Updated November 16, 2023

Homework problems for AMAT 300 (Intro to Proofs), Fall 2023. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points. General advice: try not to write too much!

Solutions will be gradually added (and may be hastily written without proofreading).

Problem 1 (due Thurs 8/31): Compute the cardinality of the set $\{\clubsuit, \{\clubsuit\}, \clubsuit\}$. Explain your answer.

Solution: Repeats don't count, so it's 2.

Problem 2 (due Thurs 8/31): Let A and B be sets such that $A \times A \subseteq B \times B$. Prove that $A \subseteq B$.

Solution: Let $a \in A$. Then $(a, a) \in A \times A$, which since $A \times A \subseteq B \times B$ implies that $(a, a) \in B \times B$. Hence $a \in B$, and so we conclude that $A \subseteq B$. $\square$

Problem 3 (due Thurs 8/31): Prove that $\{4a - 5b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Solution: ($\subseteq$): Let $x \in \{4a - 5b \mid a, b \in \mathbb{Z}\}$, so $x = 4a - 5b$ for some $a, b \in \mathbb{Z}$. Since $a, b \in \mathbb{Z}$, also $4a - 5b \in \mathbb{Z}$, i.e., $x \in \mathbb{Z}$.
($\supseteq$): Let $x \in \mathbb{Z}$. Set $a = -x$ and $b = -x$. Then $4a - 5b = -4x + 5x = x$, so $x \in \{4a - 5b \mid a, b \in \mathbb{Z}\}$. $\square$

Problem 4 (due Thurs 9/7): Let $A = \{(x, x^2 - 7) \mid x \in \mathbb{R}\}$ and $B = \{(x, -3x + 3) \mid x \in \mathbb{R}\}$. Compute $A \cap B$, and prove that your answer is right.

Solution: Let $C = \{(-5, 18), (2, -3)\}$, and we claim $A \cap B = C$.
($\subseteq$): Let $(x, y) \in A \cap B$. Then $y = x^2 - 7$ $y = -3x + 3$, so $x^2 - 7 = -3x + 3$. Solving this, we get $x = -5, 2$, so we compute that $(x, y) \in C$.
($\supseteq$): Let $(x, y) \in C$. If $(x, y) = (-5, 18)$ then we compute that it satisfies both $y = x^2 - 7$ $y = -3x + 3$, so $(x, y) \in A$. A similar argument shows $(x, y) \in B$, so we conclude that $(x, y) \in A \times B$. $\square$

Problem 5 (due Thurs 9/7): Let A and B be sets. Suppose that $A \cup B = A \cap B$. Prove that $A = B$.

Solution: ($\subseteq$): Let $a \in A$. Then also $a \in A \cup B$. Since $A \cup B = A \cap B$, we have $a \in A \cap B$, so in particular $a \in B$. This shows $A \subseteq B$. ($\supseteq$): A similar argument shows $B \subseteq A$. $\square$

Problem 6 (due Thurs 9/7): Let $X = \{\spadesuit\}$. Compute the cardinality of $\mathcal{P}(X) \cup \mathcal{P}(\mathcal{P}(X))$.

Solution: $\mathcal{P}(X) = \{\emptyset, X\}$ and $\mathcal{P}(\mathcal{P}(X)) = \{\emptyset, \{\emptyset\}, \{X\}, \{\emptyset, X\}\}$, so taking the union and throwing out one instance of the repeated element $\emptyset$, we get 5 elements.

Problem 7 (due Thurs 9/14): Let A and B be sets. Prove that $\overline{A \setminus B} = \overline{A} \cup B$.

Solution: ($\subseteq$): Let $x \in \overline{A \setminus B}$, so $x \notin A \setminus B$. If $x \in \overline{A}$ then $x \in \overline{A} \cup B$ and we are done. Now suppose $x \notin \overline{A}$. Then $x \in A$, which since $x \notin A \setminus B$ implies that $x \in B$, and we are done.
($\supseteq$): Let $x \in \overline{A} \cup B$, so $x \in \overline{A}$ or $x \in B$. First suppose $x \in \overline{A}$. Then $x \in \overline{A \setminus B}$, so we are done. Next suppose $x \in B$. Then $x \notin A \setminus B$, i.e., $x \in \overline{A \setminus B}$, so we are done. $\square$

Problem 8 (due Thurs 9/14): Let $I = \mathbb{N}$ and for each $n \in I$ let $A_n := \{p/q \in \mathbb{Q} \mid p, q \in \mathbb{Z},\ 1 \leq q \leq n\}$. Prove that $\bigcup\limits_{n \in I} A_n = \mathbb{Q}$.

Solution: ($\subseteq$): Let $x \in \bigcup\limits_{n \in I} A_n$, so $x \in A_n$ for some $n \in \mathbb{N}$. Thus, $x = p/q \in \mathbb{Q}$ for some p and q, so $x \in \mathbb{Q}$.
($\supseteq$): Let $x \in \mathbb{Q}$. Say $x = p/q$ for $p, q \in \mathbb{Z}$. If $q < 0$ then up to rewriting $p/q = (-p)/(-q)$ we can assume $g > 0$. Set $n = q$, so $n \in \mathbb{N}$. Since $1 \leq n \leq n$ we see that $x = p/q \in A_n$. Thus, $x \in \bigcup\limits_{n \in I} A_n$. $\square$

Problem 9 (due Thurs 9/14): Let $I = \mathbb{N}$ and for each $n \in I$ let $A_n := \{p/q \in \mathbb{Q} \mid p, q \in \mathbb{Z},\ 1 \leq q \leq n\}$. What does $\bigcap\limits_{n \in I} A_n$ equal? Justify your answer with a proof.

Solution: We claim that it equals $\mathbb{Z}$. Let us prove this. ($\subseteq$): Let $x \in \bigcap\limits_{n \in I} A_n$, so $x \in A_n$ for all $n \in \mathbb{N}$. In particular, $x \in A_1$, so $x = p/q$ for some $p, q \in \mathbb{Z}$ satisfying $1 \leq q \leq 1$. This implies $q = 1$, so $x = p \in \mathbb{Z}$.
($\supseteq$): Let $x \in \mathbb{Z}$. Set $p = x$ and $q = 1$. Then $x = p/q$, and for any $n \in \mathbb{N}$ we have $1 \leq 1 \leq n$, so $x \in A_n$. Hence $x \in \bigcap\limits_{n \in I} A_n$. $\square$

Problem 10 (due Thurs 9/21): Let A and B be sets. Prove (rigorously) that if $A \subseteq B$ then $A \cup B = B$.

Solution: Suppose $A \subseteq B$. ($\subseteq$): Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Since $A \subseteq B$, in either case $x \in B$. Thus $A \cup B \subseteq B$. ($\supseteq$): Let $x \in B$. Then $x \in A \cup B$, so $B \subseteq A \cup B$. We conclude that $A \cup B = B$. $\square$

Problem 11 (due Thurs 9/21): For $x, y \in \mathbb{R}$, prove that $x < y$ if and only if $-x > -y$.

Solution: ($\Leftarrow$): Suppose $x < y$. Subtracting $x + y$ from both sides yields $-y < -x$, i.e., $-x > -y$. ($\Rightarrow$): Suppose $-x > -y$. Adding $x+y$ to both sides yields $y > x$, i.e., $x < y$. $\square$

Problem 12 (due Thurs 9/21): For $x \in \mathbb{R}$, prove that $x < -5$ only if $x^2 > 25$.

Solution: Suppose $x < -5$. By Problem 11, $-x > 5$. Now $x^2 = (-x)(-x) > 5 \cdot 5 = 25$. $\square$

Problem 13 (due Thurs 9/28): Let X be a set. Prove that $|\mathcal{P}(X)| \geq 2048$ only if $|X| \geq 5$.

Solution: Suppose $|\mathcal{P}(X)| \geq 2048$. Then $2^{|X|} \geq 2^{11}$, so $|X| \geq 11$, so $|X| \geq 5$. $\square$

Problem 14 (due Thurs 9/28): Let P and Q be statements. Prove that the statement $P \Leftrightarrow Q$ is true if and only if the statement $(P \vee Q) \wedge (\sim (P \wedge Q))$ is false.

Solution: ($\Rightarrow$): Suppose $P \Leftrightarrow Q$ is true. Then either P and Q are both true, or P and Q are both false. First, if P and Q are both true, then $P \wedge Q$ is true, so $\sim (P \wedge Q)$ is false, so $(P \vee Q) \wedge (\sim (P \wedge Q))$ is false. Second, if P and Q are both false, then $P \vee Q$ is false, so $(P \vee Q) \wedge (\sim (P \wedge Q))$ is false. In either case, $(P \vee Q) \wedge (\sim (P \wedge Q))$ is false.
($\Leftarrow$): Suppose $(P \vee Q) \wedge (\sim (P \wedge Q))$ is false. Then either $P \vee Q$ is false or $\sim (P \wedge Q)$ is false. First, if $P \vee Q$ is false then P and Q are both false, so $P \Leftrightarrow Q$ is true. Second, if $\sim (P \wedge Q)$ is false then $P \wedge Q$ is true, so P and Q are both true, so $P \Leftrightarrow Q$ is true. In either case, $P \Leftrightarrow Q$ is true. $\square$

Problem 15 (due Thurs 9/28): Write out the truth table for $P \Rightarrow (Q \Rightarrow P)$.

Solution: Turns out each row has a "T".

Problem 16 (due Thurs 10/5): Let A be a set. Prove that if $A \subseteq \mathcal{P}(A)$ then $A = \emptyset$. [Hint: Use the contrapositive.]

Solution: Suppose $A \neq \emptyset$, say $a \in A$. Since a is an element of A, not a subset of A, $a \notin \mathcal{P}(A)$. [Sidenote: this actually requires an obscure axiom from set theory, but whatever.] Hence $A \not\subseteq \mathcal{P}(A)$. $\square$

Problem 17 (due Thurs 10/5): Let A and B be sets with $A \not\subseteq B$. Prove that if $A \times B = B \times A$, then $B = \emptyset$. [Hint: Use the contrapositive.]

Solution: Suppose $B \neq \emptyset$, say $b \in B$. Since $A \not\subseteq B$, we can choose $a \in A \setminus B$. Now $(a, b) \in A \times B$, but $(a, b) \neq B \times A$ since $a \notin B$, so $A \times B \neq B \times A$. $\square$

Problem 18 (due Thurs 10/5): Prove that every $x \in \mathbb{Q}$ satisfies $nx \in \mathbb{Z}$ for some $n \in \mathbb{Z}$. [First convert this into a purely "logical" statement, and then prove it.]

Solution: Logic: $\forall x \in \mathbb{Q}\ \exists n \in \mathbb{Z}$ such that $nx \in \mathbb{Z}$. Proof: Let $x \in \mathbb{Q}$. Then $x = p/q$ for some $p, q \in \mathbb{Z}$. Set $n = q$. Then $nx = qp/q = p \in \mathbb{Z}$. $\square$

Problem 19 (due Thurs 10/12): Prove (rigorously) that if $x \in \mathbb{Z}$ is odd then $x^2 + 6x - 9$ is even.

Solution: Suppose x is odd, say $x = 2n + 1$ for some $n \in \mathbb{Z}$. Then $x^2 + 6x - 9 = (2n + 1)^2 + 6(2n + 1) - 9 = 4n^2 + 16n - 2 = 2(2n^2 + 8n - 1)$. Since $2n^2 + 8n - 1 \in \mathbb{Z}$, this is even. $\square$

Problem 20 (due Thurs 10/12): Prove that for any $x \in \mathbb{Z}$ the number $x^2 - 3x$ is even.

Solution: Let $x \in \mathbb{Z}$. First suppose x is even, say $x = 2n$ for some $n \in \mathbb{Z}$. Then $x^2 - 3x = 4n^2 - 6n = 2(2n^2 - 3n)$, and $2n^2 - 3n \in \mathbb{Z}$ so this is even. Now suppose x is odd, say $x = 2n+1$ for some $n \in \mathbb{Z}$. Then $x^2 - 3x = (2n+1)^2 - 3(2n+1) = 4n^2 - 2n - 2 = 2(2n^2 - n - 1)$, and $2n^2 - n - 1 \in \mathbb{Z}$, so this is even. $\square$

Problem 21 (due Thurs 10/12): For $x \in \mathbb{N}$, prove that x divides $x + 31$ if and only if either $x = 1$ or $x = 31$.

Solution: ($\Leftarrow$): Suppose x divides $x+31$, so $nx = x+31$ for some $n \in \mathbb{N}$. Then $(n-1)x = 31$, so x divides 31. The only divisors of 31 are 1 and 31, so $x = 1$ or $x = 31$.
($\Rightarrow$): Suppose $x = 1$ or $x = 31$. If $x = 1$ then since 1 divides 32, we have that x divides $x + 31$. If $x = 31$ then since 31 divides 62, we have that x divides $x + 31$. $\square$

Problem 22 (due Thurs 10/19): Prove that if $a|b$ and $a|c$ then $a|(b + c)$.

Solution: Suppose $a|b$ and $a|c$, say $b = an$ and $c = am$ for $n, m \in \mathbb{N}$. Then $b+c = an+am = a(n + m)$, and $n + m \in \mathbb{N}$, so $a|(b + c)$. $\square$

Problem 23 (due Thurs 10/19): Prove that if $x^3 - 4x^2 + 4x \geq 0$ then $x \geq 0$.

Solution: Suppose $x < 0$. Then $x^3 - 4x^2 + 4x = x(x - 2)^2$, and $(x - 2)^2 > 0$, so $x(x - 2)^2$ is a negative number times a positive number, hence negative. This shows $x^3 - 4x^2 + 4x < 0$. $\square$

Problem 24 (due Thurs 10/19): Let $x \in \mathbb{N}$. Prove that x is even if and only if $x^2 + 2x - 4$ is even.

Solution: ($\Rightarrow$): Suppose x is even, say $x = 2n$ for some $n \in \mathbb{N}$. Then $x^2 + 2x - 4 = 4n^2 + 4n - 4 = 2(2n^2 + 2n - 2)$ is even.
($\Leftarrow$): Suppose x is odd, say $x = 2n+1$ for some $n \in \mathbb{N}$. Then $x^2+2x-4 = (2n+1)^2+4n+2-4 = 4n^2+8n-3 = 2(2n^2+4n-2)+1$, and $2n^2+4n-2 \in \mathbb{N}$, so x^2+2x-4 is odd. $\square$

Problem 25 (due Thurs 11/2): Prove that $7^{1/3}$ is irrational.

Solution: Suppose for contradiction that $7^{1/3}$ is rational, say $7^{1/3} = a/b$ in reduced form ($a, b \in \mathbb{Z}$). Then $7 = a^3/b^3$, so $a^3 = 7b^3$. This shows $7|a^3$, hence $7|a$. Say $a = 7k$ for some $k \in \mathbb{Z}$. Then $7^3k^3 = 7b^3$, so $b^3 = 7(7k^3)$, so $7|b^3$. But this implies $7|b$, and we already know $7|a$, so this contradicts that a/b was in reduced form. $\square$

Problem 26 (due Thurs 11/2): Prove that there do not exist $a, b \in \mathbb{Z}$ satisfying $2023a - 2016b = 8$. [Hint: $2023 = 7 \cdot 289$.]

Solution: Suppose not, so there do exist $a, b \in \mathbb{Z}$ satisfying $2023a - 2016b = 8$. Then $7(289a - 288b) = 8$, so $7|8$, which is false. $\square$

Problem 27 (due Thurs 11/2): Let X be a set. For each $x \in X$, let $\mathcal{Q}_x$ be the subset of $\mathcal{P}(X)$ defined via $\mathcal{Q}_x := \{S \in \mathcal{P}(X) \mid x \notin S\}$. Prove that $\bigcap_{x \in X} \mathcal{Q}_x = \{\emptyset\}$. (Why yes, you're right, this one is hard!) [But, OK, extended hint: For the ($\supseteq$) inclusion, you just need to prove that $\emptyset \in \bigcap_{x \in X} \mathcal{Q}_x$, i.e., prove that $\emptyset \in \mathcal{Q}_x$ for all $x \in X$. This is not too bad. For the ($\subseteq$) inclusion, use proof by contradiction, i.e., suppose there is some $S \neq \emptyset$ with $S \in \bigcap_{x \in X} \mathcal{Q}_x$, and then hunt for a contradiction.]

Solution: ($\subseteq$): Let $S \in \bigcap_{x \in X} \mathcal{Q}_x$. We want to prove that $S = \emptyset$, so suppose for contradiction that $S \neq \emptyset$. Say $s \in S$. Then $S \notin \mathcal{Q}_s$, which means $S \notin \bigcap_{x \in X} \mathcal{Q}_x$, a contradiction. [Hmm, so really this was contrapositive, oh well.]
($\supseteq$): We need to prove that $\emptyset \in \bigcap_{x \in X} \mathcal{Q}_x$, i.e., that $\emptyset \in \mathcal{Q}_x$ for all $x \in X$. Let $x \in X$. Since $x \notin \emptyset$, we have $\emptyset \in \mathcal{Q}_x$, as desired. $\square$

Problem 28 (due Thurs 11/9): Prove that for any $x \in \mathbb{R}$ there exists a unique $n \in \mathbb{Z}$ such that $x \in [n, n+1)$. [This has two parts: prove n exists, and then prove it's unique.]

Solution: Let $x \in \mathbb{R}$. Set n equal to x rounded down to the nearest integer. We need to show that $x \in [n, n+1)$. First, $n \leq x$ by definition. We need to prove $x < n+1$. Suppose not, so $x \geq n+1$. Then rounding x down to the nearest integer would yield $n+1$, or something larger, contradicting that we get n. We conclude that $x < n+1$. Now we need to prove that

n is unique. Suppose that $n, m \in \mathbb{Z}$ with $x \in [n, n+1)$ and $x \in [m, m+1)$. Without loss of generality $n \leq m$. If $n+1 \leq m$ then $[n, n+1) \cap [m, m+1) = \emptyset$, contradicting that x is in both sets, so we conclude that $m < n+1$. But n and m are integers, so $n \leq m < n+1$ implies $n = m$. This establishes uniqueness. $\square$

Problem 29 (due Thurs 11/9): Use induction to prove that $6|(n^3 - n)$ for all $n \in \mathbb{N}$.

Solution: Base case $n = 1$. We have $1^3 - 1 = 0 = 6 \cdot 0$, so $6|(1^3 - 1)$. Now suppose $n \geq 2$, and assume $6|((n-1)^3 - (n-1))$, say $(n-1)^3 - (n-1) = 6k$ for some $k \in \mathbb{Z}$. Then $n^3 - 3n^2 + 3n - 1 - n + 1 = 6k$, so $n^3 - n = 6k + 3n^2 - 3n = 6(k + \frac{n^2-n}{2})$. Since $\frac{n^2-n}{2}$ is an integer (if n is even then $n^2 - n$ is even, and if n is odd then $n^2 - n$ is even), this shows that $6|(n^3 - n)$. $\square$

Problem 30 (due Thurs 11/9): Use induction to prove that $3|(31^n - 4)$ for all $n \in \mathbb{N}$.

Solution: Base case $n = 1$: $31^1 - 4 = 27$, and $3|27$, so this case holds. Now suppose $n \geq 2$, and assume that $3|(31^{n-1} - 4)$, say $31^{n-1} - 4 = 3k$ for some $k \in \mathbb{Z}$. Then $31^n - 124 = 93k$, so $31^n - 4 = 93k + 120 = 3(31k + 40)$, which shows that $3|(31^n - 4)$. $\square$

Problem 31 (due Thurs 11/16): Let R be the relation on $\mathbb{R}$ defined by: xRy iff $xy > 0$. Prove that R is symmetric and transitive, but not reflexive.

Solution: Symmetric: Suppose xRy, so $xy > 0$. Then $yx > 0$, so yRx.
Transitive: Suppose xRy and yRz, so $xy > 0$ and $yz > 0$. Then x and y have the same sign (positive or negative), and y and z have the same sign, hence x and z have the same sign. We conclude that $xz > 0$, i.e., xRz.
Non-reflexive: Note that $0 \cdot 0 = 0 \not> 0$, so $0R0$ is false. $\square$

Problem 32 (due Thurs 11/16): Let S be the relation on $\mathbb{R}$ defined by: xSy iff $xy \geq 0$. Prove that S is reflexive and symmetric, but not transitive.

Solution: Reflexive: Let $x \in \mathbb{R}$. Then $xx = x^2 \geq 0$, so xSx.
Symmetric: Suppose xSy, so $xy \geq 0$. Then $yx \geq 0$, so ySx.
Non-transitive: Note that $1R0$ since $1 \cdot 0 = 0 \geq 0$, and $0R(-1)$ since $0 \cdot (-1) = 0 \geq 0$, but $1R(-1)$ is false since $1 \cdot (-1) = -1 \not\geq 0$. $\square$

Problem 33 (due Thurs 11/16): Let A be the set of all points on earth. Give an example of a relation R on A that is symmetric, but neither reflexive nor transitive.

Solution: Something like, define xRy iff x and y are 10 miles apart. This is symmetric by definition, and non-reflexive by definition. It's also non-transitive since for any x and y with xRy, we have xRy and yRx but not xRx. (Actually, even easier, just define xRy if $x \neq y$, that already works.) $\square$

For these last three problems, you can either hand them in early at the last "real" class on Tues 11/28, or email me a photo/scan of your work on Thurs 11/30 (we won't have class that day).

Problem 34 (due Thurs 11/30): Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^3 = 1\}$. Is S a function from $\mathbb{R}$ to $\mathbb{R}$? Prove or disprove.

Problem 35 (due Thurs 11/30): Let $f \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function $f(m, n) := m^n$. Prove that f is surjective but not injective.

Problem 36 (due Thurs 11/30): Let $f \colon A \to B$ be a function. For each $b \in B$, let $X_b := \{a \in A \mid f(a) = b\}$. Prove that $\{X_b \mid b \in B\}$ is a partition of A.

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end of homework