First Midterm for AMAT 327 (Elementary Abstract Algebra), Spring 2024. Thursday 2/15/24.

30 points total.

Name: ________________________________

Problem 1: Complete the following definitions:

1a (2 points):
The *order* of a group is its cardinality.

1b (2 points):
A *permutation* of a set X is a bijection from X to itself.

1c (2 points):
A group is *abelian* if every pair of elements commute.

1d (2 points):
We say that a subset H of a group G is a *subgroup* if $1 \in H$ and for all $h, h' \in H$, $h^{-1}h' \in H$.

Problem 2: Say whether the statement is true or false. You don't need to formally prove anything but **justify your answer**.

2a (3 points): True or false: The elements (12)(34) and (1234) of S_4 commute.false: (12)(34)(1234) fixes 1 but (1234)(12)(34) sends 1 to 3, so they cannot be equal.

2b (3 points): True or false: The element $[8]_{2024}$ is a zero divisor in $\mathbb{Z}_{2024}$.true: 8 times 253 is zero mod 2024.

Problem 3 (6 points): Let $\phi\colon G \to H$ be an isomorphism of groups. Prove that $\phi^{-1}\colon H \to G$ is an isomorphism. [Take for granted that it's a bijection, so you only have to prove that it's a homomorphism.]

Solution: Let $h, h' \in H$. By surjectivity we can choose $g, g' \in G$ such that $\phi(g) = h$ and $\phi(g') = h'$. Now $\phi^{-1}(hh') = \phi^{-1}(\phi(g)\phi(g')) = \phi^{-1}(\phi(gg')) = gg' = \phi^{-1}(h)\phi^{-1}(h')$. $\square$

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Problem 4 (6 points): Prove that $\mathbb{N}$ with the product $(m, n) \mapsto \gcd(m, n)$ is not a group. (Careful, this is "greatest common divisor", not "least common multiple" like on the homework.)

Solution: Suppose it is a group, say n is the identity element. then $gcd(m, n) = m$ for all $m \in \mathbb{N}$, so every m is a divisor of n. But such an n does not exist. $\square$

Problem 5 (4 points): A *monoid* is just like a group except we don't require the "inverses" axiom (so, just associativity and identity). Recall the *power set* $\mathcal{P}(X)$ of a set X is the set of all subsets of X. Prove that $\mathcal{P}(X)$ with the product $(A, B) \mapsto A \cup B$ is a monoid. Is it a group? Prove or disprove.

Solution: Associativity is clear since $(A \cup B) \cup C = A \cup (B \cup C)$. The identity element is $\emptyset$ since $A \cup \emptyset = \emptyset \cup A = A$ for all A. It is not a group (unless $X = \emptyset$) since $A \cup B = \emptyset$ is only possible when $A = \emptyset$. $\square$

BONUS (+2 points):

Let $\phi \colon G \to H$ be a homomorphism of groups. Let $K = \{g \in G \mid \phi(g) = 1\}$. Prove that if $K = \{1\}$ then ϕ is injective. Solution: see problem 3 on exam 2