$.*

Proof. Using the Bruhat decomposition, an arbitrary element of G looks like $g = unu'$ for $u, u' \in U_+$ and $n \in N$. In fact by [S3, Theorem 4′] we can choose u' such that $nu'n^{-1} \in U_-$. Thus $g = u(nu'n^{-1})n \in U_+U_-n$. Clearly $N \le N_G(T)$, so to show that $g \in N$ it suffices to show that if $u \in U_+$ and $v \in U_-$ with uv normalizing T, then $uv = 1$.

Suppose uv normalizes T, so $vTv^{-1} = u^{-1}Tu$. For each $t \in T$ choose $s_t \in T$ such that $vtv^{-1} = u^{-1}s_tu$, so $vtv^{-1}t^{-1} = u^{-1}s_tut^{-1}$. Since T normalizes U_- the left-hand side is in U_-, and the right-hand is clearly in B. By Corollary 3.1.5, $U_- \cap B = \{1\}$ and so $vtv^{-1}t^{-1} = 1$. If we instead choose s_t for each t such that $vs_tv^{-1} = u^{-1}tu$ then by a parallel argument $u^{-1}tut^{-1} = 1$. Since t was arbitrary, this tells us that u and v centralize T. It now suffices to show that $C_{U_+}(T) = C_{U_-}(T) = \{1\}$.

Suppose $1 \neq u \in C_{U_+}(T)$, say $u = \prod_{i=1}^{r} x_{\alpha_i}(\lambda_i)$ for some $\alpha_i \in \Phi_+$, $\lambda_i \in K$. Since $u \neq 1$ we can assume that the λ_i are non-zero. Now since u centralizes t, for any $\beta \in \Phi$, $\mu \in K^*$ we have $h_\beta(\mu)uh_\beta(\mu)^{-1} = u$. By (CR5) we know $h_\beta(\mu)x_{\alpha_i}(\lambda_i)h_\beta(\mu)^{-1} = x_{\alpha_i}(\lambda_i\mu^{\langle\alpha_i,\beta^\vee\rangle})$ for each i. Also by [S3, Lemma 17] this decomposition of u is unique, implying that for each i, $\mu^{\langle\alpha_i,\beta^\vee\rangle} = 1$ for all $\mu \in K^*$ and for all $\beta \in \Phi$. In particular for any μ we have $\mu^{\langle\alpha_1,\alpha_1^\vee\rangle} = \mu^2 = 1$. But since $|K| > 3$ this is impossible. We conclude that $C_{U_+}(T) = \{1\}$, and by a parallel argument $C_{U_-}(T) = \{1\}$, so indeed $N = N_G(T)$.

Now suppose $|K| > 5$. To show $T = C_G(T)$ it suffices to show $C_N(T) \leq T$. Let $n \in C_N(T)$ with $w = nT$. For any $\lambda \in K^*$ and $\alpha \in \Phi$, $nh_\alpha(\lambda)n^{-1} = h_{w\alpha}(\lambda)$ by (CR3), so in fact $h_\alpha(\lambda) = h_{w\alpha}(\lambda)$ for all α, λ. By (CR5) this implies in particular that $\lambda^{\langle\alpha,\alpha^\vee\rangle} = \lambda^{\langle\alpha,(w\alpha)^\vee\rangle}$ for all α, λ. Since w preserves the inner product, and since $\langle\alpha,\alpha^\vee\rangle = 2$, this tells us that $\lambda^2 = \lambda^{\langle w^{-1}\alpha,\alpha^\vee\rangle}$ for all α, λ.

Now, by the Cauchy-Schwarz inequality and the fact that Φ is reduced, it is easy to see that $|\langle w^{-1}\alpha,\alpha^\vee\rangle| \leq 2$ for all α, with $\langle w^{-1}\alpha,\alpha^\vee\rangle = 2$ if and only if $w^{-1}\alpha = \alpha$. Since $|K| > 5$, for any of $r = -2,-1,0,1$ we can find $\lambda \in K^*$ such that $\lambda^2 \neq \lambda^r$. We conclude that $\langle w^{-1}\alpha,\alpha^\vee\rangle = 2$ for all α, so $w^{-1}\alpha = \alpha$ for all α. Thus $w = 1$ and $n \in T$. □

We remark that we really need $|K| > 5$ to achieve the latter result. For example in $\mathrm{PSL}_2(\mathbb{F}_5)$ the subgroup N is abelian, so $C_N(T) = N$.

Now thanks to Theorem 2.2.4 and Theorem 2.1.8, we get that every Chevalley group acts strongly transitively on a canonical thick spherical building. Chevalley groups can also act canonically on affine buildings. The situation we consider is the Chevalley group $\mathfrak{g}(K)$ in the case when K is a field with discrete valuation, for example a local field.

Let $G = \mathfrak{g}(K)$ where K is a field with discrete valuation $\nu : K \twoheadrightarrow \mathbb{Z}$. For each root $\alpha \in \Phi$, let $U_\alpha = \langle x_\alpha(\lambda) \mid \lambda \in K \rangle$ be the standard root group, and let $\phi_\alpha : U_\alpha \to \mathbb{Z}$ be $\phi_\alpha(x_\alpha(\lambda)) := \nu(\lambda)$ for $\lambda \in K$ (with $\nu(0)$ understood to be ∞).

Proposition 3.3.4. *The triple $(G, (U_\alpha)_\alpha, (\phi_\alpha)_\alpha)$ is a VRGD system.*

Proof. First, $(G, (U_\alpha)_\alpha)$ is an RGD system by the preceding paragraphs, with

$$m(x_\alpha(\lambda)) := m_{-\alpha}(-\lambda^{-1}) = x_{-\alpha}(-\lambda^{-1})x_\alpha(\lambda)x_{-\alpha}(-\lambda^{-1}).$$

Since G is generated by the root groups, we are in a position to check the VRGD axioms. By our assumption that ν is surjective, (VRGD0) immediately holds.

For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, define $U_{\alpha,k} := \langle u \in U_\alpha | \phi_\alpha(u) \geq k \rangle$. To show (VRGD1) holds we need to check that this is a subgroup of U_α. Let $x_\alpha(\lambda)$ and $x_\alpha(\mu)$ be arbitrary elements of $U_{\alpha,k}$, so $\nu(\lambda) \geq k$ and $\nu(\mu) \geq k$. Then

$$\begin{aligned}\phi_\alpha(x_\alpha(\lambda)x_\alpha(\mu)^{-1}) &= \phi_\alpha(x_\alpha(\lambda - \mu)) = \nu(\lambda - \mu) \\ &\geq \max(\nu(\lambda), \nu(-\mu)) = \max(\nu(\lambda), \nu(\mu)) = k\end{aligned}$$

and so indeed $U_{\alpha,k} \leq U_\alpha$. (To reiterate, $\phi_\alpha(1)$ is considered to be ∞, so $1 \in U_{\alpha,k}$.)

We now verify (VRGD2). For each $\gamma \in (\alpha, \beta)$ let p_γ, q_γ be such that $\gamma = p_\gamma\alpha + q_\gamma\beta$. The Chevalley relation (CR2) tells us that

$$[x_\alpha(\lambda), x_\beta(\mu)] = \prod_{\gamma \in (\alpha,\beta)} x_\gamma(c_\gamma \lambda^{p_\gamma} \mu^{q_\gamma})$$

for some $c_\gamma \in \mathbb{Z}$. Now suppose $\nu(\lambda) \geq k$ and $\nu(\mu) \geq \ell$. Then

$$\nu(c_\gamma \lambda^{p_\gamma} \mu^{q_\gamma}) = \nu(c_\gamma) + p_\gamma \nu(\lambda) + q_\gamma \nu(\mu) \geq p_\gamma k + q_\gamma \ell$$

since integers have positive valuation. So (VRGD2) holds.

Next we check (VRGD3). Let $u = x_\alpha(\lambda)$ and $x = x_\beta(\mu)$. We claim that the integer given by $\phi_{s_\alpha(\beta)}(m(u)^{-1}xm(u)) - \phi_\beta(x)$ is independent of μ. We know that this quantity equals

$$\begin{aligned}\phi_{s_\alpha(\beta)}(m_{-\alpha}(-\lambda^{-1})^{-1}x_\beta(\mu)m_{-\alpha}(-\lambda^{-1})) - \nu(\mu)\\ = \phi_{s_\alpha(\beta)}(x_{s_\alpha(\beta)}(\pm\lambda^{-\langle s_\alpha(\beta), -\alpha^\vee\rangle}\mu)) - \nu(\mu)\end{aligned}$$

by Chevalley relation (CR6), since $s_{-\alpha} = s_\alpha$. This equals

$$\nu(\pm\lambda^{\langle s_\alpha(\beta),\alpha^\vee\rangle}\mu) - \nu(\mu) = \nu(\pm\lambda^{\langle s_\alpha(\beta),\alpha^\vee\rangle})$$

since $\nu(ab) = \nu(a) + \nu(b)$. This quantity is indeed independent of μ and so (VRGD3) follows.

Note that when $\alpha = \beta$ we have $s_\alpha(\beta) = s_\alpha(\alpha) = -\alpha$. Thus

$$\nu(\pm\lambda^{\langle s_\alpha(\beta),\alpha^\vee\rangle}) = \nu(\pm\lambda^{-\langle\alpha,\alpha^\vee\rangle}) = \nu(\pm\lambda^{-2}) = -2\nu(\lambda) = -2\phi_\alpha(u)$$

and (VRGD4) holds. □

By Theorem 2.3.4, we get an affine BN-pair (B_a, N) for G. If we consider G as a subgroup of $\mathrm{SL}_d(K)$, there is a lot we can say about the structure of the BN-data. Let R be the discrete valuation ring of K, with maximal ideal πR. Using the notation of Section 2.3, it is clear that since T_a normalizes each $U_{\alpha,k}$ it also normalizes each U_α. By Proposition 3.3.2 we conclude that $T_a \le T$, and so by Lemma 3.1.3, $T_a \le T_d(K)$, the subgroup of diagonal matrices.

By Corollaries 3.1.5 and 3.1.6, if $\lambda \in R$ and $\alpha \in \Phi_+$ then $x_\alpha(\lambda)$ is an upper triangular matrix with 1's on the diagonal and elements of R in the non-diagonal entries. Similarly if $\lambda \in \pi R$ and $\alpha \in \Phi_-$ then $x_\alpha(\lambda)$ is a lower triangular matrix with 1's on the diagonal and elements of πR in the non-diagonal entries.

Set

$$\widetilde{B} := \mathrm{SL}_d\begin{pmatrix} R & R & \cdots & R \\ \pi R & R & \cdots & R \\ \pi R & \pi R & \cdots & R \end{pmatrix}.$$

By the above remarks, $U \leq \widetilde{B}$. If we can show that $T_a \leq \widetilde{B}$ then we will know $B_a \leq \widetilde{B}$. We will show in the next lemma that $T_a \leq T_d(R)$, so $T_a \leq \widetilde{B}$. We also show that a very strong sort of reverse inclusion holds, namely $G \cap T_d(R) \leq T_a$. We will only need the weaker result $T \cap T_d(R) \leq T_a$, but we prove the stronger result for completeness.

Lemma 3.3.5. *$T_a = T \cap T_d(R) = G \cap T_d(R)$.*

Proof. First we show $T_a \leq T_d(R)$. Let $t = h_{\alpha_1}(\lambda_1) \cdots h_{\alpha_r}(\lambda_r) \in T_a$ for distinct α_i, and set $k_i = \nu(\lambda_i)$ for each i. By the construction of $\pi : N \twoheadrightarrow W_a$ in Section 2.3, and since $h_\alpha(\lambda) = m_\alpha(\lambda) m_\alpha(1)^{-1}$, we see that for any $v \in E$,

$$v = \pi(t)(v) = \sigma_{\alpha_1,k_1}\sigma_{\alpha_1}^{-1} \cdots \sigma_{\alpha_r,k_r}\sigma_{\alpha_r}^{-1} v = v + k_1\alpha_1^\vee + \cdots + k_r\alpha_r^\vee.$$

Thus $k_1\alpha_1^\vee + \cdots + k_r\alpha_r^\vee = 0$.

By [S3, Lemma 28(b)], T is in fact generated by the $h_\alpha(\lambda)$ for *simple* α. Thus, without loss of generality, each α_i is a simple root. By Lemma 1.4.6 the simple roots are linearly independent, so for each i we have $2k_i/\langle\alpha_i, \alpha_i\rangle = 0$, i.e., each k_i is zero. Thus each λ_i is a unit in R, and so t is diagonal with entries in R.

Now let $t \in T \cap T_d(R)$, say $t = h_{\beta_1}(\mu_1) \cdots h_{\beta_m}(\mu_m)$ for (distinct) simple roots β_i. We know t acts on V_γ^K via multiplication by $\mu_1^{\langle\gamma,\beta_1^\vee\rangle} \cdots \mu_m^{\langle\gamma,\beta_m^\vee\rangle}$, which by hypothesis is in R. Since t is nonsingular, in fact $\mu_1^{\langle\gamma,\beta_1^\vee\rangle} \cdots \mu_m^{\langle\gamma,\beta_m^\vee\rangle} \in R^\times$, i.e., has valuation 0. Thus,

$$0 = \langle\gamma, \beta_1^\vee\rangle\nu(\mu_1) + \cdots + \langle\gamma, \beta_m^\vee\rangle\nu(\mu_m) = \langle\gamma, \nu(\mu_1)\beta_1^\vee + \cdots + \nu(\mu_m)\beta_m^\vee\rangle$$

for all γ, implying that $\nu(\mu_1)\beta_1^\vee + \cdots + \nu(\mu_m)\beta_m^\vee = 0$. As before, simple roots are linearly independent and so all the μ_i have valuation 0. We conclude that $t \in \ker \pi = T_a$.

This shows that $T_a = T \cap T_d(R) \leq G \cap T_d(R)$. Also, since $T_d(R)$ is abelian and $T = C_G(T)$ by Lemma 3.3.3, we see that indeed $T_a = G \cap T_d(R)$. $\square$

As indicated, this also shows that $B_a \leq \widetilde{B}$. The next lemma establishes exactly how strict this inequality is.

Lemma 3.3.6. $B_a = \widetilde{B} \cap G$.

Proof. First let $n \in \widetilde{B} \cap N$, with $w = nT$. By [S3, Lemma 19(b)], $nV_\gamma^K = V_{w\gamma}^K$ for all γ. Thus n is "block monomial," and so has determinant equal to plus or minus the product of the determinants of each block. If $w \neq 1$ then since $n \in \widetilde{B}$ there exists a block strictly below the diagonal all of whose entries are in πR, and thus whose determinant is in πR. Since $\det n = 1$ this implies that another block must have determinant in $\pi^{-1}R$, which is impossible. So in fact $n \in T$. In particular n is diagonal in $\widetilde{B}$, and so is in $T_d(R)$. By Lemma 3.3.5 then, $n \in T_a$. We conclude that $\widetilde{B} \cap N \leq T_a$.

Now let $g \in \widetilde{B} \cap G$. By the affine Bruhat decomposition

$$G = \coprod_{w \in W_a} B_a w B_a,$$

there exist $b, b' \in B_a$, $n \in N$ such that $g = bnb'$. Since $B_a \leq \widetilde{B}$, in fact $n \in \widetilde{B} \cap N \leq T_a \leq B_a$, and so $g \in B_a$. □

Corollary 3.3.7. *B_a is open in G.*

Proof. It is clear that $\widetilde{B}$ is open in $\mathrm{SL}_d(K)$, and so by Lemma 3.3.6 we conclude B_a is open in G. □

As we have seen, a Chevalley group G over a field with discrete valuation acts canonically on an affine building Δ, and the action is strongly transitive with respect to some apartment system $\mathcal{A}$. Corollary 3.3.7, coupled with Lemma 2.4.2, thus ensures that any *dense* subgroup of G will act Weyl transitively on Δ.

Now, unlike the spherical case, $\mathcal{A}$ may not be complete. Since we will soon be looking for subgroups of G that act Weyl transitively but not weakly transitively, we want to use Lemma 2.4.3, and so need to establish whether G really acts transitively on the complete apartment system $\overline{\mathcal{A}}$.

As the terminology indicates, this will have something to do with whether the field K is complete.

Proposition 3.3.8. *Let $G = \mathfrak{g}(\Phi, K)$ be a Chevalley group over a field K with discrete valuation ν. Let R be the discrete valuation ring in K, with maximal ideal πR. Let Δ be the affine building on which G acts canonically. If K is complete with respect to ν then G acts transitively on $\overline{\mathcal{A}}$.*

The proof here is informed by the $G = \mathrm{SL}_d(K)$ case, proved in [AB2, Proposition 11.105(3)]. A more general result is proved in [W, Theorem 17.9], for any VRGD system. In the present situation we can achieve the result using only the explicit structure of Chevalley groups, with a small extra assumption, namely that the weight spaces V_γ^K are one-dimensional. This will hold for instance if V is the adjoint representation [C2, Proposition 7.22]. In the general situation we simply cite [W] for the proof.

Proof of Proposition 3.3.8. Let $k_i := R/\pi^i R$ for each $i \geq 1$ and define maps $q_i : \mathfrak{g}(R) \twoheadrightarrow \mathfrak{g}(k_i)$, induced by $R \twoheadrightarrow k_i$. Let $T(k_i)$ be the usual subgroup of $\mathfrak{g}(k_i)$, generated by the elements $h_\alpha(\lambda)$ for $\lambda \in (k_i)^\times$. Since $R^\times$ surjects onto $(k_i)^\times$ we know that $q_i(T_a) = T(k_i)$.

Let Σ_0 be the fundamental apartment of Δ and let Σ be any apartment. We want to show that some element of G maps Σ_0 to Σ. Since G acts transitively on $\mathcal{C}(\Delta)$ we can assume $\Sigma_0 \cap \Sigma$ contains the fundamental chamber C. Let $\phi : \Sigma_0 \to \Sigma$ be the canonical (type-preserving) isomorphism that fixes $\Sigma_0 \cap \Sigma$. Now choose a sequence of bounded subsets $C \subseteq F_1 \subseteq F_2 \subseteq \cdots$ of Σ_0 such that

$$\bigcup_{i=1}^{\infty} F_i = \Sigma_0$$

and for every $\alpha \in \Phi$, $h_\alpha(\pi^i)C \subseteq F_i$. Since Φ is finite, such F_i exist.

Let B_a be the usual subgroup of $\mathfrak{g}(K)$; note that $B_a \leq \mathfrak{g}(R)$ by Lemma 3.3.6. Since B_a is the stabilizer of C, we see that for each i the pointwise fixer $\mathrm{Fix}_G(F_i)$ of F_i is contained in

$$H_i := \bigcap_{\alpha\in\Phi} h_\alpha(\pi^i)B_a h_\alpha(\pi^i)^{-1} \cap B_a.$$

We claim that $q_i(H_i) \leq T(k_i)$ for each i. Let $h \in H_i$. Let $\mathcal{B} = \{v_1, \ldots, v_d\}$ be the standard basis of V^K and for each j let γ_j be the weight such that v_j lies in the weight space $V^K_{\gamma_j}$. Suppose r and s are such that $\gamma_r \neq \gamma_s$. Choose a root α such that $\langle\gamma_r, \alpha^\vee\rangle > \langle\gamma_s, \alpha^\vee\rangle$. Since $h_\alpha(\pi^i)$ is diagonal with (j, j) entry $\pi^{i\langle\gamma_j,\alpha^\vee\rangle}$ for each j, and since $h \in h_\alpha(\pi^i)\widetilde{B}h_\alpha(\pi^i)^{-1}$, we see that the (r, s) entry of h is in $\pi^{i\langle\gamma_r-\gamma_s,\alpha^\vee\rangle}R \leq \pi^i R$. Since we can choose such an α for each r, s, we get that if $h \in H_i$ then the (r, s) entry of h is in $\pi^i R$ for all r, s such that $V^K_{\gamma_r} \neq V^K_{\gamma_s}$. We conclude that the subgroup $q_i(H_i)$ in $\mathfrak{g}(k_i)$ stabilizes each weight space. But we are assuming the weight spaces are one-dimensional, so $q_i(H_i)$ is even diagonal, i.e., $q_i(g) \in T(k_i)$.

Now, by Lemma 1.3.4 we know $G\Sigma_0$ is an apartment system. By [AB2, Theorem 11.43] then, each $\phi(F_i)$ (by virtue of being bounded) is contained in some $g'\Sigma_0$ for $g' \in G$. In fact, since ϕ fixes C, we can choose for each i some $b_i \in B_a$ such that $\phi(F_i) = b_i F_i$. Note that $b_{i+1}(F_i)$ also equals $\phi(F_i)$, and so $b_{i+1}^{-1}b_i$ fixes F_i pointwise. Thus, for each i, $q_i(b_{i+1})^{-1}q_i(b_i) \in T(k_i)$.

Since T_a fixes Σ_0 pointwise, and $q_i(T_a) = T(k_i)$, we can replace b_{i+1} with $b_{i+1}t$ for a suitable $t \in T_a$ to get $q_i(b_{i+1}) = q_i(b_i)$. Thus, we can inductively construct the sequence $b_1, b_2, \ldots$ to satisfy $q_i(b_{i+1}) = q_i(b_i)$ for all i. Of course, by the nature of quotient maps, $q_j(b_{i+1}) = q_j(b_i)$ for all $j \leq i$. The sequence (b_i) is clearly Cauchy when thought of in $M_d(K)$ with topology induced by the valuation on K. Since K is complete, (b_i) converges to some b, and clearly $b\Sigma_0 = \Sigma$. We now claim that $b \in G$. As seen in Section 3.4, $G\hat{T}(K) = \hat{\mathfrak{g}}(K)$, a linear algebraic group. In particular $\hat{\mathfrak{g}}(K)$ is defined by polynomial equations and is closed. Thus at least $b \in G\hat{T}(K)$. But $\hat{T}(K)$ is diagonal, and so fixes Σ_0, and without loss of generality $b \in G$. We conclude that $G\Sigma_0 = \overline{\mathcal{A}}$. □

Since the action of G is strongly transitive with respect to *some* apartment system, it is weakly transitive, and so Proposition 3.3.8 shows that the action really is strongly transitive with respect to $\overline{\mathcal{A}}$.

3.4 Linear algebraic groups

There is a second definition of Chevalley group that is natural in light of the embedding $\mathfrak{g}(K) \leq \mathrm{SL}_d(K)$. This second definition ensures that $\mathfrak{g}(K)$ is a linear algebraic group (LAG) for any (infinite) K, which is not true in general under the first definition. Note that we have already made use of this notion in the proof of Proposition 3.3.8, and so in particular will justify this result. We will denote a Chevalley group in the second, linear algebraic sense by $\hat{\mathfrak{g}}(K)$, and will refer to it not as a Chevalley group, but as the group of K-rational points, to avoid ambiguity. In the following K is an infinite field and $\overline{K}$ is a fixed algebraically closed field containing K.

Definition 3.4.1. Let $\mathfrak{g}(K)$ be a Chevalley group for K infinite. The *group of K-rational points* $\hat{\mathfrak{g}}(K)$ is defined to be $\hat{\mathfrak{g}}(K) := \mathfrak{g}(\overline{K}) \cap \mathrm{SL}_d(K)$, where $\mathfrak{g}(K) \leq \mathrm{SL}_d(K)$ in the canonical way.

It is a fact that $\mathfrak{g}(\overline{K})$ is a semisimple linear algebraic group [S3, Theorem 6(a)]. We will not elaborate much here on the definitions and properties of semisimple LAGs; the books by Borel [B2] and Humphreys [H1] provide a comprehensive overview, and we will reference these as needed. In particular, Theorem 13.18 of Borel collects many important results.

Clearly $\mathfrak{g}(K) \leq \hat{\mathfrak{g}}(K)$, and as we will eventually see the groups are in some sense not too different. Let $B(\overline{K}) = U_+(\overline{K})T(\overline{K})$ and $N(\overline{K})$ (resp. $B(K) = U_+(K)T(K)$ and $N(K)$) denote the usual subgroups of $\mathfrak{g}(\overline{K})$ (resp. $\mathfrak{g}(K)$) and set $\hat{T}(K) :=$

$T(\overline{K}) \cap \mathrm{SL}_d(K)$. In case there is no ambiguity we will write T and $\hat{T}$ for $T(K)$ and $\hat{T}(K)$. Note that the Weyl group W does not depend on the field, so $W = N(\overline{K})/T(\overline{K}) = N(K)/T(K)$.

A naive guess is that $T(K) = \hat{T}(K)$, but this is in general false. Let Λ be the weight lattice of the representation V. Let $X(\Lambda)$ be the K-character group of Λ, i.e., the group of homomorphisms from $(\Lambda, +)$ to K^*. If the field needs to be specified we will use the notation $X_K(\Lambda)$. For each $\chi \in X(\Lambda)$ define $h(\chi)$ to be the automorphism of V^K given by: if v is in the weight space V_γ^K then $h(\chi)v := \chi(\gamma)v$. Note that if we let $\chi_{\alpha,\lambda}$ be the character given by $\chi_{\alpha,\lambda}(\gamma) = \lambda^{\langle \gamma, \alpha^\vee \rangle}$ then $h(\chi_{\alpha,\lambda})$ can be identified with $h_\alpha(\lambda)$ in $\mathfrak{g}(K)$ by Lemma 3.1.3. See [C1, Section 7.1] for more details about $h(\chi)$.

Proposition 3.4.2. $\hat{T} = \langle h(\chi) \mid \chi \in X(\Lambda) \rangle$.

Proof. By the above argument, $T(\overline{K}) \leq \langle h(\chi) \mid \chi \in X_{\overline{K}}(\Lambda) \rangle$. Let $\chi \in X_{\overline{K}}(\Lambda)$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ denote the simple roots. By Lemma 1.4.6, Π is a basis of E, and since $\Lambda_r \leq \Lambda$ by Remark 3.1.2, we can choose a set of weights $\{\gamma_1, \ldots, \gamma_\ell\}$ that generate Λ. Let $x_1, \ldots, x_\ell$ be indeterminants and consider the system of equations

$$\begin{aligned}
\chi(\gamma_1) &= x_1^{\langle \gamma_1, \alpha_1^\vee \rangle} \cdots x_\ell^{\langle \gamma_1, \alpha_\ell^\vee \rangle} \\
\chi(\gamma_2) &= x_1^{\langle \gamma_2, \alpha_1^\vee \rangle} \cdots x_\ell^{\langle \gamma_2, \alpha_\ell^\vee \rangle} \\
&\vdots \\
\chi(\gamma_\ell) &= x_1^{\langle \gamma_\ell, \alpha_1^\vee \rangle} \cdots x_\ell^{\langle \gamma_\ell, \alpha_\ell^\vee \rangle}.
\end{aligned}$$

For each i, j set $a_{ij} = \langle \gamma_i, \alpha_j^\vee \rangle$, and let A denote the matrix $A = (a_{ij})_{1 \leq i,j \leq \ell}$. This is invertible since $\{\gamma_1, \ldots, \gamma_\ell\}$ and $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ are both bases of E, so we can let b_{ij} denote the (i, j) entry of A^{-1}. Since the a_{ij} are integers the b_{ij} are rational, and since $\overline{K}$ is algebraically closed it makes sense to use the b_{ij} as exponents.

For each $1 \leq j \leq \ell$ set $\lambda_j = (\chi(\gamma_1))^{b_{j1}} \cdots (\chi(\gamma_\ell))^{b_{j\ell}}$. It is easy to verify that $(\lambda_1, \ldots, \lambda_\ell)$ is a solution to the above system of equations. In particular for each

$1 \le i \le \ell$, $\chi(\gamma_i) = \chi_{\alpha_1,\lambda_1}(\gamma_i)\cdots\chi_{\alpha_\ell,\lambda_\ell}(\gamma_i)$, and since the γ_i generate Λ we conclude that $\chi = \chi_{\alpha_1,\lambda_1}\cdots\chi_{\alpha_\ell,\lambda_\ell}$. Since $h(\chi_1)h(\chi_2) = h(\chi_1\chi_2)$ for any χ_1, χ_2, we get that $T(\overline{K}) = \langle h(\chi) \mid \chi \in X_{\overline{K}}(\Lambda)\rangle$.

Now we intersect with $\mathrm{SL}_d(K)$. On the left we get $\hat{T}$. It remains to show that $\langle h(\chi) \mid \chi \in X_{\overline{K}}(\Lambda)\rangle \cap \mathrm{SL}_d(K) \le \langle h(\chi) \mid \chi \in X(\Lambda)\rangle$ (the reverse inclusion is clear). Since $h(\chi_1)h(\chi_2) = h(\chi_1\chi_2)$, if we let χ be an arbitrary character in $X_{\overline{K}}(\Lambda)$ then $h(\chi)$ is an arbitrary element of $\langle h(\chi) \mid \chi \in X_{\overline{K}}(\Lambda)\rangle$. If in addition $h(\chi) \in \mathrm{SL}_d(K)$, then $\chi(\gamma) \in K$ for all weights γ and in fact $\chi \in X(\Lambda)$, proving the proposition. □

In particular $T \le \hat{T}$. It is worth noting that if $\Lambda = \Lambda_w$, then $T = \hat{T}$. This is a consequence of [S3, Lemma 27] and the proof of Proposition 3.4.2. Namely, in the proof we can choose our generators $\gamma_1, \dots, \gamma_\ell$ to be the *fundamental weights*, i.e., $\langle \gamma_i, \alpha_j^\vee\rangle = \delta_{ij}$, the Kronecker delta function, and so we can take $\lambda_i = \chi(\gamma_i)$ for each i. Clearly this no longer depends on K being algebraically closed, and so in this case $T(K) = \langle h(\chi) \mid \chi \in X_K(\Lambda)\rangle = \hat{T}(K)$.

We return to Λ being arbitrary. Our next goal is to relate $\hat{\mathfrak{g}}(K)$ and $\mathfrak{g}(K)$ explicitly.

Proposition 3.4.3. $\hat{\mathfrak{g}}(K) = \mathfrak{g}(K)\hat{T}(K)$

Proof. First note that $\mathfrak{g}(K)\hat{T}(K)$ is a group; indeed, $\hat{T}(K)$ normalizes each $U_\alpha(\overline{K})$ and has entries in K, so normalizes each $U_\alpha(K)$. By construction $\hat{\mathfrak{g}}(K) \ge \mathfrak{g}(K)\hat{T}(K)$. Let $x \in \hat{\mathfrak{g}}(K)$. In particular $x \in \mathfrak{g}(\overline{K})$. Using the Bruhat decomposition we can find $w \in W$ such that $x \in B(\overline{K})wU_+(\overline{K})$, and we can choose a representative n of w in $N(K)$. It suffices now to show that $xn^{-1} \in \mathfrak{g}(K)\hat{T}(K)$. Choose $u, u' \in U_+(\overline{K})$ and $t \in T(\overline{K})$ such that $x = utnu'$. By [S3, Theorem 4′(a)] we can choose u' such that $nu'n^{-1} =: v \in U_-(\overline{K})$. Then $xn^{-1} = utv$. Now, $x, n \in \mathrm{SL}_d(K)$, so $utv \in \mathrm{SL}_d(K)$. By the proof of [S3, Theorem 7(b)], in fact $u, t, v \in \mathrm{SL}_d(K)$. Thus by construction $t \in \hat{T}(K)$, and again by the proof of [S3, Theorem 7(b)] in fact $u \in U_+(K)$ and $v \in U_-(K)$. Thus $utv \in \mathfrak{g}(K)\hat{T}(K)$ and the result follows. □

Note that if $\Lambda = \Lambda_w$ then we even have $\mathfrak{g}(K) = \hat{\mathfrak{g}}(K)$. We now slightly modify some Chevalley relations, namely (CR3) and (CR5), since we will need them in this context later. For each weight γ define $\hat{\gamma} : \hat{T} \to K^*$ via $\hat{\gamma}(h(\chi)) := \chi(\gamma)$. This is clearly a character; if we fix a basis $v_1, \dots, v_d$ with weights γ_i such that $v_i \in V_{\gamma_i}^K$ for each i then $\hat{\gamma_i}$ essentially just picks out the (i, i) entry.

We replace (CR3) by the more general

(CR3′): $m_\alpha(\lambda)h(\chi)m_\alpha(\lambda)^{-1} = h(\chi \circ s_\alpha)$

and generalize (CR5) to

(CR5′): $h(\chi)x_\beta(\mu)h(\chi)^{-1} = x_\beta(\chi(\beta)\mu)$.

These relations obviously hold in $\mathfrak{g}(\overline{K})$ since $\hat{T}(\overline{K})$ is already generated by the $h_\alpha(\lambda)$, and thus the relations must also hold in the subgroup $\hat{\mathfrak{g}}(K)$. Also see [S3, page 60].

We can also show that the Weyl group doesn't change. Since $\hat{T}$ normalizes N we can define $\hat{N} := N\hat{T}$. It is clear by Lemma 3.3.3 that $N \cap T_d(K) = T$, so $N \cap \hat{T} = T$ and we get that $\hat{N}/\hat{T} = N\hat{T}/\hat{T} \cong N/T \cong W$. Thus, the Weyl group of a Chevalley group is the same as the Weyl group $\hat{N}/\hat{T}$ of the group of K-rational points. If we are in the affine situation, with R the valuation ring of K, we also define $\hat{T}_a$ to be $\hat{T} \cap \mathrm{SL}_d(R)$.

It is easy to see that $(\hat{\mathfrak{g}}(K), (U_\alpha)_{\alpha \in \Phi})$ still satisfies the RGD axioms, with "T" now $\hat{T}$. Since our definition of VRGD system only considered groups that are generated by their root groups, it is not entirely accurate to say that $\hat{\mathfrak{g}}(K)$ admits a VRGD system, though as the next theorem implies, we really only care that $\mathfrak{g}(K)$ does.

Theorem 3.4.4. *Let $G = \mathfrak{g}(K)$ be a Chevalley group and $\hat{G} = \hat{\mathfrak{g}}(K) = G\hat{T}$ the group of K-rational points. Let B be the standard spherical Tits subgroup of G and, if appropriate, let B_a be the standard affine Tits subgroup. Let $\Delta = (G, B)$ and $\Delta_a = (G, B_a)$. The action of G on these buildings extends to an action of $\hat{G}$, inducing BN-pairs $(B\hat{T}, \hat{N})$ and $(B_a\hat{T}_a, \hat{N})$.*

Proof. First we consider the spherical case, and the affine case will follow by a parallel argument. An arbitrary element of $\hat{\mathfrak{g}}(K)$ looks like gt for $g \in G$, $t \in \hat{T}$. If C is the fundamental chamber, an arbitrary chamber looks like hC for $h \in G$. Then gt acts naturally via $gt(hC) = gtht^{-1}C$, which makes sense since $tht^{-1} \in G$. This obviously extends the action of G, with $\hat{T}$ fixing the fundamental apartment chamber-wise. Thus we get the expected BN-pair $(B\hat{T}, \hat{N})$. The affine results follow similarly. □

The upshot of all this is that as far as the buildings are concerned, it doesn't matter if we talk about $\mathfrak{g}(K)$ or $\hat{\mathfrak{g}}(K)$.

Chapter 4

Chevalley groups and buildings

4.1 Torsion properties of Weyl group representatives

Having constructed canonical actions of Chevalley groups on buildings, we turn our attention to transitivity properties. We know that in order for the action of a subgroup to be weakly transitive, some conjugate of the subgroup must represent every coset in $N/T = W$. In this section we produce a family of cosets that have a very restrictive torsion property. In the next section we will exploit this to produce examples of Weyl transitive group actions that are not strongly transitive with respect to any apartment system.

Let K be a field complete with respect to a discrete valuation, and let $G = \mathfrak{g}(\Phi, K)$. Let N, T, and T_a be the subgroups described in Section 3.3, so $T_a \leq T$, and let Δ_a be the associated affine building. The spherical and affine Weyl groups are, respectively, $W = N/T$ and $W_a = N/T_a$. We claim that there is a coset in N/T, all of whose representatives in N have the same finite order. Since $T_a \leq T$ this will imply the same result concerning cosets in N/T_a, which will be a key step in showing that certain subgroups of G do not act weakly transitively on Δ_a.

Let

$$N \geq N_0 := \langle m_\alpha(1) \mid \alpha \in \Phi \rangle$$

and let

$$T_0 := \langle h_\alpha(-1) \mid \alpha \in \Phi \rangle.$$

Since $m_\alpha(-1) = m_\alpha(1)^{-1}$, we know $h_\alpha(-1) = m_\alpha(1)^{-2}$, so $T_0 \leq N_0$. Also, by (CR3), $m_\alpha(1)h_\beta(-1)m_\alpha(1)^{-1} = h_{s_\alpha(\beta)}(-1)$, so $T_0 \lhd N_0$.

Lemma 4.1.1. *$W \cong N_0/T_0$.*

Proof. Construct the homomorphism $\phi_0 : W \to N_0/T_0$ via $\phi_0(s_\alpha) = T_0 m_\alpha(1)$. This coincides precisely with the map ϕ from Lemma 3.1.7. The injectivity and surjectivity follow by the exact arguments used in the proof of that lemma; we need only check well definedness. This proof also follows almost identically, but we state it for completeness.

Note that $T_0 m_\alpha(1) = T_0 h_\alpha(-1)m_\alpha(1) = T_0 m_\alpha(-1) = T_0 m_\alpha(1)^{-1}$, so for any α we have $s_\alpha^2 \mapsto T_0 m_\alpha(1)m_\alpha(1)^{-1} = T_0$. Also,

$$s_\alpha s_\beta s_\alpha^{-1} s_{s_\alpha(\beta)}^{-1} \mapsto T_0 m_\alpha(1)m_\beta(1)m_\alpha(1)^{-1}m_{s_\alpha(\beta)}(1)^{-1}.$$

But this is just T_0 since $m_\alpha(1)m_\beta(1)m_\alpha(1)^{-1} = m_{s_\alpha(\beta)}(c)$ by (CR4), where $c = \pm 1$. If $c = -1$ we must also again use the fact that $T_0 m_\alpha(-1) = T_0 m_\alpha(1)$.

These relations define W, so ϕ_0 is well-defined, and is an isomorphism by the same proof as in Lemma 3.1.7. □

As before, we identify W with N/T and N_0/T_0. Let E be the Euclidean space spanned by Φ, on which W acts by orthogonal transformations. The following theorem establishes a criterion whereby the representatives of $w \in W$ in N will all have finite order. In fact, the proof establishes that in this case the representatives all have *the same* finite order.

Theorem 4.1.2. *Let $W = N/T$ be the spherical Weyl group corresponding to the Chevalley group $\mathfrak{g}(K)$. Let $w \in W$ have order m. Then the following are equivalent:*

(i) *As an orthogonal transformation of E, w does not have eigenvalue 1.*

(ii) *For any field K, every representative of w in N has the same order in N, namely m or $2m$.*

(iii) *For any field K, every representative of w in N has finite order.*

Proof of the forward implication. For any $v \in E$ we have

$$w(v + w(v) + \cdots + w^{m-1}(v)) = v + w(v) + \cdots + w^{m-1}(v),$$

so the hypothesis forces $1 + w + \cdots + w^{m-1}$ to be zero. By Lemma 4.1.1 there exists a representative $n_0 \in N_0$ of w. Since $w^m = 1$, $n_0^m \in T_0$. Now, $h_\alpha(-1)^2 = 1$ for all α and T_0 is abelian, so T_0 is 2-torsion. Since m divides the order of n_0, we know that n_0 has order m or $2m$.

Now let h be any element of T. Say

$$n_0 = m_{\alpha_1}(\epsilon_1) \cdots m_{\alpha_k}(\epsilon_k),$$

where each ϵ_i is either 1 or -1, and

$$h = h_{\beta_1}(\lambda_1) \cdots h_{\beta_\ell}(\lambda_\ell).$$

Since n_0 represents w, we have $w = s_{\alpha_1} \cdots s_{\alpha_k}$.

Using (CR3) and the fact that $m_\alpha(1)^{-1} = m_\alpha(-1)$, we see that

$$m_\alpha(1) h_\beta(\lambda) m_\alpha(1)^{-1} = m_\alpha(1)^{-1} h_\beta(\lambda) m_\alpha(1) = h_{s_\alpha(\beta)}(\lambda)$$

. Thus, $n_0 h = h_{w(\beta_1)}(\lambda_1) \cdots h_{w(\beta_\ell)}(\lambda_\ell) n_0$. Repeating this, we get that

$$(n_0 h)^m = \left(\prod_{i=1}^{m} \prod_{j=1}^{\ell} h_{w^i(\beta_j)}(\lambda_j) \right) n_0^m = \left(\prod_{j=1}^{\ell} \prod_{i=1}^{m} h_{w^i(\beta_j)}(\lambda_j) \right) n_0^m.$$

The last equality holds because T is abelian.

Now let γ be any weight in the weight lattice Λ, and let j be an arbitrary index. Then

$$\prod_{i=1}^{m} \lambda_j^{\langle \gamma, w^i(\beta_j)^\vee \rangle} = \lambda_j^{\sum_{i=1}^{m} \langle \gamma, w^i(\beta_j)^\vee \rangle}.$$

Since w is an orthogonal transformation and $\langle , \rangle$ is bilinear,

$$\begin{aligned}\sum_{i=1}^{m} \langle \gamma, w^i(\beta_j)^\vee \rangle &= \sum_{i=1}^{m} 2\langle \gamma, w^i(\beta_j)\rangle / \langle w^i(\beta_j), w^i(\beta_j)\rangle \\ &= \sum_{i=1}^{m} 2\langle \gamma, w^i(\beta_j)\rangle / \langle \beta_j, \beta_j \rangle \\ &= \frac{2}{\langle \beta_j, \beta_j\rangle} \left\langle \gamma, \sum_{i=1}^{m} w^i(\beta_j) \right\rangle = 0\end{aligned}$$

for each j.

Thus, $\prod_{i=1}^{m} \lambda_j^{\langle \gamma, w^i(\beta_j)^\vee \rangle} = 1$, and this holds regardless of the field K. But by Lemma 3.1.3, $h_\alpha(\lambda)$ acts on the weight space V_γ^K via multiplication by $\lambda^{\langle \gamma, \alpha^\vee \rangle}$, and so we get that $\prod_{i=1}^{m} h_{w^i(\beta_j)}(\lambda_j) = 1$ for each j. Thus, $(n_0 h)^m = n_0^m$ for any $h \in T$. Since $n_0 h$ represents w, it must have order a multiple of m, and so we conclude that $n_0 h$ has the same order as n_0. Thus, all representatives of w in N have the same order, namely m or $2m$. □

It is a quick exercise to check that such a w exists, in fact any Coxeter element of W will work, as seen in [C1, Proposition 10.5.6; H2, Lemma 3.16]. In general an element w with no eigenvalue 1 is called a *generalized Coxeter element*; see [DW]. We will also call m a *generalized Coxeter number* if m is the order of some generalized Coxeter element.

Proof of reverse implication. Let $K = \mathbb{Q}$. Suppose $0 \neq v \in E$ is a 1-eigenvector of w. Then $v + w(v) + \cdots + w^{m-1}(v) = mv \neq 0$, and so $1 + w + \cdots + w^{m-1} \neq 0$ as a linear transformation. Since the roots span E, there exists a root β such that $\beta + w(\beta) + \cdots + w^{m-1}(\beta) \neq 0$. Choose a representative $n_0 \in N_0$ of w as before, so $n_0^m \in T_0$, say $n_0^m = h_{\alpha_1}(-1) \cdots h_{\alpha_k}(-1)$. Then for any $\ell \in \mathbb{N}$,

$$(n_0 h_\beta(2))^{\ell m} = \left(\prod_{i=1}^{\ell m} h_{w^i(\beta)}(2) \right) n_0^{\ell m}.$$

(We chose $K = \mathbb{Q}$ but in fact, any K that is not an algebraic extension of a finite field will work; we just need an element with infinite multiplicative order; for $K = \mathbb{Q}$ we have used the number 2.) Suppose this equals 1 for some ℓ. Then by Lemma 3.1.3, for any weight γ we have

$$1 = \prod_{i=1}^{\ell m} 2^{\langle \gamma, w^i(\beta)\rangle} \prod_{j=1}^{k} (-1)^{\ell\langle \gamma, \alpha_k\rangle} = \pm \prod_{i=1}^{\ell m} 2^{\langle \gamma, w^i(\beta)\rangle}.$$

By the same argument as before, this equals

$$\pm 2 \wedge \frac{2}{\langle \beta, \beta\rangle} \left\langle \gamma, \sum_{i=1}^{\ell m} w^i(\beta) \right\rangle = \pm 2 \wedge \frac{2}{\langle \beta, \beta\rangle} \left\langle \gamma, \ell \sum_{i=1}^{m} w^i(\beta) \right\rangle$$

where the caret notation indicates exponentiation. The only way this can equal 1 is if $\left\langle \gamma, \ell \sum_{i=1}^{m} w^i(\beta) \right\rangle = 0$. But since $\ell \sum_{i=1}^{m} w^i(\beta) \neq 0$, this is impossible, since we can always choose a weight γ to be not orthogonal to $\ell \sum_{i=1}^{m} w^i(\beta)$. Since ℓm are the only candidates for a finite order of $n_0 h_\beta(2)$, in fact it has infinite order. Since $n_0 h_\beta(2)$ is a representative of w, the theorem follows. □

We now describe how Theorem 4.1.2 translates when $G = \hat{\mathfrak{g}}(K)$ rather than $\mathfrak{g}(K)$. The coset of interest is now $n_0\hat{T}$ rather than $n_0 T$. This new coset has potentially more elements, so the reverse implication of the theorem clearly still holds. It is also true, though not immediately obvious, that the forward implication still holds.

Theorem 4.1.3. *Let $W = \hat{N}/\hat{T}$ be the spherical Weyl group corresponding to the group $\mathfrak{g}(K)$ of K-rational points. Let $w \in W$ have order m, such that as an orthogonal transformation of E, w does not have eigenvalue 1. Then all representatives of w in $\hat{N}$ have the same order, namely m or $2m$.*

Proof. As before choose a representative n_0 of w in N_0, so n_0 has order m or $2m$. We want to show that $(n_0 t)^m = n_0^m$ for any $t \in \hat{T}$. Let $t = h(\chi_1)\cdots h(\chi_k)$. By (CR3′), $n_0 h(\chi) n_0^{-1} = h(\chi \circ w)$ for any χ, and since $\hat{T}$ is abelian, we get that

$$(n_0 t)^m = \prod_{i=1}^{k} h(\chi_i \circ w) h(\chi_i \circ w^2) \cdots h(\chi_i \circ w^m) n_0^m.$$

To show this equals n_0^m we must show that $\prod_{i=1}^{k}\prod_{j=1}^{m} h(\chi_i \circ w^j) = 1$.

Considering the action of $h(\chi)$ on V^K, it suffices to show that $\prod_{i=1}^{k}\prod_{j=1}^{m} \chi_i \circ w^j(\gamma) = 1$ for any weight γ. Indeed, for any character χ,

$$\prod_{j=1}^{m} \chi(w^j(\gamma)) = \chi\left(\sum_{j=1}^{m} w^j(\gamma)\right) = \chi(0) = 1.$$

This last step relies on the fact that 1 is not an eigenvalue of w, and so $\sum_{j=1}^{m} w^j(\gamma) = 0$ for any γ. Thus $\prod_{i=1}^{k}\prod_{j=1}^{m} \chi_i \circ\ w^j(\gamma) = 1$ and $(n_0 t)^m = n_0^m$ for any $t \in \hat{T}$, which proves the theorem. $\square$

Remark 4.1.4. If we just want to prove that any representative of w has *some* finite order, there is a more elegant proof due to A. Rapinchuk [R1]. With the same setup as in Theorem 4.1.3, we know $1 + w + \cdots + w^{m-1} = 0$ as a transformation on E, so for any $v \in E$ we have $v + w(v) + \cdots + w^{m-1}(v) = 0$. We now associate each weight γ to a character on $\hat{T}$, via $\hat{\gamma}(h(\chi)) := \chi(\gamma)$, as in Section 3.4. In light of (CR3′), $nh(\chi)n^{-1} = h(\chi \circ w)$ where $w = n\hat{T}$ for $n \in \hat{N}$. Thus

$$\widehat{w\gamma}(h(\chi)) = \chi(w\gamma) = \hat{\gamma}(h(\chi \circ w)) = \hat{\gamma}(nh(\chi)n^{-1}),$$

so $\widehat{w\gamma}(t) = \hat{\gamma}(ntn^{-1})$ for all weights γ and all $t \in \hat{T}$.

By hypothesis $\gamma + w\gamma + \cdots + w^{m-1}\gamma = 0$, so $\hat{\gamma} + \widehat{w\gamma} + \cdots + \widehat{w^{m-1}\gamma}$ is the trivial character. Thus for any $t \in \hat{T}$, $\hat{\gamma}(t) + \hat{\gamma}(ntn^{-1}) + \cdots + \hat{\gamma}(n^{m-1}tn^{-(m-1)}) = 1$, i.e., $\hat{\gamma}(tntn^{-1} \cdots n^{m-1}tn^{-(m-1)}) = 1$. If we let t equal n^m, which is in $\hat{T}$ since $w^m = 1$, we get that $\hat{\gamma}(n^m n^m \cdots n^m) = 1$, i.e., $\hat{\gamma}(n^{m^2}) = 1$ for all γ. Thus every diagonal entry of n^{m^2} is 1, and indeed $n^{m^2} = 1$ in $\hat{N}$.

This also improves on the "m or $2m$" disjunction. Namely, if m is odd, the representatives must have order m, since the order is either m or $2m$ and must divide

m^2. Also note that if we are working in $\mathfrak{g}(K)$ instead of $\hat{\mathfrak{g}}(K)$ this proof still applies, since $T \leq \hat{T}$.

4.2 Weyl transitive actions that are not weakly transitive

We now have the tools to produce examples of Weyl transitive group actions on buildings that are not weakly transitive, and thus not strongly transitive with respect to any apartment system. Let K be a field complete with respect to a discrete valuation, and let $G = \mathfrak{g}(\Phi, K)$. Let (B_a, N) be the affine BN-pair of $G = \mathfrak{g}(K)$ as described in Section 3.3, and let Δ_a be the canonical affine building on which G acts strongly transitively with respect to $\overline{\mathcal{A}}$ (see Proposition 3.3.8). Then B_a is open in G by Corollary 3.3.7, and so by Lemma 2.4.2 any dense subgroup $H \leq G$ acts Weyl transitively on Δ. By Lemma 2.4.3 and Theorem 4.1.2 however, if H is to act weakly transitively on Δ it must contain elements of order m, for any m that is a generalized Coxeter number of W.

Thus, to find examples of groups acting Weyl transitively but not weakly transitively on affine buildings, it suffices to exhibit dense subgroups of G containing no elements of order m. The condition of being m-torsionfree is of course not a necessary condition, and we will see in Chapter 5 other examples that don't rely on such a strong, global condition. In this section, however, that is our goal.

First consider the case when $K = \mathbb{Q}_p$, so $G = \mathfrak{g}(\mathbb{Q}_p)$. Let $\Gamma = \mathfrak{g}(\mathbb{Z}[\frac{1}{p}])$. As usual think of G as a subgroup of $\mathrm{SL}_d(\mathbb{Q}_p)$ for some d. In the same way we can think of Γ as a subgroup of $\mathrm{SL}_d(\mathbb{Z}[\frac{1}{p}])$. Now let q be any nonzero integer prime to p, so it makes sense to reduce the entries of matrices in Γ mod q. We define the *congruence subgroup* Γ_q to be $\Gamma_q := \{g \in \Gamma \mid g \equiv I_d \mod q\}$, where matrices are taken mod q entry-wise. This is the kernel of the restriction to Γ of the natural group homomorphism $\mathrm{SL}_d(\mathbb{Z}[\frac{1}{p}]) \to \mathrm{SL}_d((\mathbb{Z}/q\mathbb{Z}))$, so it really is a subgroup. We will show that for any $q > 2$, Γ_q is both torsionfree (in particular m-torsionfree for any m) and dense in G.

Lemma 4.2.1. *For any nonzero q in $\mathbb{Z}$, Γ_q is dense in G.*

Proof. Since the topological closure of $\mathbb{Z}[\frac{1}{p}]$ contains $\mathbb{Z}_p$ and $1/p$, $\mathbb{Z}[\frac{1}{p}]$ is dense in $\mathbb{Q}_p$. Also $\mathbb{Q}_p = q\mathbb{Q}_p$, so $q\mathbb{Z}[\frac{1}{p}]$ is dense in $\mathbb{Q}_p$. Thus the set $\{x_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in q\mathbb{Z}[\frac{1}{p}]\}$ is dense in $\{x_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in \mathbb{Q}_p\}$. Since the latter set generates $\mathfrak{g}(\mathbb{Q}_p)$, it now suffices to show that $x_\alpha(\lambda) \in \Gamma_q$ for any $\alpha \in \Phi$, $\lambda \in q\mathbb{Z}[\frac{1}{p}]$. Clearly $x_\alpha(\lambda) \in \Gamma$, and by Corollaries 3.1.5 and 3.1.6 the diagonal entries of $x_\alpha(\lambda)$ are 1 and all other entries are congruent to $0 \mod q$, so it is also clear that $x_\alpha(\lambda) \equiv I_d \mod q$. Thus indeed $x_\alpha(\lambda) \in \Gamma_q$. □

We will see later, in the more general setup of Theorem 4.2.3, that for any q with $|q| > 2$, Γ_q is torsionfree. Assuming this, we get the following building-theoretic result.

Theorem 4.2.2. *Let $G = \mathfrak{g}(\mathbb{Q}_p)$ be a Chevalley group. Let Δ_a be the canonical affine building on which G acts strongly transitively with respect to $\overline{\mathcal{A}}$. Let $q \in \mathbb{Z}\backslash p\mathbb{Z}$ such that $|q| > 2$. Then the action of the congruence subgroup $\Gamma_q \leq G$ on Δ is Weyl transitive but not strongly transitive with respect to any apartment system.*

We next consider a field with positive characteristic p, $K = \mathbb{F}_p((t))$. This is local and so $\mathfrak{g}(K)$ acts strongly transitively on an affine building with respect to the complete apartment system. As before we wish to find dense subgroups that have no m-torsion for m a generalized Coxeter number. Unlike the characteristic 0 case, where we could produce a recipe independent of $\mathfrak{g}$ and W, here the recipe for one Chevalley group may not work for another. We need not know very much, however; just the order of the spherical Weyl group. When choosing p we just need to ensure that p does not divide $2|W|$, and as we will now see this yields the examples we want.

Let $G = \mathfrak{g}(\mathbb{F}_p((t)))$ with p not dividing $2|W|$. Let $\Gamma = \mathfrak{g}(\mathbb{F}_p[t, t^{-1}])$ and let f be an irreducible element of $\mathbb{F}_p[t, t^{-1}]$. Set $\Gamma_f := \{g \in \Gamma \mid g \equiv 1 \mod f\}$, where as before we think of the elements g as matrices. Then, similar to the characteristic 0 case, $f\mathbb{F}_p[t, t^{-1}]$ is dense in K, so Γ_f is dense in G. It also turns out that Γ_f only has p-torsion, that is the only elements of finite order in Γ_f have p-power order. We will show this, in more generality, in Theorem 4.2.3, which is partially inspired by Lemma 17.5 and Proposition 17.6 in [B1]. See also Minkowski's Lemma, [PR, Lemma 4.19], which is a special case of our theorem but has a general proof.

Theorem 4.2.3. *Let K be a topological field and $\mathfrak{g}(K)$ a Chevalley group. Let $L \leq K$ be a noetherian integral domain that is dense in K. For any proper nonzero ideal I of L, define $\Gamma_I := \ker(\mathfrak{g}(L) \twoheadrightarrow \mathfrak{g}(L/I))$. Let P be any nonzero prime ideal of L. If K has characteristic $p > 0$, then the subgroup Γ_P of $\mathfrak{g}(K)$ is dense and has only p-torsion. If K has characteristic 0 and L/P has characteristic $q > 2$, and if $qL = P$, then the subgroup Γ_P of $\mathfrak{g}(K)$ is dense and torsionfree.*

Proof. We first claim that Γ_I is dense in $\mathfrak{g}(K)$ for any proper nontrivial ideal I of L. Let $x = x_\alpha(\mu)$ for $\alpha \in \Phi$, $\mu \in I$. By Corollary 3.1.6, $x \in \Gamma_I$. Since L is dense in K and $\mathfrak{g}(K)$ is generated by the $x_\alpha(\lambda)$ we conclude that Γ_I is dense in $\mathfrak{g}(K)$.

Now we analyze torsion properties, thinking of $\mathfrak{g}(L)$ as a subgroup of $\mathrm{SL}_d(L)$. Let P be a prime ideal of L. Let A be a matrix in Γ_P with finite order r, and suppose $r > 1$. Replacing A with an appropriate power, we may assume r is prime. Since $A \equiv_P I_d$ there exists $B \in M_d(L)$ such that $A = I_d + B$ and $B \equiv_P 0$, i.e. every entry of B is in P. Then

$$I_d = (I_d + B)^r = I_d + rB + \binom{r}{2}B^2 + \binom{r}{3}B^3 + \cdots + B^r$$

and so

$$-rB = \binom{r}{2}B^2 + \binom{r}{3}B^3 + \cdots + B^r.$$

Call this equation $(*)$.

Now choose $s \geq 1$ such that $B \equiv_{P^s} 0$ but $B \not\equiv_{P^{s+1}} 0$. Since L is a noetherian domain and P is a proper ideal, this is possible by the Krull Intersection Formula [E, Corollary 5.4]. The right-hand side of $(*)$ is congruent to 0 mod $P^{2s} \subseteq P^{s+1}$, which looking at the left-hand side implies that $r \cdot 1_L \in P$. Since r is prime, $r = \operatorname{char} L/P$. If K has characteristic $p > 0$ then immediately we get $r = p$, i.e., Γ_P can only have p-torsion.

Now suppose K has characteristic 0 and L/P has characteristic $q > 2$, so $r = q$ is odd. Then r divides $\binom{r}{2}$, and the right-hand side of $(*)$ is congruent to zero mod P^{s+2}, which is rP^{s+1} since $rL = P$. But the left-hand side of $(*)$ is not congruent to zero mod rP^{s+1} since L has characteristic 0. In this case we conclude that our assumption $r > 1$ was impossible and in fact Γ_P is torsionfree. □

Corollary 4.2.4. *Let K be a field complete with respect to a discrete valuation, $G = \mathfrak{g}(K)$ a Chevalley group with spherical Weyl group W such that* $\operatorname{char} K$ *does not divide $2|W|$, and Δ_a the affine building associated to G. Then G has "many" subgroups H that act Weyl transitively on Δ but do not act strongly transitively with respect to any apartment system.* □

In particular this finishes the examples of $K = \mathbb{Q}_p$ and $K = \mathbb{F}_p((t))$. The word "many" that we have been using is justifiable. For $K = \mathbb{Q}_p$ and $L = \mathbb{Z}[\frac{1}{p}]$, any $q \in \mathbb{Z}\backslash p\mathbb{Z}$ that satisfies $|q| > 2$ produces such a subgroup. For $K = \mathbb{F}_p((t))$ and $L = \mathbb{F}_p[t, t^{-1}]$, any irreducible $f \in \mathbb{F}_p[t, t^{-1}]$ produces such a subgroup. The general case works similarly.

4.3 Classical groups

There are some classes of Chevalley groups that are easy to explicitly realize as subgroups of $\mathrm{SL}_d(K)$. In this section we examine the implications of the previous two sections in the cases when G is one of $\mathrm{SL}_n(K)$, $\mathrm{Sp}_{2n}(K)$, $\mathrm{SO}_{2n}(K)$, or $\mathrm{SO}_{2n+1}(K)$.

The $\mathrm{SL}_n(K)$ case will be particularly important in Chapter 5. In this section we will only consider fields K that contain elements of infinite multiplicative order, for reasons similar to the situation that arose in the proof of the reverse implication of Theorem 4.1.2. This section will also serve to precisely identify the generalized Coxeter elements in Coxeter groups of the classical types; see [DW] for more results in this vein.

Example 4.3.1. Let $\mathfrak{g} = \mathfrak{sl}_n$ $(n \geq 2)$ and let V be the universal representation. It is an easy exercise to check that $\mathfrak{g}(K) = \mathrm{SL}_n(K)$, $B = B_n(K)$, and N is the subgroup of monomial matrices; see [AB2, Section 6.5; C1, Section 11.3; S3, page 36]. Thus $T = T_n(K)$ and $W = N/T \cong S_n$, the symmetric group.

We can check directly which cosets in N/T have the property that all their representatives have finite order. First we note that if

$$x = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where the a_i are any elements of K^*, then since $1 = \det x = \pm \prod_{i=1}^{n} a_i$ we get immediately that

$$x^n = \begin{pmatrix} \pm 1 & 0 & 0 & \cdots & 0 \\ 0 & \pm 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \pm 1 \end{pmatrix}.$$

The coset xT corresponds in S_n to an n-cycle. This implies that all representatives of xT in N have order n or $2n$. More specifically, they all have order n if n is odd and $2n$ if n is even. Of course since all n-cycles are conjugate in S_n we see that if xT is *any* n-cycle, a similar result holds.

Now, are the n-cycles the *only* cosets with this property? It is not difficult to see that they are. Suppose $w \in S_n$ is not an n-cycle; say it decomposes into disjoint cycles $w = \sigma_1 \cdots \sigma_r$ where σ_i is a k_i-cycle and each k_i is strictly less than n. We adopt the convention that for any fixed points of w in $\{1, \ldots, n\}$ there is a "1-cycle," $\sigma_i = id$, in the decomposition. Then since w is not an n-cycle, r is necessarily greater than 1.

Now, an element $x \in N$ represents w if and only if it is conjugate to a block diagonal matrix $x = \mathrm{diag}(x_1, \ldots, x_r)$, where each x_i is of the form

$$x_i = \begin{pmatrix} 0 & {}_ia_1 & 0 & \cdots & 0 \\ 0 & 0 & {}_ia_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & {}_ia_{k_i-1} \\ {}_ia_{k_i} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that $\mathrm{diag}(x_1, \ldots, x_r)$ has determinant 1 if and only if $\displaystyle\prod_{i=1}^{r} \det x_i = 1$. Recall our assumption that K has an element a of infinite multiplicative order. Choose matrices x_i of form above for each i such that $\det x_1 = a$, $\det x_2 = a^{-1}$, and $\det x_i = 1$ for $2 < i \leq r$. Then indeed $x = \mathrm{diag}(x_1, \ldots, x_r)$ has determinant 1, and so represents w in N. However, since $a = \det x_1 = \pm \prod_{i=1}^{n} {}_1a_i$, we see that $x_1^{k_i} = \pm a$ and so x cannot have finite order.

For two concrete examples, consider the case $n = 4$, $K = \mathbb{Q}$, $w = (243)$ and $w' = (12)(34)$. Then the matrices

$$x = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1/2 & 0 & 0 \end{pmatrix} \text{ and } x' = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}$$

represent w and w' respectively, and have infinite order.

We now recover a well-known result.

Corollary 4.3.2. *If W is a Coxeter group of type A_{n-1}, then the generalized Coxeter elements of W are precisely the Coxeter elements, which are the n-cycles.* □

As this example shows, an n-torsionfree subgroup H of $\mathrm{SL}_n(K)$ cannot represent the n-cycles of the Weyl group S_n. In general though, we cannot discount the possibility that H represents other elements of S_n. In Section 5.4 we will see that the $\mathrm{SL}_n(K)$ case allows for a completely different approach that produces subgroups H representing "very few" elements of S_n. In particular we will see examples where such H are dense, yielding Weyl transitive actions that are in some sense not even close to being weakly transitive. For now though, we move on to another classical group.

Example 4.3.3. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ $(n \geq 2)$ and let V be the universal representation. As in the previous example, it is easy to check that $\mathfrak{g}(K) = \mathrm{Sp}_{2n}(K)$, $B = B_{2n}(K) \cap \mathrm{Sp}_{2n}(K)$, and N is the subgroup of monomial matrices; see [AB2, Section 6.6, C1, Section 11.3, S3, page 38]. These definitions of B and N rely on a choice of basis, so it is germane now to discuss the construction of $\mathrm{Sp}_{2n}(K)$ before moving on.

Denote the standard basis of K^{2n} by $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. Define a bilinear form on K^{2n} in the following way on the basis vectors:

$$\langle e_i, f_i \rangle = 1 \text{ for all } i$$

$$\langle f_i, e_i \rangle = -1 \text{ for all } i$$

$$\langle v, v' \rangle = 0 \text{ for all other pairs of basis vectors } v,\ v'.$$

Definition 4.3.4. The group $\mathrm{Sp}_{2n}(K) := \{g \in \mathrm{GL}_{2n}(K) \mid \langle gv, gv' \rangle = \langle v, v' \rangle \text{ for all } v, v' \in K^{2n}\}$ is called the *symplectic group*. In fact $\mathrm{Sp}_{2n}(K) \leq \mathrm{SL}_{2n}(K)$.

We have defined N to be the stabilizer of $\{[e_1], \ldots, [f_n]\}$, where we denote span by square brackets. For $g \in N$ the symplectic condition tells us the following *symplectic monomial condition* **(SMC)**:

- If $ge_i = \lambda e_j$ then $gf_i = \lambda^{-1} f_j$.
- If $ge_i = \lambda f_j$ then $gf_i = -\lambda^{-1} e_j$.
- If $gf_i = \lambda f_j$ then $ge_i = \lambda^{-1} e_j$.
- If $gf_i = \lambda e_j$ then $ge_i = -\lambda^{-1} f_j$.

We can also characterize the Weyl group relatively easily. Think of S_{2n} as permuting the set $\{1, \ldots, n, -1, \ldots, -n\}$. Then the action of N on $\{[e_1], \ldots, [f_n]\}$ induces an action of N/T on $\{1, \ldots, n, -1, \ldots, -n\}$, so the Weyl group is a subgroup of S_{2n}. In particular, by (SMC) W is precisely the group of permutations $\sigma \in S_{2n}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in \{1, \ldots, n, -1, \ldots, -n\}$.

We now can ask which cosets in $W = N/T$ have the property that any representative has finite order, or equivalently which elements of W are generalized Coxeter elements. As in the previous example, any $2n$-cycle will work, by virtue of symplectic matrices having determinant 1. For example any representative of $(1\ 2 \cdots n\ {-1}\ {-2} \cdots {-n})$ in N has order $4n$. Unlike the special linear case, in the symplectic case there exist generalized Coxeter elements that are not Coxeter elements. Consider the permutation $w = (1\ {-1})(2\ {-2}) \cdots (n\ {-n})$. By (SMC) if $ge_i = \lambda_i f_i$ for each i then $gf_i = -\lambda_i^{-1} e_i$, so g^2 acts as multiplication by -1. We see immediately that every representative has order 4, and so w is a generalized Coxeter element.

More generally, any permutation w such that each i shares its w-orbit with $-i$ will work. By (SMC) any representative g of such a w will have finite order: in particular if ℓ is the least common multiple of the cycle types in w then $g^{2\ell} = I_{2n}$. Conversely, if w is a permutation such that there exists i *not* sharing a w-orbit with $-i$, then since we are assuming K contains an element of infinite multiplicative order it is easy to construct a representative g of w in N with infinite order.

Lastly, we note that if m is the order of the generalized Coxeter element w then we can determine precisely whether representatives have order m or $2m$. Let w have cycle decomposition $w = w_1 \cdots w_r$ where each w_j is a k_j-cycle. Let n_j denote the number of positive i such that $w_j(i) < 0$. If w_j has order m_j, (SMC) ensures that representatives of w_j have order m_j if n_j is even and order $2m_j$ if n_j is odd. This completely determines the order of representatives of w.

Corollary 4.3.5. *Let W be a Coxeter group of type C_n. Think of W as a subgroup of S_{2n} acting on $\{1, \cdots, n, -1, \cdots, -n\}$, as above. Then $w \in W$ is a generalized Coxeter element if and only if every i shares its w-orbit with $-i$.* □

Example 4.3.6. This example is very similar to the previous one. We inspect the special orthogonal group $\mathrm{SO}_{2n+1}(K)$. Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$ $(n \geq 1)$ and let V be the adjoint representation. Then $\mathfrak{g}(K) = \mathrm{SO}_{2n+1}(K)$, $B = B_{2n+1}(K) \cap \mathrm{SO}_{2n+1}(K)$, and N is the subgroup of monomial matrices; see [AB2, Section 6.7; C1, Section 11.3; S3, pages 38,45].

Denote the standard basis of K^{2n+1} by $\{e_1, \ldots, e_n, f_1, \ldots, f_n, e_0\}$. Define a symmetric bilinear form on K^{2n+1} by:

$$\begin{aligned} \langle e_i, f_i \rangle &= 1 \text{ for } 1 \leq i \leq n \\ \langle e_0, e_0 \rangle &= 1 \\ \langle v, v' \rangle &= 0 \text{ for all other pairs of basis vectors } v,\ v'. \end{aligned}$$

Definition 4.3.7. The group $\mathrm{SO}_{2n+1}(K) := \{g \in \mathrm{SL}_{2n+1}(K) \mid \langle gv, gv' \rangle = \langle v, v' \rangle$ for all $v, v' \in K^{2n+1}\}$ is called the *special orthogonal group* of odd dimension.

N is the stabilizer of $\{[e_1], \ldots, [f_n], [e_0]\}$, so for $g \in N$ the orthogonal condition gives us the following *orthogonal monomial condition* **(OMC)**:
For $1 \leq i \leq n$,

- If $ge_i = \lambda e_j$ then $gf_i = \lambda^{-1} f_j$.
- If $ge_i = \lambda f_j$ then $gf_i = \lambda^{-1} e_j$.
- If $gf_i = \lambda f_j$ then $ge_i = \lambda^{-1} e_j$.

- If $gf_i = \lambda e_j$ then $ge_i = \lambda^{-1} f_j$.

- $ge_0 = \pm e_0$.

We can also characterize the Weyl group. As in the previous example we think of S_{2n} as permuting the set $\{1, \ldots, n, -1, \ldots, -n\}$. Since N fixes $[e_0]$, the action of N on $\{[e_1], \ldots, [f_n], [e_0]\}$ induces an action of N/T on $\{1, \ldots, n, -1, \ldots, -n\}$, so the Weyl group is a subgroup of S_{2n}. In particular, by (OMC) W is precisely the group of permutations $\sigma \in S_{2n}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in \{1, \ldots, n, -1, \ldots, -n\}$. (Note that $\mathrm{SO}_{2n+1}(K)$ has the same Weyl group as $\mathrm{Sp}_{2n}(K)$, a fact that is reflected in the Dynkin diagrams of type B_n and C_n [C2].)

Since W is the same as in the previous example, by Corollary 4.3.5 we already know the generalized Coxeter elements of W. The situation is slightly different though. For example any representative of the permutation $w = (1\ -1)(2\ -2)\cdots(n\ -n)$ in N now has order 2 instead of 4. This is due to the slight difference between (SMC) and (OMC). In fact, if w is a generalized Coxeter element of order m and g a representative of w in N, (OMC) tells us that g really has order m, not $2m$. This is a nice strengthening of Theorem 4.1.2 in the B_n case.

Corollary 4.3.8. *Let W be a Coxeter group of type B_n. Think of W as a subgroup of S_{2n} acting on $\{1, \cdots, n, -1, \cdots, -n\}$, as above. Then $w \in W$ is a generalized Coxeter element if and only if every i shares its w-orbit with $-i$.* □

Example 4.3.9. The last example is $\mathrm{SO}_{2n}(K)$. Let $\mathfrak{g} = \mathfrak{so}_{2n}$ $(n \geq 2)$. With an appropriate choice of representation V, $\mathfrak{g}(K) = \mathrm{SO}_{2n}(K)$, $B = B_{2n}(K) \cap \mathrm{SO}_{2n}(K)$, and N is the subgroup of monomial matrices; see [AB2, Section 6.7; C1, Section 11.3; S3, pages 38,45].

Denote the standard basis of K^{2n} by $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. Define a symmetric bilinear form on K^{2n} by:

$$\langle e_i, f_i \rangle = 1 \text{ for } 1 \le i \le n$$
$$\langle v, v' \rangle = 0 \text{ for all other pairs of basis vectors } v,\ v'.$$

Definition 4.3.10. The group $\mathrm{SO}_{2n}(K) := \{g \in \mathrm{SL}_{2n}(K) \mid \langle gv, gv' \rangle = \langle v, v' \rangle$ for all $v, v' \in K^{2n}\}$ is called the *special orthogonal group* of even dimension.

As before N is the stabilizer of $\{[e_1], \ldots, [f_n]\}$, and for $g \in N$ we have the same condition (OMC), but with an added condition (E) that ensures $\det g = 1$:

- If $ge_i = \lambda e_j$ then $gf_i = \lambda^{-1} f_j$.
- If $ge_i = \lambda f_j$ then $gf_i = \lambda^{-1} e_j$.
- If $gf_i = \lambda f_j$ then $ge_i = \lambda^{-1} e_j$.
- If $gf_i = \lambda e_j$ then $ge_i = \lambda^{-1} f_j$.
- (E) The cardinality of $\{g[e_1], \ldots, g[e_n]\} \cap \{[f_1], \ldots, [f_n]\}$ is even.

Again we want to characterize the Weyl group. As in the previous examples we think of S_{2n} as permuting the set $\{1, \ldots, n, -1, \ldots, -n\}$, with the Weyl group a subgroup of S_{2n}. In particular, by (OMC) and (E) W is precisely the group of permutations $\sigma \in S_{2n}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in \{1, \ldots, n, -1, \ldots, -n\}$, and the number of positive i such that $\sigma(i)$ is negative is even.

This shows that W is a subgroup of the Weyl group from the previous example, which we will now call W'. Also, an element w of W is a generalized Coxeter element in W if and only if it is one in W'. Of course this also means that if w has order m, then as in the B_n case all its representatives have order m, not $2m$. We can exhibit a few explicit examples: if n is even, then the permutation $w = (1\ {-1})(2\ {-2}) \cdots (n\ {-n})$ from before satisfies (E) and so works in this context. If n is odd, this no longer satisfies (E). Instead consider $w = (1\ 2\ {-1}\ {-2})(3\ {-3}) \cdots (n\ {-n})$, which is a legitimate construction since $n \geq 2$ and n is odd. Now w satisfies (E) and still satisfies (OMC). As one last example, note that the $2n$-cycles from before don't satisfy (E), but $w = (1\ -1)(2\ 3 \cdots n\ -2\ -3 \cdots -n)$ does.

Corollary 4.3.11. *Let W be a Coxeter group of type D_n. Think of W as a subgroup of S_{2n} acting on $\{1, \cdots, n, -1, \cdots, -n\}$, as above. Then $w \in W$ is a generalized Coxeter element if and only if every i shares its w-orbit with $-i$.* $\square$

Chapter 5

Division algebras

The examples in the previous chapter certainly work, but are somewhat *ad hoc*. Having to pass to congruence subgroups leaves one wondering whether Weyl- but not strongly transitive actions are very natural. Indeed, Tits' original thought was not to pass to congruence subgroups but rather to inspect anisotropic groups over global fields. These groups embed into Chevalley groups over local fields, and so act on affine buildings, but as Tits suggested the anisotropic groups themselves should already have the desired transitivity properties. There should be no need to pass to subgroups, or to impose any further criteria on the situation.

The simplest example of an anisotropic group over a global field is the norm-1 group G of a quaternion division algebra over $\mathbb{Q}$. This case was analyzed in [AB1] with the surprising result that G acts strongly transitively on an associated tree roughly "half the time." This ran contrary to Tits' prediction, but the quaternion case was expected to exhibit some "accidents of low dimension." Indeed, as we show in this chapter, for central F-division algebras of degree greater than 2, the norm-1 group fails to act weakly transitively on the appropriate building, with no further restrictions. In the case F is a topological field, we can produce actions on affine buildings that are Weyl transitive but not strongly transitive with respect to any apartment system. Also, the failure to act weakly transitively is in some sense quite "dramatic."

It should be mentioned that there is a relatively easy proof due to A. Rapinchuk that non-split algebraic groups over global fields in general do provide examples of Weyl- but not strongly transitive group actions on affine buildings [R1]. In particular one can construct such actions using any non-split group that does not split over any quadratic extension of the base field. The anisotropic case, being at the opposite end of the spectrum from the split case, is thus "especially striking," as expressed in [T], though of course certain isotropic groups work as well. The present chapter is thus less interesting for proving such actions exist than for showing that, at least in the division algebra case, the failure to act weakly transitive is "dramatic."

The final part of this chapter deals with a slightly different, though related question. Instead of asking about weak transitivity, i.e., transitivity on an arbitrary apartment, we analyze the action just on the fundamental apartment. In this context we can achieve a complete, precise description of the action of the stabilizer of Σ_0 in $D^\times$ on Σ_0, for "most" division algebras D. This description also lets us say something about the action restricted to $\mathrm{SL}_1(D)$, and we show that in general a complete description in this context is at least as difficult as the unsolved problem of relating exponent and index of a division algebra. See Section 5.5 for details.

5.1 Central simple algebras

Before we can talk about the multiplicative and norm-1 groups of a division algebra we need some background on the algebras themselves. Central division algebras are a special case of *central simple algebras*, and in some sense completely classify them as seen in Theorem 5.1.2.

Let A be a ring with unity. If F is a subring of A that is a field we say F is a *subfield* of A. If F is contained in the center of A we say that A is an *F-algebra*. More specifically, if F *equals* the center of A we call A a *central F-algebra*. Lastly, if A has no nontrivial two-sided ideals we say that it is *simple*. The following then will be our working definition of a central simple algebra (CSA).

Definition 5.1.1. Let F be a field. A *central simple F-algebra* is a finite dimensional F-algebra A, with $Z(A) = F$, that is simple as a ring. If there is no ambiguity, a central simple F-algebra will just be called a central simple algebra, or CSA.

The only non-tautological aspect to this definition is the assumption of finite dimensionality, but we will never make use of any infinite dimensional algebras, so we encode this into the definition.

The starting point for the theory of CSAs is the well-known Wedderburn Structure Theorem, quoted below. This can be found in any number of books, in particular [FD, L1, L2].

Theorem 5.1.2. *(Wedderburn) Let A be a central simple F-algebra. Then there exists a central F-division algebra D and a natural number r such that $A \cong M_r(D)$ as F-algebras. Also, r is uniquely determined and D is unique up to isomorphism.*

We thus see that every CSA is associated to a central division algebra, and this association establishes an equivalence relation on the class of CSAs. If A_1, A_2 are CSAs over F, we say A_1 and A_2 are *Brauer equivalent* $A_1 \sim A_2$ if there exist r_1, r_2 natural numbers and a central F-division algebra D such that $A_i \cong M_{r_i}(D)$ for $i = 1, 2$. This is clearly an equivalence relation since D is uniquely determined up to isomorphism.

A crucial fact in the present theory is that CSAs are closed under the tensor product. This is very standard and we will not prove it here; see [FD, Corollary 3.6; L2, Theorem 29.8]. In this chapter, all tensor products are over F, unless otherwise specified.

Proposition 5.1.3. *Let A, B be CSAs over F. Then $A \otimes B$ is a CSA.*

What's more, the tensor product is Brauer invariant. The proof of this is constructive, and so we will present it here. We will take for granted the fact that if A is any F-algebra, we have $A \otimes M_r(F) \cong M_r(A)$, and that $M_r(F) \otimes M_s(F) \cong M_{rs}(F)$; see [FD, Lemma 4.1; L1, Corollary 15.5; L2, Theorem 29.9].

Proposition 5.1.4. *Let A, A', B, and B' be CSAs. If $A \sim A'$ and $B \sim B'$ then $A \otimes B \sim A' \otimes B'$.*

Proof. Suppose $A \cong M_r(D)$, $A' \cong M_{r'}(D)$, $B \cong M_s(E)$, and $B' \cong M_{s'}(E)$, for D and E division algebras. Then

$$\begin{aligned} A \otimes B \cong M_r(D) \otimes M_s(E) &\cong D \otimes M_r(F) \otimes E \otimes M_s(F) \\ &\cong (D \otimes E) \otimes M_{rs}(F) \cong M_{rs}(D \otimes E). \end{aligned}$$

An analogous statement holds in the primed situation, and the result follows. □

For a CSA A, let $[A]$ denote the equivalence class of A under the Brauer equivalence relation. By Propositions 5.1.3 and 5.1.4, the operation $[A][B] := [A \otimes B]$ is well defined. It is also clearly associative and commutative, and has an identity element $[F]$. If $\mathrm{Br}(F)$ denotes the set of equivalence classes of CSAs over F, this proves that $\mathrm{Br}(F)$ is a commutative monoid. It turns out to actually be a group. The inversion operation is given by $[A]^{-1} = [A^{op}]$, where A^{op} is the *opposite algebra*; see [FD, Proposition 3.12; L2, Section 29.F14] for details.

Definition 5.1.5. For a field F, the abelian group $\mathrm{Br}(F)$ is called the *Brauer group of F.*

Another important aspect to the theory of CSAs is *extension of scalars*. As seen, the tensor product of two CSAs is again a CSA. But it is an important fact that given a CSA A and a simple (not necessarily central) algebra K, the tensor product $A_K := A \otimes_F K$ is again simple, and is central over K. See [FD, Theorem 3.5; L2, Section 29.F15]. In particular, if $[A] \in \mathrm{Br}(F)$ and $K|F$ is any field extension, we have that $[A_K] \in \mathrm{Br}(K)$.

Definition 5.1.6. Let $[A] \in \mathrm{Br}(F)$, $K|F$ a field extension. If $[A_K] = [K]$ in $\mathrm{Br}(K)$ we say that A *splits* over K.

Note that $M_r(D) \otimes K \cong M_r(D \otimes K)$, so extension of scalars is Brauer invariant and this definition makes sense. Also note that extension of scalars, realized as the map $\mathrm{Br}(F) \to \mathrm{Br}(K)$ given by $[A] \mapsto [A_K]$, is a homomorphism. This motivates the following:

Definition 5.1.7. The kernel $\mathrm{Br}(K|F)$ of the map $\mathrm{Br}(F) \to \mathrm{Br}(K)$ defined above is called the *relative Brauer group*. This consists precisely of equivalence classes $[A]$ such that A splits over K.

Every CSA splits over some field, in particular every CSA splits over any algebraically closed extension of F. This is because any central division algebra over an algebraically closed field must just be the field itself. Let A be a CSA that splits over K. Since $\dim_F A = \dim_K A_K$ and $A_K \cong M_d(K)$ for some d, we get that $\dim_F A = d^2$.

Definition 5.1.8. We define the *degree* of a CSA A to be $d = \sqrt{\dim_F A}$. If $[A] = [D]$ for a division algebra D, we define the *index* or *Schur index* $\mathrm{ind}(A)$ of A to be the degree of D. Note that if A has degree d then $1 \leq \mathrm{ind}(A) \leq d$, with $\mathrm{ind}(A) = 1$ if and only if A is split and $\mathrm{ind}(A) = d$ if and only if A is a division algebra.

To close this chapter we establish some results regarding subfields of division algebras. The genesis of all these results is the well-known Centralizer Theorem (or Double Centralizer Theorem), the relevant part of which we quote below; see [FD, Theorem 3.15; L1, Theorem 15.4; L2, Theorem 29.14].

Theorem 5.1.9. *Let A be a CSA and K a subfield of A containing F. Let $C_A(K) := \{a \in A \mid ax = xa \text{ for all } x \in K\}$. Then $\dim_F A = [K : F] \dim_F C_A(K)$.*

Let K be a subfield of a central F-division algebra D and let $x \in C_D(K)$. Then $K(x)$ is also a subfield of D. This tells us that a subfield K of D is maximal (with respect to inclusion of fields) if and only if $C_D(K) = K$.

Lemma 5.1.10. *Let D be a central F-division algebra of degree d and let K be a maximal subfield of D. Then $[K : F] = d$.*

Proof. By Theorem 5.1.9, $d^2 = [K : F] \dim_F C_D(K) = [K : F] \dim_F K = [K : F]^2$, so $[K : F] = d$. □

Corollary 5.1.11. *Let D be a central F-division algebra of degree d and let K be a subfield of D containing F. Then $[K : F] \mid d$.*

Proof. Choose a maximal subfield L of D such that $K \leq L$. Then $[K : F] \mid [L : F] = d$. □

We state the final result without proof. See Corollaries 3.17 and 4.7 of [FD], Exercise 15.6 of [L1], and Theorem 29.19 of [L2].

Proposition 5.1.12. *Let D be a central F-division algebra of degree d and let $K|F$ be a splitting field of D. Then $d \mid [K : F]$. Also, any maximal subfield of D splits D.*

5.2 The Brauer group of local and global fields

As in the previous chapter, we are particularly interested in the cases when the base field is local or global. The Brauer group of a local field is well-understood, and the Brauer group of a global field is completely determined by the corresponding local theory. Refer to Section 3.2 for the relevant background.

Let F be a non-archimedean local field. We can always decompose $\mathrm{Br}(F)$ as a union of relative Brauer groups,

$$\mathrm{Br}(F) = \bigcup_{K|F} \mathrm{Br}(K|F),$$

but in the local case we can do better than this. Namely, we claim that

$$\mathrm{Br}(F) = \bigcup_{\substack{K|F \\ \text{unramified}}} \mathrm{Br}(K|F).$$

For this we could just quote [L2, Theorem 31.1], but since the proof is quite easy modulo [L2, Lemma 31.1] we give it here.

Proposition 5.2.1. *For F a local field and D a central F-division algebra, there exists a maximal subfield K that is unramified.*

Proof. We know that D contains at least one subfield unramified over F, namely F itself, so we may choose K to be maximal among the collection of unramified subfields of D. It now suffices to show that K is actually a maximal subfield of D, i.e., that $C_D(K) = K$. We know that $C_D(K)$ is a central K-division algebra. Also, since K is maximal among the unramified subfields, every $K \subsetneq L \subseteq C_D(K)$ is ramified. Thus, by [L2, Lemma 31.1], $K = C_D(K)$. □

In particular we have that

$$\mathrm{Br}(F) = \bigcup_{\substack{K|F \\ \text{unramified}}} \mathrm{Br}(K|F).$$

We now inspect the group $\mathrm{Br}(K|F)$ for an unramified extension $K|F$.

It can be shown that every element of $\mathrm{Br}(K|F)$ can be represented by a *cyclic algebra* $(K/F, \sigma, a)$. A proof of this would require a long digression into the theory of cyclic algebras, which we skip here. See [L1, L2] for details, including an explanation of the notation. The upshot is that every $\alpha \in \mathrm{Br}(L|K)$ can be associated to an element $a \in K^\times$. Since $(K/F, \sigma, a) \otimes (K/F, \sigma, b) \cong (K/F, \sigma, ab)$ and $(K/F, \sigma, a)$ splits if and only if a is a norm (see Remark 2 on page 198 of [L2]), we actually get a homomorphism $\mathrm{Br}(K|F) \to F^\times/N_{K|F}(K^\times)$ given by $\alpha \mapsto aN_{K|F}(K^\times)$. This map is bijective, and so $\mathrm{Br}(K|F) \cong F^\times/N_{K|F}(K^\times)$.

This *norm residue group* behaves nicely in the local case. We quote [L2, Theorem 31.2′] as the following

Proposition 5.2.2. *Let $K|F$ be an unramified extension of local fields. Then the norm residue group $F^\times/N_{K|F}(K^\times)$ is cyclic of order $[K : F]$.*

In particular every relative Brauer group is cyclic. Now let ν denote the valuation on F, and consider a cyclic algebra $(K/F, \sigma, a)$ representing some element of $\mathrm{Br}(K|F)$.

If we set $k := \nu(a)$ and $n := [K : F]$, then we obtain a natural map

$$\operatorname{inv}_{K|F} : \operatorname{Br}(K|F) \to \mathbb{Q}/\mathbb{Z}$$

via $[(K/F, \sigma, a)] \mapsto \frac{k}{n} + \mathbb{Z}$. It turns out this map is well-defined, and induces an isomorphism from $\operatorname{Br}(K|F)$ to $(\frac{1}{n}\mathbb{Z})/\mathbb{Z}$; see [L2, Section 31.4].

Now since

$$\operatorname{Br}(F) = \bigcup_{\substack{K|F \\ \text{unramified}}} \operatorname{Br}(K|F)$$

we can define the *Hasse invariant* map $\operatorname{inv}_F : \operatorname{Br}(F) \to \mathbb{Q}/\mathbb{Z}$ as follows. Given $\alpha \in \operatorname{Br}(F)$ choose an unramified $K|F$ such that $\alpha \in \operatorname{Br}(K|F)$, and set $\operatorname{inv}_F(\alpha) := \operatorname{inv}_{K|F}(\alpha)$. It is a fact that this is a well-defined isomorphism. It is also very well-behaved under finite extensions. These statements are justified in [L2, Theorem 31.4], which we summarize in the following

Theorem 5.2.3. *Let F be a non-archimedean local field. The map $\operatorname{inv}_F : \operatorname{Br}(F) \to \mathbb{Q}/\mathbb{Z}$ is an isomorphism. If L is a finite extension of F and $f_{L|F} : \operatorname{Br}(F) \to \operatorname{Br}(L)$ is the map taking $[D]$ to $[D \otimes_F L]$, then we have $\operatorname{inv}_L \circ f_{L|F} = [L : F] \operatorname{inv}_F$. That is, a field extension in $\operatorname{Br}(F)$ corresponds to multiplication by $[L : F]$ in $\mathbb{Q}/\mathbb{Z}$.*

This is sufficient setup to turn our attention to the global case. Let F be a global field, and let S be the set of valuations on F. For each $\nu \in S$ let F_ν be the completion of F with respect to ν. For a central F-division algebra D, let $D_\nu = D \otimes_F F_\nu$. Let *sum* denote the map

$$\bigoplus_{\nu \in S} \operatorname{Br}(F_\nu) \to \mathbb{Q}/\mathbb{Z}$$

given by $sum((q_\nu)_\nu) = \sum_\nu q_\nu$, where we think of $\operatorname{Br}(F_\nu)$ as $\mathbb{Q}/\mathbb{Z}$ via the Hasse invariant, and q_ν denotes an element of $\mathbb{Q}/\mathbb{Z}$. Now, any element of $\operatorname{Br}(F)$ maps into $\bigoplus_{\nu \in S} \operatorname{Br}(F_\nu)$ via $[D] \mapsto ([D_\nu])_\nu$. In fact, this is a monomorphism, whose image is precisely the kernel of *sum* [R2, Section 6.2]. We summarize this result in the following proposition.

Proposition 5.2.4. *For a global field F, the Brauer group* $\mathrm{Br}(F)$ *is defined by the short exact sequence*

$$1 \to \mathrm{Br}(F) \to \bigoplus_{\nu \in S} \mathrm{Br}(F_\nu) \stackrel{sum}{\to} \mathbb{Q}/\mathbb{Z} \to 1.$$

In particular, the central F-division algebras are in one-to-one correspondence with sequences (q_ν) in $\mathbb{Q}/\mathbb{Z}$ such that $q_\nu = 0$ for all but finitely many ν and $\sum_\nu q_\nu = 0$ in $\mathbb{Q}/\mathbb{Z}$. For each central F-division algebra D, set $\mathrm{inv}_F(D) = (\mathrm{inv}_\nu(D_\nu))_\nu$, where $\mathrm{inv}_\nu := \mathrm{inv}_{F_\nu}$. Note that if ν is an archimedean valuation, then F_ν is either $\mathbb{C}$ or $\mathbb{R}$ and so $\mathrm{Br}(F_\nu)$ is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus for archimedean ν, $\mathrm{inv}_\nu(D_\nu)$ is either $0 + \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$. This will be important in Lemma 5.4.10.

5.3 Invariants of division algebras

As we've seen, every CSA A can be embedded in a matrix algebra over a field extension of F, in particular in $M_d(\overline{F})$. Thus it is reasonable to define the characteristic polynomial, trace, and norm of an element of A. There are some things to check, however, in particular the definitions should not depend on a choice of embedding.

We define a *representation* of A to be an F-algebra homomorphism $\rho : A \to M_n(K)$ for some n and for some extension K of F. Note that we do not require $K = F$. In the case that $n = d$ we will call ρ a *splitting representation*, à la [P, Section 16.1].

Definition 5.3.1. Let A be a CSA over F. Let $\rho : A \to M_n(K)$ be any representation. We define the *ρ-characteristic polynomial* on A to be $\chi_\rho(x, t) = \chi_{\rho(x)}(t)$ for $x \in A$. In particular we get the *ρ-trace* $\mathrm{trace}_\rho(x) = \mathrm{trace}(\rho(x))$ and *ρ-norm* $N_\rho(x) = \det(\rho(x))$.

We would like these to all be independent of K and ρ, at least in the case that ρ is a splitting representation. We establish the following lemma and proposition, both from Section 16.1 of [P]. Also see [L2, Section 29.F23].

Lemma 5.3.2. *Let A be a CSA of degree d over F. Let $\rho : A \to M_d(K)$ be a splitting representation and let $\lambda : A \to M_n(K)$ be any representation. Then $n = dm$ for some m and $\chi_\lambda = \chi_\rho^m$.*

Proof. First extend ρ and λ to K-algebra homomorphisms $\rho : A_K \to M_d(K)$ and $\rho : A_K \to M_n(K)$, using the universal property of tensor products. Since A splits over K we can in particular extend ρ such that it is an isomorphism of K-algebras, and λ so that it is injective. The injectivity tells us that d divides n, so $n = dm$ for some m.

Now define a new map $\tau : A_K \hookrightarrow M_n(K)$ that "enlarges" ρ, i.e., $\tau(z) = \rho(z) \otimes I_m$. Here we use the identification of $M_d(K) \otimes M_m(K)$ with $M_n(K)$. By the well-known Skolem-Noether Theorem [FD, Theorem 3.14; L2, Theorem 29.20; P, Theorem 12.6], λ and τ are conjugate, and so have the same characteristic polynomial. But $\chi_\tau(z,t) = \det(tI_n - \tau(z)) = \det(t(I_d \otimes I_m) - \rho(z) \otimes I_m) = \det((tI_d - \rho(z)) \otimes I_m) = \chi_\rho(z,t)^m$ for all $z \in A_K$. □

Proposition 5.3.3. *Let A be a CSA of degree d over F. Let $\rho : A \to M_d(K)$ and $\lambda : A \to M_d(L)$ be splitting representations. Then $\chi_\rho \in F[t]$ and $\chi_\rho = \chi_\lambda$.*

Proof. First choose a field E containing both K and L, as in the proof of [P, Proposition 16.1]. We can extend ρ and λ to maps into $M_d(E)$ without changing χ_ρ or χ_λ. But then by Lemma 5.3.2, χ_ρ and χ_λ are equal. This shows that the characteristic polynomial is independent of the choice of splitting representation. It remains to show that the coefficients are really in F.

We reference without proof the fact that A has some splitting field that is (finite) Galois over F. See [L1, Theorem 15.12] or [L2, Section 29.F22]. Thus we may assume without loss of generality that $K|F$ is Galois. Let $\sigma \in \mathrm{Gal}(K|F)$. By considering entry-wise action, we can think of σ as an automorphism of $M_d(K)$. Then of course $\sigma \circ \rho$ is also a splitting representation. By Lemma 5.3.2, $\chi_\rho = \chi_{\sigma\circ\rho}$. The latter equals $\sigma(\chi_\rho)$, where σ acts on polynomials via their coefficients, fixing t. Thus σ fixes all the coefficients of χ_ρ and indeed $\chi_\rho \in F[t]$. □

The upshot is that an element of a CSA has a well-defined characteristic polynomial with coefficients in F, and thus a trace and norm taking values in F.

Definition 5.3.4. We define the *reduced characteristic polynomial*, *reduced trace*, and *reduced norm* on the central simple F-algebra A to be $\chi = \chi_\rho$, $\mathrm{trace} = \mathrm{trace}_\rho$, and $N = N_\rho$ respectively for any splitting representation ρ of A. We define the *norm-1 group* of A to be $\mathrm{SL}_1(A) = \{x \in A \mid N(x) = 1\}$.

Note that if A is already split over F, then the identity map is a splitting representation and $\mathrm{SL}_1(A) = \mathrm{SL}_d(F)$. In the same vein, if K splits A then the natural embedding $A \hookrightarrow M_d(K)$ is a splitting representation and so $\mathrm{SL}_1(A) \leq \mathrm{SL}_d(K)$. Of course, the groups $\mathrm{SL}_d(K)$ are standard examples of Chevalley groups, and so norm-1 groups can be made to act on buildings. Also, the multiplicative group $A^\times$ can be thought of as a subgroup of $\mathrm{GL}_d(K)$, which corresponds to the same spherical building as $\mathrm{SL}_d(K)$; see [AB2, Section 6.5].

5.4 Not weakly transitive actions

Throughout this section, D is a central F-division algebra of degree $d > 2$. For any $K|F$ splitting D, we can consider $D^\times$ as a subgroup of $(D_K)^\times \cong \mathrm{GL}_d(K)$, and can thus refer to the action of $D^\times$ on an appropriate spherical building Δ. We will later also consider the action of $\mathrm{SL}_1(D)$ on Δ, and also on the appropriate affine building for certain K. As usual N denotes the group of monomial matrices in $\mathrm{GL}_d(K)$ and T denotes the group of diagonal matrices. Note that the spherical Weyl group is $N/T \cong S_d$.

It is actually quite easy to see that $D^\times$ fails to act weakly transitively on Δ. We will get this out of the way now, and the rest of this section will inspect the question of how extreme this failure is.

Theorem 5.4.1. *Think of* $D^\times \leq \mathrm{GL}_d(K)$ *via some splitting representation* $D \hookrightarrow M_d(K)$. *Then there exists a coset* X *in* N/T *such that for any* $g \in \mathrm{GL}_d(K)$, $gD^\times g^{-1} \cap X = \emptyset$.

Proof. Let $w \in S_d$ be any permutation that fixes precisely one element of $\{1, \ldots, d\}$. Since $d > 2$ such a w exists, and without loss of generality we may suppose that fixed point is d. Let X denote the coset in N/T corresponding to w, so X consists only of matrices of the form $A = \mathrm{diag}(B, c)$, where B is a $(d-1)$-by-$(d-1)$ monomial matrix with zeros on its diagonal.

Now let $z \in D^\times$, $g \in \mathrm{GL}_d(K)$ and suppose $gzg^{-1} \in X$. Say $gzg^{-1} = A = \mathrm{diag}(B, c)$. Clearly z has trace c, so by Proposition 5.3.3, $c \in F$. In particular this means that $c \in D$, and so also $z - c \in D$. Thus, the matrix $g(z-c)g^{-1}$ must either be zero or invertible. Since $gzg^{-1} = A$ and $c = cI_d$ is central in $\mathrm{GL}_d(K)$, we have $g(z-c)g^{-1} = A - cI_d = \mathrm{diag}(B - cI_{d-1}, 0)$. This is not invertible since the bottom row consists of zeros. Also, if $B - cI_{d-1}$ is zero then by construction of B, we have $c = 0$, which contradicts the assumption that z is invertible. □

Corollary 5.4.2. *Let* F *be any field,* D *a central* F*-division algebra of degree* $d > 2$. *Consider* $D^\times \leq \mathrm{GL}_d(K)$ *and* $\mathrm{SL}_1(D) \leq \mathrm{SL}_d(K)$ *for splitting field* $K|F$. *If* Δ *is the spherical building for* $\mathrm{GL}_d(K)$ *and* $\mathrm{SL}_d(K)$, *then the action of* $D^\times$ *on* Δ *is not weakly transitive. If* K *is complete with respect to a discrete valuation and* Δ_a *is the affine building for* $\mathrm{SL}_d(K)$ *then the action of* $\mathrm{SL}_1(D)$ *on* Δ_a *is not weakly transitive, and thus not strongly transitive with respect to any apartment system.*

Proof. This is immediate from Lemma 2.4.3 and Theorem 5.4.1. □

Example 5.4.3. The method used to prove Theorem 5.4.1 becomes especially clear in the $d = 3$ case. Suppose gzg^{-1} equals the matrix

$$A = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & c \end{pmatrix},$$

for $a, b, c \in K^\times$. Then z has reduced trace c, so $c \in F$, and $z - c \in D$. But since $gzg^{-1} = A$, we have $g(z-c)g^{-1} = A - cI_3$, which equals

$$\begin{pmatrix} -c & a & 0 \\ b & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is neither zero nor invertible, a contradiction.

The result of Corollary 5.4.2 is ultimately not surprising. A. Rapinchuk has a short, elegant proof that proves this result in much more generality, in particular the conclusion holds for any non-split algebraic group that doesn't split over a quadratic extension [R1]. However, as we will see the failure of $D^\times$ and $\mathrm{SL}_1(D)$ to act weakly transitively is especially stark. The rest of this section is devoted to justifying the claim that this failure is "dramatic." Namely, we will exhibit a large collection of cosets X in N/T that have empty intersection with any conjugate of $D^\times$. From now on we equate N/T with S_d, and may refer to cosets by their cycle decomposition. This next proof is a direct generalization of the "unique fixed point" situation from Theorem 5.4.1.

Theorem 5.4.4. *Let X be any coset in N/T other than a d-cycle, with cycle decomposition featuring a unique cycle of minimum length, where a 1-cycle represents a fixed point. Let $z \in D^\times$, $g \in \mathrm{GL}_d(K)$. Then $gzg^{-1} \notin X$.*

Proof. Suppose $gzg^{-1} = A \in X$ for some g, z. By adjusting g as necessary we may assume A is of the form $A = \mathrm{diag}(B_1, \ldots, B_r, C)$ where

$$B_i = \begin{pmatrix} 0 & {}_1b_i & 0 & \cdots & 0 \\ 0 & 0 & {}_2b_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & {}_{k_i-1}b_i \\ {}_{k_i}b_i & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & c_1 & 0 & \cdots & 0 \\ 0 & 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & c_{\ell-1} \\ c_\ell & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $\ell < k_i$ for all $1 \leq i \leq r$. Let $b_i = ({}_1b_i) \cdots ({}_{k_i}b_i)$ for each i and $c = c_1 \cdots c_\ell$.

Consider the characteristic polynomial $\chi_A(t) = \chi_{B_1}(t) \cdots \chi_{B_r}(t)\chi_C(t)$. Of course $\chi_A(t)$ equals $\chi_z(t)$ and so by Proposition 5.3.3 its coefficients lie in F. The constant term is clearly $\pm b_1 \cdots b_r c$, so $b_1 \cdots b_r c \in F$. The t^ℓ term is also of interest. Since $\chi_{B_i}(t) = t^{k_i} - b_i$ and $\chi_C(t) = t^\ell - c$, and since $\ell < k_i$ for all i, we see that the t^ℓ term of $\chi_A(t)$ must be $\pm b_1 \cdots b_r t^\ell$. Thus, $b_1 \cdots b_r \in F$ and we conclude that $c \in F$.

In particular $z^\ell - c \in D$, so $A^\ell - cI_d$ is either invertible or is the zero matrix. Since $C^\ell = cI_\ell$, it is impossible that $A^\ell - cI_d$ can be invertible. But $\ell < k_i$ for all i, so the $B_i^\ell - cI_{k_i}$ are all nonzero. We conclude that in fact $gzg^{-1} \notin X$ for any g, z. □

Theorem 5.4.4 discounts any coset featuring a "unique smallest" cycle, and we can also discount cosets featuring a "big" cycle, as the next theorem will show. First we need a simple lemma.

Lemma 5.4.5. *The matrix*

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_{d-1} \\ a_d & 0 & 0 & \cdots & 0 \end{pmatrix}$$

has minimal polynomial $t^d - a$ *where* $0 \neq a := a_1 \dots a_d$.

Proof. Clearly A satisfies this polynomial. Also, a quick calculation shows that $I_d, A, A^2, \dots, A^{d-1}$ are linearly independent, so the minimal polynomial of A cannot have degree less than d. We conclude that $t^d - a$ is the minimal polynomial. □

Theorem 5.4.6. *Let X be any coset in N/T whose cycle decomposition features a k-cycle, with $d/2 < k < d$. Let $z \in D^\times$, $g \in \mathrm{GL}_d(K)$. Then $gzg^{-1} \notin X$.*

Proof. Suppose $gzg^{-1} = A \in X$ for some g, z. By adjusting g as necessary we may assume A is of the form $A = \mathrm{diag}(A', B)$ where

$$A' = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_{k-1} \\ a_k & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and B is a $d-k$ by $d-k$ matrix. Set $a = a_1 \cdots a_k$ and $j = d-k$, so $0 < j < k$. Then the characteristic polynomial $\chi_A(t)$ equals $(t^k - a)\chi_B(t)$, and $\chi_B(t)$ has degree j. As before, the coefficients of $\chi_A(t)$ lie in F. In particular the coefficient on the t^j term is in F. But this term must be $-at^j$, since $k > j$ and $\chi_B(t)$ is monic of degree j. Thus, $a \in F$. Also note that the minimal polynomial of z must have degree dividing d, by Proposition 5.1.11, but by Lemma 5.4.5 it also must have degree at least k. Since $k > d/2$ we conclude that $\chi_A(t)$ is the minimal polynomial of z.

Now, since $a \in F$, $z^k - a \in D$ and $g(z^k - a)g^{-1} = A^k - aI_d$ is either 0 or invertible. It's clearly not invertible, we in fact $z^k - a = 0$. But $\chi_A(t)$ is the minimal polynomial of z and $k < d$ so this is impossible. □

We can generalize these two situations simultaneously with another criterion we call "lonely cycles." Let σ be a permutation with cycle decomposition $\sigma = \sigma_1 \cdots \sigma_m$ for each σ_i a k_i-cycle (as usual we account for fixed points with "1-cycles"). We will call σ_i *lonely* if k_i satisfies the following two conditions:

1. For any $\epsilon_1, \ldots, \epsilon_r \in \{0, 1\}$, if $k_i = \sum_{j=1}^{r} \epsilon_j k_j$ then $\epsilon_j = 0$ for all $j \neq i$.
2. If k_i is maximal among all k_j then k_i does not divide d.

For example the cycle (1 2 3) is lonely in the permutation (1 2 3)(4 5)(6 7) since 3 cannot be written as a sum involving 2 and 2, and 3 does not divide 7, but the cycle (4 5) is *not* lonely since it *can* be written as a sum involving 3 and 2, namely 2=2.

Note that the second condition in particular ensures that d-cycles themselves are not lonely. However, the d-cycles are still an interesting case, which we will consider later. For now we claim that any permutation featuring a lonely cycle cannot be represented by any conjugate of $D^\times$.

Theorem 5.4.7. *Let X be any coset in N/T whose cycle decomposition features a lonely k-cycle. Let $z \in D^\times$, $g \in \mathrm{GL}_d(K)$. Then $gzg^{-1} \notin X$.*

Proof. Suppose $gzg^{-1} = A \in X$ for some g, z. Let the cycle type of X be $j_1, \ldots, j_r, k$, so no collection of distinct j_i can sum to k. By adjusting g as necessary we may assume A is of the form $A = \mathrm{diag}(B_1, \ldots, B_r, C)$ where

$$B_i = \begin{pmatrix} 0 & {}_ib_1 & 0 & \cdots & 0 \\ 0 & 0 & {}_ib_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & {}_ib_{j_i-1} \\ {}_ib_{j_i} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & c_1 & 0 & \cdots & 0 \\ 0 & 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & c_{k-1} \\ c_k & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Set $b_i = ({}_ib_1) \cdots ({}_ib_{j_i})$ for each i and set $c = c_1 \cdots c_k$. Then the characteristic polynomial $\chi_A(t)$ equals $\chi_{B_1}(t) \cdots \chi_{B_r}(t)\chi_C(t)$. The coefficients of $\chi_A(t)$ lie in F, so in particular the coefficient on the t^k term is in F. But this term must be $\pm(b_1 \cdots b_r)t^k$, since $\chi_{B_i}(t) = t^{j_i} - b_i$, $\chi_C(t) = t^k - c$, and no collection of j_i can sum to k. Thus, $b_1 \cdots b_r \in F$. Of course the constant term of $\chi_A(t)$ is $\pm b_1 \cdots b_r c$, so we also know that $c \in F$.

In particular, $z^k - c \in D$ and $g(z^k - c)g^{-1} = A^k - cI_d$ is either 0 or invertible. It's clearly not invertible, so in fact $z^k - c = 0$. This immediately implies that each j_i

divides k, and so k is maximal among $j_1, \ldots, j_r, k$. Then by the second criterion in the definition of lonely cycles, k does not divide d. However, by the proof of Lemma 5.4.5 the minimal polynomial of z cannot have degree smaller than k, so in fact $z^k - c$ is the minimal polynomial of z. Since the degree of the subfield $F(z)$ in D must divide d by Proposition 5.1.11, this is a contradiction. □

As an example, consider the permutation $\sigma = (1\ 2\ 3)(4\ 5)(6\ 7)$ from earlier. Here $(1\ 2\ 3)$ is lonely, so no conjugate of $D^\times$ can represent σ. Note that σ does not feature a "big" cycle, nor a "unique smallest" cycle, so we have really gained ground by considering lonely cycles. Also note that any big or unique smallest cycle is clearly lonely, so this is really a generalization.

We now discuss the case of d-cycles with regard to $\mathrm{SL}_1(D)$.

Theorem 5.4.8. *Suppose d is not a power of 2. For any $z \in \mathrm{SL}_1(D)$, $g \in \mathrm{GL}_d(K)$, gzg^{-1} cannot be of the form*

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_{d-1} \\ a_d & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Proof. Suppose $gzg^{-1} = A$. By Lemma 5.4.5, z has minimal polynomial $t^d - a$ where $a = a_1 \ldots a_d$. Since z has norm 1, $a = (-1)^{d+1}$ so z has minimal polynomial $t^d + (-1)^d$. Since D is a division algebra and $z \in D$, $t^d + (-1)^d$ is irreducible.

This clearly is impossible if d is odd, since 1 is a root of $t^d - 1$, so assume d is even. Let $d = 2^e m$ for odd m. Then $t^{2^e} + 1$ divides $t^d + 1$, so by irreducibility we know that d must be a power of 2. But we discounted this possibility in the hypothesis. □

In particular no conjugate of $\mathrm{SL}_1(D)$ can represent a d-cycle in S_d (assuming that d is not a power of 2).

As seen in [AB1] the case when $F = \mathbb{Q}$ is already quite interesting, and it turns out that in this case we can achieve the above result regarding d-cycles with no restriction on d. By Theorem 4.1.2 and Example 4.3.1, all representatives in N of a d-cycle in S_d have order d or $2d$. Specifically if d is odd they all have order d and if d is even they all have order $2d$. To show that no conjugate of $\mathrm{SL}_1(D)$ represents a d-cycle, it thus suffices to show that D does not contain certain roots of unity.

Theorem 5.4.9. *If d is odd then $D^\times$ contains no elements of order d, and if d is even then $D^\times$ contains no elements of order $2d$.*

Before proving the theorem we will need the following lemma. Recall that $D_p := D \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Lemma 5.4.10. *Suppose $d > 2$ is a power of a prime, $d = s^m$. Then there exists an odd prime p such that D_p is a division algebra.*

Proof. Since D has index $d = s^m$ we know $[D]$ has order s^m in $\mathrm{Br}(\mathbb{Q})$, as discussed in [R2, Section 5.4.4]. Thus by the construction of the Hasse invariant map we know that there exists some valuation ν such that $\mathrm{inv}_\nu(D)$ has reduced denominator s^m. Also, any archimedean valuation yields an invariant of order 1 or 2, so we can choose ν non-archimedean, i.e., associated to some prime p. Since the invariants sum to zero, we actually know that there are at least *two* p_1, p_2 such that $\mathrm{inv}_{p_1}(D)$ and $\mathrm{inv}_{p_2}(D)$ have denominator s^m. Thus, we can choose an *odd* prime p such that D_p has index d and so is a division algebra. □

Proof of Theorem 5.4.9. First suppose $d = 2^e m$ for odd $m > 1$, i.e., d is not a power of 2. Suppose D contains a primitive r_{th} root of unity, ζ_r. Then D contains the subfield $L := \mathbb{Q}(\zeta_r)$ of degree $\varphi(r)$, so by Corollary 5.1.11 $\varphi(r)$ must divide d. If $r = d$ and d is odd, then $\varphi(r)$ cannot divide d, since $\varphi(r)$ is even for $r > 2$. If $r = 2d$ and d is even, then $\varphi(r) = \varphi(2^{e+1}m) = 2^e\varphi(m)$. Since $m > 2$, in this case $2^{e+1}|\varphi(r)$ so $\varphi(r)$ cannot divide d.

The only remaining case we care about is when $d = 2^e$. In this case $\varphi(2d) = d$, so $L = \mathbb{Q}(\zeta_{2d})$ is a maximal subfield of D and D splits over L, by Proposition 5.1.12. For prime p, let $L_p := \mathbb{Q}_p(\zeta)$ where ζ is a primitive $2d_{th}$ root of unity. Note that there is (at least one) embedding of $\mathbb{Q}(\zeta_{2d})$ into L_p, sending ζ_{2d} to ζ. Since D splits over L, we have that

$$\begin{aligned} D_p \otimes_{\mathbb{Q}_p} L_p &= (D \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} L_p = D \otimes_{\mathbb{Q}} L_p \\ &= (D \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{2d})) \otimes_{\mathbb{Q}(\zeta_{2d})} L_p = M_d(\mathbb{Q}(\zeta_{2d})) \otimes_{\mathbb{Q}(\zeta_{2d})} L_p \\ &= M_d(L_p), \end{aligned}$$

so D_p splits over L_p. By Proposition 5.1.12 again, for each p the index of D_p divides the degree of L_p. We claim that $[L_p : \mathbb{Q}_p] < d$ for any odd prime p. This will contradict Lemma 5.4.10, proving the theorem. By [S2, Section 4.4, Corollary 1], $[L_p : \mathbb{Q}_p]$ equals the order of p in $(\mathbb{Z}/2^{e+1}\mathbb{Z})^\times$. Since $e > 1$, this group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{e-1}\mathbb{Z}$, and so p has order strictly less than 2^e. Thus, $[L_p : \mathbb{Q}_p] < d$. □

This is an interesting result in its own right, and also proves that no conjugate of $\mathrm{SL}_1(D)$ can represent any d-cycle in $N/T \cong S_d$, even in the case that d is a proper power of 2. There was nothing special about $F = \mathbb{Q}$, except for the fact that $[\mathbb{Q}(\zeta_r) : \mathbb{Q}] = \varphi(r)$. It is easy to see that these results hold for global fields F other than $\mathbb{Q}$, provided the relevant cyclotomic extensions have the same degree as in the $\mathbb{Q}$ case.

Remark 5.4.11. As mentioned earlier, the actions on the buildings in all these examples are "not even close" to being weakly transitive. To justify this statement, let us estimate how many elements of S_d feature a "big" cycle, i.e., a k-cycle with $d/2 < k < d$. The number of k-cycles in S_d is $\dfrac{d!}{(d-k)!k}$. Thus, the number of permutations that *feature* a k-cycle is $\dfrac{d!}{(d-k)!k}(d-k)! = \dfrac{d!}{k}$. Since $d/2 < k$ there is no risk that we have over-counted; a permutation cannot contain more than one k-cycle. Now we calculate the number $b(d)$ of elements of S_d featuring a "big" cycle.

This is

$$b(d) = \sum_{d/2<k<d} \frac{d!}{k} = d! \sum_{d/2<k<d} \frac{1}{k}$$
$$= d! \left(H(d-1) - H\left(\left\lfloor \frac{d}{2} + 1 \right\rfloor \right) \right)$$

where

$$H(m) := \sum_{k=1}^{m} \frac{1}{k}$$

is the m_{th} harmonic number. For large m, $H(m)$ grows like $\ln m$. Thus for large d,

$$b(d) \approx d!(\ln d - \ln d/2) = d! \ln 2.$$

Since $\ln 2 \approx .7$, we see that for large enough d about 70% of the elements of S_d feature a "big" cycle. Since conjugates of $D^\times$ cannot represent any elements of this form, we see that the failure to act weakly transitively is indeed quite dramatic.

One could try to estimate the number of permutations featuring a "lonely" cycle, but it is doubtful that the estimate of 70% would change much. Similarly, ruling out the d-cycles in case either d is not a power of 2 or F is an appropriate global field will not affect this estimate, since for large d the d-cycles make up an infinitesimal percent of all permutations. It would be interesting to see whether as d grows, the percent of cosets that cannot be represented by $D^\times$ actually approaches 100%. M. Kassabov has suggested that it seems likely $D^\times$ can only represent a very narrow class of elements of S_d, namely those whose cycle decomposition features cycles all of the same size. Proving this for arbitrary base field F would require new methods however. For example in S_8, we have at present no way of ruling out the possibility that $D^\times$ represents a 4-2-2 cycle.

We have barely mentioned affine buildings, but of course since the action of $\mathrm{SL}_1(D)$ is not weakly transitive in the spherical case this also holds in the affine case. We now claim that this action may nonetheless be Weyl transitive. Suppose D is a central

F-division algebra for some topological field F, dense in a field $K|F$ splitting D. For example we could consider F global and $K = F_\nu$ for F_ν splitting D. (In fact all but finitely many ν will work.) Suppose further that $\operatorname{char} F$ does not divide the degree d of D. Let $G = \mathrm{SL}_1(D)$ and $G_K = \mathrm{SL}_1(D_K) \cong \mathrm{SL}_d(K)$. The following lemma is an analog of [AB1, Lemma 2.1].

Lemma 5.4.12. *The group G is dense in G_K.*

Proof. For any $x \in D^\times$, since $N(x) \in F$ we know $x^d/N(x)$ has norm 1. Thus the closure $\overline{G}$ of G in G_K contains all elements of the form $y^d/N(y)$ for $y \in D_K^\times$. In particular, considering all y of norm 1, we get that $G_K^d \subseteq \overline{G}$. It now suffices to show that $G_K^d = G_K$, that is that $\mathrm{SL}_d(F_K)$ is generated by d_{th} powers. Indeed $\mathrm{SL}_d(F_K)$ is generated by elementary matrices, and since $\operatorname{char} F$ does not divide d these are all d_{th} powers. $\square$

Together with Lemma 2.4.2, and the fact that affine chamber stabilizers in $\mathrm{SL}_d(K)$ are open, this proves that the action is indeed Weyl transitive. Between this section and Section 4.2, we have established a broad class of examples of group actions on affine buildings that are Weyl transitive but not strongly transitive with respect to any apartment system. In particular the examples in the present section are, as discussed, very far from being weakly transitive.

5.5 Action on the fundamental apartment

As we have seen, the action of $D^\times$ on the relevant building is far from weakly transitive. We now analyze a related question, namely, we know that $D^\times$ does not represent the whole Weyl group, but how much exactly *does* it represent? Thinking of the Weyl group as the stabilizer modulo the fixer of an *arbitrary* apartment, this problem seems difficult. If we consider the *fundamental* apartment however, we can achieve a precise description, at least for “most” division algebras D.

Let D be a division algebra of degree d, and suppose D has a maximal subfield K that is Galois over the center F of D. (Division algebras lacking this property exist, but are difficult to construct; the first examples were discovered only in the 1970's, by Amitsur [A]. Also for certain F, e.g., F global, all F-division algebras have this property, and so really our restriction is not very severe.) Let $\Gamma = \mathrm{Gal}(K|F)$. Using the notion of a crossed product, we can construct a (right) K-basis $\{x_\sigma\}_{\sigma\in\Gamma}$ for D, with multiplication given by $x_\sigma x_\tau = x_{\sigma\tau}a_{\sigma,\tau}$ for some $a_{\sigma,\tau} \in K^\times$ and $bx_\sigma = x_\sigma\sigma(b)$ for $b \in K$; see [L2, Theorem 30.1.1] for details. The action of $D^\times$ by left multiplication on the d-dimensional right K-vector space D is K-linear, and so induces an injective homomorphism $D^\times \hookrightarrow \mathrm{GL}_d(K)$. We will suppress this map and just think of $D^\times$ as a subgroup of $\mathrm{GL}_d(K)$.

Let Δ be the standard spherical building for $\mathrm{GL}_d(K)$. Let Σ_0 denote the fundamental apartment, with stabilizer N the group of monomial matrices and fixer T the group of diagonal matrices. Note that the Weyl group $W = N/T$ is isomorphic to S_d. Consider the action of the subgroup $D^\times$ on Δ. We claim that we can completely describe the subgroup $W_{D^\times} := \mathrm{Stab}_{D^\times}(\Sigma_0)/\,\mathrm{Fix}_{D^\times}(\Sigma_0)$ of W. Define $\phi : \Gamma \to W_{D^\times}$ to be $\sigma \mapsto x_\sigma \,\mathrm{Fix}_{D^\times}(\Sigma_0)$. Since $a_{\sigma,\tau} \in K^\times$ fixes Σ_0, ϕ is a homomorphism. Also, if x_σ fixes Σ_0 then in particular $x_\sigma x_\sigma$ lies in the K-span of x_σ, implying that $\sigma^2 = \sigma$, so $\sigma = 1$. So ϕ is injective. Lastly, since our choice of standard basis is unique up to K-span [L2], ϕ is a canonical map.

Proposition 5.5.1. *The map ϕ defined above is surjective, and so is a canonical isomorphism.*

Proof. Suppose that $z \in D^\times$ is a monomial matrix. Then for any basis element x_τ, zx_τ is again in the K-span of some basis element. Say $z = \sum_{\sigma\in\Gamma} x_\sigma b_\sigma$, so $zx_\tau = \sum_{\sigma\in\Gamma} x_{\sigma\tau}a_{\sigma,\tau}\tau(b_\sigma)$. For this to lie in the span of a single basis element, we must have that all but one of the b_σ are in fact zero. Thus z is of the form $x_\sigma b$ for some $b \in K^\times$, $\sigma \in \Gamma$. We conclude that the stabilizer of Σ_0 in $D^\times$ is made up precisely of elements of this form. Of course $x_\sigma b \operatorname{Fix}_{D^\times}(\Sigma_0) = x_\sigma \operatorname{Fix}_{D^\times}(\Sigma_0)$, and so indeed ϕ is surjective. $\square$

This yields a precise description of the action of $D^\times$ on Σ_0, given by the subgroup $W_{D^\times} \cong \Gamma$ of W. In fact the map $\phi : \Gamma \hookrightarrow W_{D^\times}$ is explicit. Since $W = S_d$ and $|\Gamma| = d$ we can think of W as the symmetric group on the set Γ. Then ϕ is just induced by the left multiplication of Γ on itself. If we fix $\sigma \in \Gamma$ with order ℓ and choose $\tau_1, \ldots, \tau_r$ such that $\Gamma = \bigcup_{i=1}^{r} \langle\sigma\rangle\tau_i$, where $r = d/\ell$, then $\phi(\sigma)$ has cycle decomposition $(\tau_1\ \sigma\tau_1\ \sigma^{\ell-1}\tau_1)\cdots(\tau_r\ \sigma\tau_r\ \sigma^{\ell-1}\tau_r)$, which in particular consists only of ℓ-cycles. This verifies the suggestion of M. Kassabov that $D^\times$ should only be able to represent $w \in W$ if the cycle decomposition of w features cycles all of the same length, at least for the case of the fundamental apartment.

As an aside, we note that the building and fundamental apartment depend on the choice of K, and so the fact that this description depends on Γ is not surprising. That is, if D contains some other Galois maximal subfield K' with Galois group $\Gamma' \ncong \Gamma$, then the resulting building and fundamental apartment are different, and so $W_{D^\times}$ will be different. It would thus be most precise to use the notation $W_{D^\times,K,\Sigma_0}$ but for brevity we will just write $W_{D^\times}$.

Thanks to the description of $W_{D^\times}$ we can also say something about the action of $\mathrm{SL}_1(D)$ on Σ_0. Namely, $\mathrm{SL}_1(D)$ represents $\phi(\sigma)$ in $W_{D^\times}$ if and only if there exists $b \in K$ such that $x_\sigma b$ has reduced norm 1. This in turn will happen if and only if the reduced norm of x_σ is already a Galois norm of something in K. In general the question of whether the reduced norm of x_σ is a Galois norm is a difficult one, and so the precise determination of $W_{\mathrm{SL}_1(D)} := \operatorname{Stab}_{\mathrm{SL}_1(D)}(\Sigma_0)/\operatorname{Fix}_{\mathrm{SL}_1(D)}(\Sigma_0)$ is difficult. We now specialize to the case of cyclic algebras and see that already this question is at least as difficult as a well-known problem that remains unsolved in many cases.

Let D be a cyclic algebra $D = (K/F, \sigma, a)$ with $\mathrm{Gal}(K|F) = \langle\sigma\rangle$ and the above standard K-basis now given by $x_{\sigma^i} = x^i$ for $0 \le i \le d-1$, where $x := x_\sigma$ and $x^d = a \in F^\times$; for details about cyclic algebras see [L1, Section 14; L2, Chapter 31].

In particular for $0 \leq i, j \leq d-1$, a_{σ^i,σ^j} is 1 if $i+j < d$ and is a otherwise. This allows us to explicitly calculate the reduced norm of the basis elements, namely x^i has reduced norm $(-1)^{i(d-1)}a^i$ for each i. Thus for each i, $\phi(\sigma^i)$ is represented by $\mathrm{SL}_1(D)$ if and only if $(-1)^{i(d-1)}a^i$ is a Galois norm of something in $K^\times$.

Consider the (σ-dependent) isomorphism $\mathrm{Br}(K|F) \to F^\times/N(K^\times)$ under which $[D] \mapsto aN(K^\times)$, described in [L2, Theorem 30.4.4]. Thanks to this isomorphism we see that for any i, a^i is a norm if and only if $e(D)|i$, where $e(D)$ is the order of $[D]$ in $\mathrm{Br}(K|F)$. This shows that determining $W_{\mathrm{SL}_1(D)}$ is essentially equivalent to determining $e(D)$, and in general it is an open question to determine $e(D)$ for arbitrary D. We now consider the global case, where we can say much more, and in particular can precisely calculate $W_{\mathrm{SL}_1(D)}$.

Suppose F is global. Then D is automatically cyclic, say $D = (K/F, \sigma, a)$, and $e(D) = d$. Both of these facts are results of the well-known Albert-Brauer-Hasse-Noether theorem [R2; S1, Theorem 10.7(a)]. If $\mathrm{SL}_1(D)$ represents $\phi(\sigma^i)$, then the element $(-1)^{i(d-1)}a^i$ is a Galois norm, and so its square a^{2i} is as well. This tells us that i can only possibly be d, or $d/2$ if d is even. The $i = d$ case corresponds to the trivial Weyl group element, which is of course in $W_{\mathrm{SL}_1(D)}$. Suppose now that $i = d/2$. Since a^i is not a Galois norm, $(-1)^{i(d-1)}$ must be -1, i.e., i is odd and d is congruent to 2 mod 4. We now have a complete characterization of $W_{\mathrm{SL}_1(D)}$ for the case when F is global, described in the following:

Proposition 5.5.2. *Let F be a global field. Let $D = (K/F, \sigma, a)$ be as above, with degree d. If d is not congruent to 2 mod 4, or if it is but $-a^{d/2}$ is not in $N_{K|F}(K^\times)$, then $W_{\mathrm{SL}_1(D)} = \{1\}$. If d is congruent to 2 mod 4 and $-a^{d/2}$* is *in $N_{K|F}(K^\times)$, then $W_{\mathrm{SL}_1(D)} = \{1, \phi(\sigma^{d/2})\}$.* □

We observe that when $d = 2$ this proposition says that $\mathrm{Stab}_{\mathrm{SL}_1(D)}(\Sigma_0)$ acts transitively on $\mathcal{C}(\Sigma_0)$ if and only if $-a \in N_{K|F}(K^\times)$. In [AB1], it is shown that if $d = 2$ and $F = \mathbb{Q}$ then $\mathrm{SL}_1(D)$ acts weakly transitively if and only if $-1 \in D^2$. The conditions $-a \in N_{K|F}(K^\times)$ and $-1 \in D^2$ would thus seem to be related. Indeed if $-a \in N_{K|F}(K^\times)$, say $-a = b\sigma(b)$ for $b \in K$, then $(xb^{-1})^2 = x^2\sigma(b^{-1})b^{-1} = a(-a)^{-1} = -1$ so -1 is a square. The converse need not be true, since $\mathrm{SL}_1(D)$ can act weakly transitively without doing so via Σ_0. For instance if $K = \mathbb{Q}(i)$ and $a = -3$ then of course $-1 \in D^2$ but $-a = 3$ is not a Galois norm. In this case then $\mathrm{SL}_1(D)$ acts weakly transitively on Δ via some apartment other than Σ_0.

We conclude with a few words regarding affine buildings. Let F be a global field and $D = (K/F, \sigma, a)$ as above, with degree d. For technical reasons we assume the characteristic of F does not divide d. Suppose K is embedded in F_ν for some non-archimedean valuation ν of F. Let F_ν have valuation ring R and residue field k. Let Δ_a be the affine building on which $\mathrm{SL}_d(F_\nu)$ acts strongly transitively with respect to $\overline{\mathcal{A}}$, and let X_0 denote the fundamental affine apartment.

Let W, N, and T be the standard spherical data for $\mathrm{SL}_d(F_\nu)$, and let W_a, T_a be the standard affine data. Then $W_a = N/T_a$ and $W = N/T$, so if $Q := T/T_a$ we have the short exact sequence

$$1 \to Q \to W_a \to W \to 1.$$

In fact the inclusion $W \hookrightarrow W_a$ provides a splitting, and we get $W_a = W \ltimes Q$ [AB2, Section 6.9.2; B3, Section 6.2.1]. We call Q the group of *translations*; note that $Q = T/T_a \cong (F_\nu^\times/R^\times)^{d-1} \cong \mathbb{Z}^{d-1}$. We claim that $\mathrm{SL}_1(D)$ represents every element of Q, i.e., Q is contained in $(W_a)_{\mathrm{SL}_1(D)} := \mathrm{Stab}_{\mathrm{SL}_1(D)}(X_0)/\,\mathrm{Fix}_{\mathrm{SL}_1(D)}(X_0)$.

First note that T_a is open in T, since $T_a = T_d(R)$ and $T = T_d(F_\nu)$. It thus suffices to show that $\mathrm{SL}_1(D) \cap T$ is dense in T. By construction, if $z \in \mathrm{SL}_1(D)$ is diagonal then as a matrix it looks like $\mathrm{diag}(b, \sigma(b), \ldots, \sigma^{d-1}(b))$ for some $b \in K^\times$, and $N_{K|F}(b) = 1$. Thus, as an element of D, $z = b$. Also, for any $x \in K$, if $b = x/\sigma(x)$ then clearly $N_{K|F}(b) = 1$. We want to show that the collection of matrices $\mathrm{diag}(b, \sigma(b), \ldots, \sigma^{d-1}(b))$ with $b = x/\sigma(x)$ as above is dense in T. Let $\{e_1, \ldots, e_d\}$ be an F-basis of K, and for each i, j set $e_j^{(i)} = \sigma^i(e_j)$. Let $\tilde{\sigma} : K \otimes_F F_\nu \to K \otimes_F F_\nu$ be the map induced by $\sigma : K \to K$ and $id_{F_\nu} : F_\nu \to F_\nu$. Consider $b = x/\sigma(x)$ where

$x = x_1e_1 + \cdots + x_de_d$ for $x_j \in F$, so

$$b = \frac{x_1e_1 + \cdots + x_de_d}{x_1e_1^{(1)} + \cdots + x_de_d^{(1)}}.$$

The matrix $\operatorname{diag}(b, \sigma(b), \ldots, \sigma^{d-1}(b))$ looks like

$$\operatorname{diag}\left(\frac{x_1e_1 + \cdots + x_de_d}{x_1e_1^{(1)} + \cdots + x_de_d^{(1)}}, \ldots, \frac{x_1e_1^{(d-1)} + \cdots + x_de_d^{(d-1)}}{x_1e_1^{(d)} + \cdots + x_de_d^{(d)}}\right).$$

The closure of the set of such matrices includes all matrices of the same form but with the x_j coming from F_ν. Thus if we can show every element of T is of that form, we'll know that $\mathrm{SL}_1(D) \cap T$ is dense in T.

Proposition 5.5.3. *Every element of T is of the above form for some $x_j \in F_\nu$.*

Proof. Let $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ be an arbitrary element of T, so $\lambda_d = (\lambda_1 \cdots \lambda_{d-1})^{-1}$. We want to solve the system $(*)$ of d equations

$$\lambda_i = \frac{x_1e_1^{(i-1)} + \cdots + x_de_d^{(i-1)}}{x_1e_1^{(i)} + \cdots + x_de_d^{(i)}}$$

for $1 \leq i \leq d$. This system can be made linear, rewritten as the system $(\dagger)$ given by

$$(e_1^{(i-1)} - \lambda_i e_1^{(i)})x_1 + (e_2^{(i-1)} - \lambda_i e_2^{(i)})x_2 + \cdots + (e_d^{(i-1)} - \lambda_i e_d^{(i)})x_d = 0.$$

If we find a solution $(x_1, \ldots, x_d) \in F_\nu^d$ to $(\dagger)$ and set $x = x_1e_1 + \cdots + x_de_d$, then we'll have $\tilde{\sigma}^{i-1}(x) = \lambda_i \tilde{\sigma}^i(x)$ for each i. Since the λ_i are all invertible, we see that either all the $\tilde{\sigma}^i(x)$ are zero, or none of them are. Thus to show that $(*)$ has a solution, it suffices to show that $(\dagger)$ has a solution $(x_1, \ldots, x_d)$ for which $0 \neq x = x_1e_1 + \cdots + x_de_d$.

First note that since $\lambda_d = (\lambda_1 \cdots \lambda_{d-1})^{-1}$, any solution to the first $d-1$ equations of $(\dagger)$ will automatically satisfy the d_{th} equation. Since each of these equations is homogeneous and linear, and there are $d-1$ of them, the set of solutions is a subspace of F_ν^d of dimension at least 1, in particular non-zero solutions $(x_1, \ldots, x_d)$ exist. We now need to ensure that $0 \neq x_1e_1 + \cdots + x_de_d$.

Suppose $(x_1, \dots, x_d)$ is a non-zero solution to (†) with $0 = x_1 e_1 + \cdots + x_d e_d$. As explained above we also have $0 = x_1 e_1^{(i)} + \cdots + x_d e_d^{(i)}$ for all i, and so in particular $(x_1, \dots, x_d)$ is a solution to the matrix $A := (e_j^{(i)})_{i,j=1}^d$. Thus A is singular, and since it has entries in K it already must have a non-zero solution $(x_1, \dots, x_d)$ in K^d. Note that some entry x_k is non-zero, so we can replace each x_j by x_j/x_k and assume without loss of generality that $x_k = 1$. Now, by the construction of A, $(\sigma^i(x_1), \dots, \sigma^i(x_d))$ is also a solution to A for any i. Thus if t_j denotes the Galois trace of x_j, $(t_1, \dots, t_d) \in F^d$ is a solution to A. But the e_j are F-linearly independent, so all the x_j must have trace 0. Since $x_k = 1$ and the characteristic of F does not divide d, this is impossible. □

We conclude that $\mathrm{SL}_1(D) \cap T$ is dense in T, and so $\mathrm{SL}_1(D)$ represents every element of Q. This shows that indeed Q is contained in $(W_a)_{\mathrm{SL}_1(D)}$, and so $(W_a)_{\mathrm{SL}_1(D)} = W_{\mathrm{SL}_1(D)} \ltimes Q$. As we have seen $W_{\mathrm{SL}_1(D)}$ is usually trivial, and if not then it has order 2. Thus despite $\mathrm{SL}_1(D)$ acting Weyl transitively on Δ_a, it acts on X_0 only by translations in most cases, and only by translations and a single transposition in all other cases.

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