Take-home midterm for AMAT 540B (Topology II), Spring 2022. Distributed Weds 3/9, due Weds 3/30.

You can use the textbook and your notes from class, and take as much time as you need. So it's kind of just like doing homework, except obviously you can't work together or use materials other than the textbook and your notes. (These are probably in increasing order of difficulty, and especially #5 might be really hard.)

Problem 1: Let $p\colon \widetilde{X} \to X$ be a covering map, K a compact subset of $\widetilde{X}$, and x an element of X. Prove that $K \cap p^{-1}(\{x\})$ is finite.

Problem 2: View S^1 as the boundary of the 2-disk $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Fix $x_0 \in S^1$, and let $X = (S^1 \times S^1) \cup (D^2 \times \{x_0\})$. Compute $\pi_1(X)$. [Note: All these spaces are sufficiently nice that the version of Van Kampen using closed subspaces instead of open spaces is valid. Feel free to just take this as given.]

Problem 3: Let $K \leq F_2 = \langle a, b \rangle$ be the kernel of the map $\phi\colon F_2 \to \mathbb{Z}$ sending a to 2 and b to 3. Draw a cover of $S^1 \vee S^1$ whose fundamental group maps isomorphically to K under the homomorphism induced by the covering map.

Problem 4: Let $p\colon \widetilde{X} \to X$ be a covering map. Assume that $\widetilde{X}$ is path connected, and that X is path connected and locally path connected. Assume that $\widetilde{X}$ is homotopy equivalent to a space of the form $Y \times Z$, for neither Y nor Z simply connected. Prove that X is not homotopy equivalent to any graph.

Problem 5: Let X be a path connected, locally path connected, semilocally simply connected space, and let $\widetilde{X}$ be its universal cover. Prove that if $\widetilde{X}$ is homeomorphic to $\mathbb{R}^2$ then $\pi_1(X)$ has no non-trivial elements of finite order. [Note: I'm not totally sure how to prove this without using more advanced tools. I'll be curious if anyone can figure it out!]