Updated April 17, 2022

Homework problems for AMAT 540B (Topology II), Spring 2022. Over the course of the semester I'll add problems to this list, with each problem's due date specified.

Problem 1 (due Weds 2/23): Let X be a topological space. Let A and B be simply connected open subspaces such that $X = A \cup B$ and such that $A \cap B$ is path connected. Prove that X is simply connected.

Problem 2 (due Weds 2/23): Let $U = \{(x,0,0) \mid x \in \mathbb{R}\} \cup \{(0,y,0) \mid y \in \mathbb{R}\} \cup \{(0,0,z) \mid z \in \mathbb{R}\}$ be the union of the coordinate axes in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \setminus U)$.

Problem 3 (due Weds 2/23): Let $K \leq F_2 = \langle a, b \rangle$ be the kernel of the map $\phi\colon F_2 \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ sending a to $(1+2\mathbb{Z}, 0+3\mathbb{Z})$ and b to $(0+2\mathbb{Z}, 1+3\mathbb{Z})$. Draw a cover of $S^1 \vee S^1$ whose fundamental group maps isomorphically to K under the homomorphism induced by the covering map.

Problem 4 (due Weds 3/9): Call a group G *almost co-Hopfian* if every injective homomorphism $\phi\colon G \to G$ whose image has finite index in G is bijective (so the image has index 1). Prove that the free group F_n is almost co-Hopfian for all $n \geq 2$.

Problem 5 (due Weds 3/9): Let X be the torus with one point removed. Compute the universal cover of X. (Draw a picture, explain what's going on, etc.)

Problem 6 (due Weds 3/9): Let X be a (path connected, locally path connected, semilocally simply connected) space with fundamental group $\mathbb{Z}$. Let $x \in X$. Let $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x)$ be covering maps of X, with $\widetilde{X}_1$ and $\widetilde{X}_2$ non-simply connected. Prove that there exists a non-simply connected space $\widetilde{Y}$, a point $\widetilde{y} \in Y$, and covering maps $q_1\colon (\widetilde{Y}, \widetilde{y}) \to (\widetilde{X}_1, \widetilde{x}_1)$, $q_2\colon (\widetilde{Y}, \widetilde{y}) \to (\widetilde{X}_2, \widetilde{x}_2)$, and $r\colon (\widetilde{Y}, \widetilde{y}) \to (X, x)$ such that $p_1 \circ q_1 = p_2 \circ q_2 = r$.

Problem 7 (due Weds 4/27): Compute the simplicial homology of $S^1 \vee (S^1 \times S^1)$ in all dimensions.

Problem 8 (due Weds 4/27): Use Mayer–Vietoris to compute the singular homology of $S^1 \vee (S^1 \times S^1)$ in all dimensions. (You should get the same answer as in Problem 7, but this should show you how useful Mayer–Vietoris is.)