Take-home final for AMAT 540B (Topology II), Spring 2022. Distributed Mon 4/25, due Mon 5/9.

You can use the textbook and your notes from class, and take as much time as you need. So it's kind of just like doing homework, except obviously you can't work together or use materials other than the textbook and your notes.

Problem 1: Let X be path connected and locally path connected, and fix $x_0 \in X$. Let $f\colon X \to S^1$ be continuous, set $s_0 = f(x_0)$, and let $p\colon \mathbb{R} \to S^1$ be the universal covering map. Prove that f lifts to a map $\widetilde{f}\colon X \to \mathbb{R}$ (i.e., $f = p \circ \widetilde{f}$) if and only if the induced homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(S_1, s_0)$ is trivial.

Problem 2: View S^1 as the boundary of the 2-disk $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Fix $x_0 \in S^1$, and let $X = (S^1 \times S^1) \cup (D^2 \times \{x_0\})$. Compute $H_n(X)$ for all n. [Note: All these spaces are sufficiently nice that Mayer–Vietoris applies.]

Problem 3: Let X be a subspace of $\mathbb{R}^2$ with simply connected open subspaces $U_1, U_2, U_3 \subseteq X$ such that $X = U_1 \cup U_2 \cup U_3$. Suppose that $U_i \cap U_j$ is simply connected for all i and j, but that $U_1 \cap U_2 \cap U_3 = \emptyset$. Compute $\pi_1(X)$.

Problem 4: Prove that if X is simply connected then $H_1(X) = 0$. (Don't just say, "H_1 is the abelianization of π_1," we never actually proved that!)

Problem 5: A group G is of *type F* if there exists a finite CW-complex X with $\pi_1(X) \cong G$ such that the universal cover $\widetilde{X}$ is contractible. Prove that if G and H are of type F then so is the free product $G * H$.