Second Midterm for AMAT 327 (Elementary Abstract Algebra), Spring 2024. Thursday 3/14/24.

30 points total.

Name: ________________________________

Problem 1: Complete the following definitions:

1a (2 points):
We say that a subgroup H of a group G is *normal* if $\underline{gHg^{-1} = H \text{ for all } g \in G}$.

1b (2 points):
The *kernel* of a homomorphism $\phi\colon G \to H$ is $\underline{\{g \in G \mid \phi(g) = 1\}}$.

1c (2 points): For G a group and $S \subseteq G$, the subgroup *generated by* S, denoted by $\langle S \rangle$, is defined to be $\underline{\{s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \mid s_i \in S, \epsilon_i \in \{1, -1\}\}}$.

1d (2 points):
The *order* of an element g of a group G is $\underline{\text{the order of } \langle g \rangle}$.

Problem 2: Say whether the statement is true or false. You don't need to formally prove anything but **justify your answer**.

2a (3 points): True or false: The permutation $(1\ 2\ 5\ 6)(2\ 3\ 7)(4\ 5\ 6\ 7)$ in S_7 is even. True: it decomposes as $(12)(25)(56)(23)(37)(45)(56)(67)$, which is a product of an even number (eight) of 2-cycles.

2b (3 points): True or false: A group of order 2024 can have a subgroup isomorphic to $\mathbb{Z}_3$. false: 3 does not divide 2024, so Lagrange's Theorem says false.

Problem 3 (6 points): Let $\psi\colon G \to H$ be a homomorphism of groups. Prove that if $\ker(\psi)$ is trivial then ψ is injective.

Solution: Suppose $\ker(\psi) = \{1\}$. Let $g, g' \in G$ with $\psi(g) = \psi(g')$. Then $\psi(g^{-1}g') = 1$, so $g^{-1}g' \in \ker(\psi)$, i.e., $g^{-1}g' = 1$. Thus, $g = g'$. $\square$

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Problem 4 (6 points): Let G be a group. The *center* $Z(G)$ of G is the subset

$$Z(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\},$$

i.e., the subset of all elements that commute with every element. Prove that $Z(G)$ is a subgroup, and in fact a normal subgroup, of G. [For context, this is very similar to your homework problem proving that every subgroup of an abelian group is normal.]

Solution: Since $1g = g1$ for all $g \in G$, $1 \in Z(G)$. Now let $z, z' \in Z(G)$, so $zg = gz$ and $z'g = gz'$ for all $g \in G$. Rewriting $zg = gz$ to $gz^{-1} = z^{-1}g$, we get $(z^{-1}z')g = z^{-1}gz' = g(z^{-1}z')$ for all $g \in G$, which shows that $z^{-1}z' \in Z(G)$. Hence $Z(G)$ is a subgroup. Finally, $gzg^{-1} = zgg^{-1} = z$ for all $g \in G$, so $gZ(G)g^{-1} = Z(G)$, so $Z(G)$ is normal. $\square$

Problem 5 (4 points): Let $G = \mathbb{R}$ with the group operation $+$, and let H be any subgroup of G containing 1 and π. Prove that H is not cyclic. [Hint: Since the operation is $+$, the notation $\langle a \rangle$ means $\{na \mid n \in \mathbb{Z}\}$.]

Solution: Suppose H is cyclic, say $H = \langle a \rangle$. Since 1 and π lie in H, we have $1 = na$ and $\pi = ma$ for some $n, m \in \mathbb{Z}$. Thus, $\pi = m/n$ is rational, a contradiction. $\square$

BONUS (+2 points): Prove that the only normal subgroups of S_5 are $\{\text{id}\}$, A_5, and S_5.this is actually pretty hard/tedious, not going to type it all out!