Updated May 5, 2022

Homework problems for AMAT 299 (Intro to Proofs), Spring 2022. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points. General advice: try not to write too much!

Problem 1 (due Weds 2/2): Compute the cardinality of the set $\{\emptyset, \{\emptyset\}, \emptyset\}$. Explain your answer.

Problem 2 (due Weds 2/2): Let A and B be sets such that $A \times A = B \times B$. Prove that $A = B$.

Problem 3 (due Weds 2/2): Prove that $\{5a - 3b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Problem 4 (due Weds 2/9): Let $A = \{(x, 3 - x^2) \mid x \in \mathbb{R}\}$ and $B = \{(x, 2x + 4) \mid x \in \mathbb{R}\}$. Compute $A \cap B$.

Problem 5 (due Weds 2/9): Let $X = \{\heartsuit\}$. Compute the cardinality of $X \cup \mathcal{P}(X)$.

Problem 6 (due Weds 2/9): Let A and B be sets. Prove that $A \setminus B = (A \cup B) \setminus B$.

Problem 7 (due Weds 2/16): Let X be a set and $A_\alpha \subseteq X$ a family of subsets of X, indexed by $\alpha \in I$ for some indexing set I. Suppose that $\bigcup_{\alpha \in I} A_\alpha = X$. Prove that $\bigcap_{\alpha \in I} \overline{A_\alpha} = \emptyset$.

Problem 8 (due Weds 2/16): For each $\alpha \in [-1, 1]$, let $A_\alpha = \{(x, y) \in \mathbb{R}^2 \mid y = \alpha + x\}$. Prove that $\bigcup_{\alpha \in [-1,1]} A_\alpha = \{(x, y) \in \mathbb{R}^2 \mid x - 1 \leq y \leq x + 1\}$.

Problem 9 (due Weds 2/16): For each $\alpha \in [-1, 1]$, let $B_\alpha = \{(x, y) \in \mathbb{R}^2 \mid y = \alpha x\}$. Prove that $\bigcap_{\alpha \in [-1,1]} B_\alpha = \{(0, 0)\}$.

Problem 10 (due Weds 2/23): Let $\alpha \in \mathbb{R}$. Prove that if $\alpha > 0$ then the intersection $\{(x, y) \in \mathbb{R} \mid y = x^2 - \alpha\} \cap \{(x, 0) \mid x \in \mathbb{R}\}$ has cardinality 2.

Problem 11 (due Weds 2/23): For $x \in \mathbb{R}$, prove that $x^2 > x$ if and only if $x < 0$ or $x > 1$.

Problem 12 (due Weds 2/23): For a set X, prove that $|\mathcal{P}(X)| = 1$ only if $X = \emptyset$.

Problem 13 (due Weds 3/2): Let A and B be finite sets. Use the contrapositive to prove that if $A \times B$ has odd cardinality then A and B both have odd cardinality.

Problem 14 (due Weds 3/2): Write out the truth table for $(P \vee Q) \Rightarrow (P \wedge Q)$.

Problem 15 (due Weds 3/2): Prove that $P \Rightarrow (P \Rightarrow Q)$ is logically equivalent to $P \Rightarrow Q$.

No homework due Weds 3/9; the midterm is that day. (Also nothing Weds 3/16, that's spring break.)

Problem 16 (due Weds 3/23): Prove (rigorously) that if $x \in \mathbb{Z}$ is odd then $x^2 - 4x + 17$ is even.

Problem 17 (due Weds 3/23): Prove that for any $n \in \mathbb{N}$, the number $n^2 - n$ is even.

Problem 18 (due Weds 3/23): For $m \in \mathbb{N}$, prove that if $m + 7$ is a multiple of m then either $m = 1$ or $m = 7$.

Problem 19 (due Weds 3/30): Prove that if $a|b$ and $c|d$ then $ac|bd$.

Problem 20 (due Weds 3/30): Prove that every odd integer is a difference of two squares of integers. [For example, $11 = 6^2 - 5^2$, $-7 = 3^2 - 4^2$, etc.]

Problem 21 (due Weds 3/30): Prove that for $a, b \in \mathbb{N}$, if $lcm(a, b)|(a + b)$ then $a = b$.

Problem 22 (due Weds 4/6): Prove that if $x^3 - 6x^2 + 9x \geq 0$ then $x \geq 0$.

Problem 23 (due Weds 4/6): Prove that there do not exist $a, b \in \mathbb{Z}$ satisfying

$$2022a - 1685b = 338.$$

Problem 24 (due Weds 4/6): Prove that $5^{1/7}$ is irrational.

Nothing due Weds 4/13.

Problem 25 (due Weds 4/20): Prove that there exists $x \in \mathbb{N}$ such that x has exactly seven divisors. [You don't need to show your scratch work, just prove the statement.]

Problem 26 (due Weds 4/20): Prove that $x = 5$ is the unique solution to $x^3 - 15x^2 + 75x - 125 = 0$. [Don't just prove it's a solution, prove it's the *unique* solution!]

Problem 27 (due Weds 4/20): Prove that $(\mathbb{R} \times \mathbb{Q}) \cap (\mathbb{Q} \times \mathbb{R}) = \mathbb{Q} \times \mathbb{Q}$.

Problem 28 (due Weds 4/27): Use (regular) induction to prove that $6|(7^n + 5)$ for all $n \in \mathbb{N}$.

Problem 29 (due Weds 4/27): Use (regular) induction to prove that $6|(n^3 - n)$ for all $n \in \mathbb{N}$.

Problem 30 (due Weds 4/27): Let $X = \{2a + 3b \mid a, b \in \mathbb{N}\}$. Use (strong) induction to prove that $X = \{n \in \mathbb{N} \mid n \geq 2\}$. [The point is to show that every natural number bigger than 1 can be written in the form $2a + 3b$.]

Problem 31 (due Weds 5/4): Let R be the relation on $\mathbb{R}$ defined by: xRy iff $xy > 0$. Prove that R is symmetric and transitive, but not reflexive.

Problem 32 (due Weds 5/4): Let S be the relation on $\mathbb{R}$ defined by: xSy iff $xy \geq 0$. Is S an equivalence relation? Prove or disprove.

Problem 33 (due Weds 5/4): Let $f \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function $f(a, b) = a^b$. Prove that f is surjective but not injective.

End of homework.

Solutions (typed hastily and without proofreading):

Problem 1 (due Weds 2/2): Compute the cardinality of the set $\{\emptyset, \{\emptyset\}, \emptyset\}$. Explain your answer.

Solution: It's 2, because "repeats don't matter".

Problem 2 (due Weds 2/2): Let A and B be sets such that $A \times A = B \times B$. Prove that $A = B$.

Solution: Let $a \in A$. Then $(a, a) \in A \times A$. Since $A \times A = B \times B$, this means $(a, a) \in B \times B$. Hence $a \in B$. This shows $A \subseteq B$ and a parallel argument shows $B \subseteq A$, so we conclude $A = B$. $\square$

Problem 3 (due Weds 2/2): Prove that $\{5a - 3b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Solution: Since products and sums of integers are integers, $\{5a - 3b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}$. Now let $x \in \mathbb{Z}$. Set $a = 2x$ and $b = 3x$. Then $5a - 3b = 10x - 9x = x$, so $x \in \{5a - 3b \mid a, b \in \mathbb{Z}\}$. We conclude $\{5a - 3b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$. $\square$

Problem 4 (due Weds 2/9): Let $A = \{(x, 3 - x^2) \mid x \in \mathbb{R}\}$ and $B = \{(x, 2x + 4) \mid x \in \mathbb{R}\}$. Compute $A \cap B$.

Solution: We claim it is $\{(-1, 2)\}$. First, it is easy to check that $(-1, 2) \in A \cap B$. Now let $(x, y) \in A \cap B$, so $y = 3 - x^2$ and $y = 2x + 4$. This means $3 - x^2 = 2x + 4$, so $x^2 + 2x + 1 = 0$, so $x = -1$, and thus $y = 2$, so $(x, y) = (-1, 2)$. $\square$

Problem 5 (due Weds 2/9): Let $X = \{\heartsuit\}$. Compute the cardinality of $X \cup \mathcal{P}(X)$.

Solution: We have $X \cup \mathcal{P}(X) = \{\heartsuit\} \cup \{\emptyset, \{\heartsuit\}\} = \{\heartsuit, \emptyset, \{\heartsuit\}\}$, so it's 3.

Problem 6 (due Weds 2/9): Let A and B be sets. Prove that $A \setminus B = (A \cup B) \setminus B$.

Solution: Let $x \in A \setminus B$, so $x \in A$ and $x \notin B$. Since $x \in A$, also $x \in A \cup B$. Hence $x \in (A \cup B) \setminus B$. Now let $x \in (A \cup B) \setminus B$, so $x \in A \cup B$ and $x \notin B$. If $x \in A$, then $x \in A \setminus B$. If $x \in B$, then $x \in B$ and $x \notin B$, a contradiction. We conclude $x \in A \setminus B$. $\square$

Problem 7 (due Weds 2/16): Let X be a set and $A_\alpha \subseteq X$ a family of subsets of X, indexed by $\alpha \in I$ for some indexing set I. Suppose that $\bigcup_{\alpha \in I} A_\alpha = X$. Prove that $\bigcap_{\alpha \in I} \overline{A_\alpha} = \emptyset$.

Solution: Since $\bigcup_{\alpha \in I} A_\alpha = X$, DeMorgan's Law implies $\overline{\bigcup_{\alpha \in I} A_\alpha} = \overline{X}$, which then implies $\bigcap_{\alpha \in I} \overline{A_\alpha} = \emptyset$. $\square$

Problem 8 (due Weds 2/16): For each $\alpha \in [-1,1]$, let $A_\alpha = \{(x,y) \in \mathbb{R}^2 \mid y = \alpha + x\}$. Prove that $\bigcup_{\alpha \in [-1,1]} A_\alpha = \{(x,y) \in \mathbb{R}^2 \mid x - 1 \leq y \leq x + 1\}$.

Solution: Let $(x,y) \in \bigcup_{\alpha \in [-1,1]} A_\alpha$, so $(x,y) \in A_\alpha$ for some $\alpha \in [-1,1]$. This means $y = \alpha + x$, so $-1 + x \leq \alpha + x \leq 1 + x$ implies $-1 + x \leq y \leq 1 + x$, which shows that (x,y) lies in the right-hand side set. Now suppose (x,y) is in the right-hand side set, so $x - 1 \leq y \leq x + 1$. Set $\alpha = y - x$, so $-1 \leq y - x \leq 1$ implies $\alpha \in [-1,1]$. Also, $(x,y) \in A_\alpha$, so we get $(x,y) \in \bigcup_{\alpha \in [-1,1]} A_\alpha$ as desired. $\square$

Problem 9 (due Weds 2/16): For each $\alpha \in [-1,1]$, let $B_\alpha = \{(x,y) \in \mathbb{R}^2 \mid y = \alpha x\}$. Prove that $\bigcap_{\alpha \in [-1,1]} B_\alpha = \{(0,0)\}$.

Solution: Let $(x,y) \in \bigcap_{\alpha \in [-1,1]} B_\alpha$, so $(x,y) \in B_\alpha$ for all $\alpha \in [-1,1]$. This means $y = \alpha x$ for all $\alpha \in [-1,1]$, in particular for $\alpha = 1$ and $\alpha = -1$. Hence $y = x$ and $y = -x$, which means $x = -x$, so $x = 0$, and then also $y = 0$, so $(x,y) \in \{(0,0)\}$. Now let $(x,y) \in \{(0,0)\}$, so $(x,y) = (0,0)$. Then $y = 0 = \alpha 0 = \alpha x$ for all $\alpha \in [-1,1]$, so $(x,y) \in \bigcap_{\alpha \in [-1,1]} B_\alpha$ and we are done. $\square$

Problem 10 (due Weds 2/23): Let $\alpha \in \mathbb{R}$. Prove that if $\alpha > 0$ then the intersection $\{(x,y) \in \mathbb{R} \mid y = x^2 - \alpha\} \cap \{(x,0) \mid x \in \mathbb{R}\}$ has cardinality 2.

Solution: Suppose $\alpha > 0$. This intersection consists of all (x,y) such that $y = x^2 - \alpha$ and $y = 0$, that is, all $(x,0)$ such that $x^2 = \alpha$. Since $\alpha > 0$ there are two such x, namely $x = \sqrt{\alpha}$ and $x = -\sqrt{\alpha}$, so the intersection equals $\{(\sqrt{\alpha},0),(-\sqrt{\alpha},0)\}$, and so has cardinality 2. $\square$

Problem 11 (due Weds 2/23): For $x \in \mathbb{R}$, prove that $x^2 > x$ if and only if $x < 0$ or $x > 1$.

Solution: ($\Rightarrow$): First suppose $x^2 > x$. Also suppose $x < 0$ does not hold, so $x \geq 0$. Since $x^2 > x$ we know $x \neq 0$, so we can divide by it. Since $x > 0$, dividing by x does not change the direction of the inequality, so $x^2 > x$ becomes $x > 1$, as desired. ($\Leftarrow$): Now suppose $x < 0$ or $x > 1$. If $x < 0$ then $x^2 > 0 > x$ so $x^2 > x$. If $x > 1$ then $x^2 > x \cdot 1 = x$. In either case $x^2 > x$. $\square$

Problem 12 (due Weds 2/23): For a set X, prove that $|\mathcal{P}(X)| = 1$ only if $X = \emptyset$.

Solution: Suppose $|\mathcal{P}(X)| = 1$. Note that $\emptyset \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$, which since $|\mathcal{P}(X)| = 1$ tells us that these must be the same element, i.e., $X = \emptyset$. $\square$

Problem 13 (due Weds 3/2): Let A and B be finite sets. Use the contrapositive to prove that if $A \times B$ has odd cardinality then A and B both have odd cardinality.

Solution: Suppose A or B has even cardinality. We have $|A \times B| = |A| \cdot |B|$, and a product of an even number with any number is even, so $|A \times B|$ is even. $\square$

Problem 14 (due Weds 3/2): Write out the truth table for $(P \vee Q) \Rightarrow (P \wedge Q)$.

Solution: Hard to type out - you get T F F T in the last column.

Problem 15 (due Weds 3/2): Prove that $P \Rightarrow (P \Rightarrow Q)$ is logically equivalent to $P \Rightarrow Q$.

Solution: Hard to type out - you get T F T T in the last column for both things, so they're logically equivalent.

Problem 16 (due Weds 3/23): Prove (rigorously) that if $x \in \mathbb{Z}$ is odd then $x^2 - 4x + 17$ is even.

Solution: Suppose $x \in \mathbb{Z}$ is odd. Say $x = 2n + 1$ for some $n \in \mathbb{Z}$. Then $x^2 - 4x + 17 = (2n+1)^2 - 4(2n+1) + 17 = 4n^2 - 4n + 14 = 2(2n^2 - 2n + 7)$, and $2n^2 - 2n + 7 \in \mathbb{Z}$, so this is even. $\square$

Problem 17 (due Weds 3/23): Prove that for any $n \in \mathbb{N}$, the number $n^2 - n$ is even.

Solution: Let $n \in \mathbb{N}$. First suppose n is even, say $n = 2k$ for some $k \in \mathbb{N}$. Then $n^2 - n = (2k)^2 - (2k) = 4k^2 - 2k = 2(2k^2 - k)$, and $2k^2 - k \in \mathbb{N}$, so this is even. Now suppose n is odd, say $n = 2k - 1$ for some $k \in \mathbb{N}$. Then $n^2 - n = (2k-1)^2 - (2k-1) = 4k^2 - 6k + 2 = 2(2k^2 - 3k + 1)$, and $2k^2 - 3k + 1 \in \mathbb{Z}$, so this is even. $\square$

Problem 18 (due Weds 3/23): For $m \in \mathbb{N}$, prove that if $m + 7$ is a multiple of m then either $m = 1$ or $m = 7$.

Solution: Let $m \in \mathbb{N}$. Suppose $m + 7$ is a multiple of m, say $m + 7 = km$ for some $k \in \mathbb{N}$. Then $(k-1)m = 7$. Since $m, k \in \mathbb{N}$ and 7 is prime, the only options are $m = 1$ (and $k = 8$) or $m = 7$ (and $k = 2$). $\square$

Problem 19 (due Weds 3/30): Prove that if $a|b$ and $c|d$ then $ac|bd$.

Solution: Suppose $a|b$ and $c|d$. Say $b = ak$ and $d = c\ell$ for some $k, \ell \in \mathbb{N}$. Then $bd = akc\ell = (ac)(k\ell)$, and $k\ell \in \mathbb{N}$, so $ac|bd$. $\square$

Problem 20 (due Weds 3/30): Prove that every odd integer is a difference of two squares of integers. [For example, $11 = 6^2 - 5^2$, $-7 = 3^2 - 4^2$, etc.]

Solution: Let $n \in \mathbb{Z}$ be odd, say $n = 2k+1$. We observe that $(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n$, so n is the difference of the squares $(k+1)^2$ and k^2. $\square$

Problem 21 (due Weds 3/30): Prove that for $a, b \in \mathbb{N}$, if $lcm(a, b)|(a + b)$ then $a = b$.

Solution: Suppose $lcm(a, b)|(a + b)$, say $a + b = lcm(a, b)m$ for some $m \in \mathbb{N}$. Since $lcm(a, b)$ is a multiple of both a and b, we have $lcm(a, b) = ak = b\ell$ for some $k, \ell \in \mathbb{N}$. Now $a + b = akm = b\ell m$. This shows that $b = a(km - 1)$ and $a = b(\ell m - 1)$, so $a|b$ and $b|a$. We conclude $a = b$. $\square$

Problem 22 (due Weds 4/6): Prove that if $x^3 - 6x^2 + 9x \geq 0$ then $x \geq 0$.

Solution: Suppose for the contrapositive that $x < 0$. Since $(x - 3)^2 > 0$, this implies $x(x - 3)^2 < 0$, i.e., $x^3 - 6x^2 + 9x < 0$. $\square$

Problem 23 (due Weds 4/6): Prove that there do not exist $a, b \in \mathbb{Z}$ satisfying

$$2022a - 1685b = 338.$$

Solution: Suppose there do, so $2022a - 1685b = 338$. Then $337(6a - 5b) = 338$, so $337|338$, which is a contradiction. $\square$

Problem 24 (due Weds 4/6): Prove that $5^{1/7}$ is irrational.

Solution: Suppose it is rational, say $5^{1/7} = a/b$ in reduced form. Then $5 = a^7/b^7$, so $a^7 = 5b^7$. This implies a^7 is divisible by 5, so a is as well, say $a = 5k$ for some $k \in \mathbb{N}$. Now $5^7k^7 = 5b^7$, so $5^6k^7 = b^7$, which means b^7 and hence b is divisible by 5. But a and b cannot both be divisible by 5 since a/b is in reduced form, a contradiction. $\square$

Problem 25 (due Weds 4/20): Prove that there exists $x \in \mathbb{N}$ such that x has exactly seven divisors. [You don't need to show your scratch work, just prove the statement.]

Solution: Set $x = 64$. Then we note that the divisors of x are $1, 2, 4, 8, 16, 32, 64$, so there are exactly seven. $\square$

Problem 26 (due Weds 4/20): Prove that $x = 5$ is the unique solution to $x^3 - 15x^2 + 75x - 125 = 0$. [Don't just prove it's a solution, prove it's the *unique* solution!]

Solution: First note that $x^3 - 15x^2 + 75x - 125 = (x-5)^3$. Thus, we easily check that $x = 5$ is a solution to $(x-5)^3 = 0$. Now suppose $x = a$ is another solution, so $(a-5)^3 = 0$. This implies that $a - 5 = 0$, so $a = 5$. We conclude $x = 5$ is the unique solution. □

Problem 27 (due Weds 4/20): Prove that $(\mathbb{R} \times \mathbb{Q}) \cap (\mathbb{Q} \times \mathbb{R}) = \mathbb{Q} \times \mathbb{Q}$.

Solution: $(\subseteq)$: Let $(x, y) \in (\mathbb{R} \times \mathbb{Q}) \cap (\mathbb{Q} \times \mathbb{R})$. Then $(x, y) \in \mathbb{R} \times \mathbb{Q}$ and $(x, y) \in \mathbb{Q} \times \mathbb{R}$, so in particular $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$. Thus, $(x, y) \in \mathbb{Q} \times \mathbb{Q}$. $(\supseteq)$: Let $(x, y) \in \mathbb{Q} \times \mathbb{Q}$, so $x, y \in \mathbb{Q}$. Since $\mathbb{Q} \subseteq \mathbb{R}$, we get $(x, y) \in \mathbb{R} \times \mathbb{Q}$ and $(x, y) \in \mathbb{Q} \times \mathbb{R}$. Thus, $(x, y) \in (\mathbb{R} \times \mathbb{Q}) \cap (\mathbb{Q} \times \mathbb{R})$. □

Problem 28 (due Weds 4/27): Use (regular) induction to prove that $6|(7^n + 5)$ for all $n \in \mathbb{N}$.

Solution: Base case: $n = 1$. We have $7^1 + 5 = 12$, and $6|12$, so the base case holds. Now let $n \geq 2$ and assume for induction that $6|(7^{n-1} + 5)$, say $7^{n-1} + 5 = 6k$ for some $k \in \mathbb{N}$. Then $7^n + 5 = 7 \cdot 7^{n-1} + 5 = 6 \cdot 7^{n-1} + 7^{n-1} + 5 = 6(7^{n-1} + k)$. Since $7^{n-1} + k \in \mathbb{N}$, we conclude that $6|(7^n + 5)$. □

Problem 29 (due Weds 4/27): Use (regular) induction to prove that $6|(n^3 - n)$ for all $n \in \mathbb{N}$.

Solution: Base case: $n = 1$. We have $1^3 - 1 = 0$, and $6|0$, so the base case holds. Now let $n \geq 2$ and assume for induction that $6|((n-1)^3 - (n-1))$, say $(n-1)^3 - (n-1) = 6k$ for some $k \in \mathbb{N}$. Then $6k = (n^3 - 3n^2 + 3n - 1) - (n - 1) = n^3 - n - 3n^2 + 3n$, so $n^3 - n = 6k + 3(n^2 - n)$. By something from class, $n^2 - n$ is even, say $n^2 - n = 2\ell$ for some $\ell \in \mathbb{N}$. Now $n^3 - n = 6(k + \ell)$, and $k + \ell \in \mathbb{N}$, so $6|(n^3 - n)$. □

Problem 30 (due Weds 4/27): Let $X = \{2a + 3b \mid a, b \in \mathbb{N}\}$. Use (strong) induction to prove that $X = \{n \in \mathbb{N} \mid n \geq 2\}$. [The point is to show that every natural number bigger than 1 can be written in the form $2a + 3b$.]

[oops need to pretend $0 \in \mathbb{N}$ for this to be literally right] Solution: Bases cases: $n = 2, 3$. We check that $2 = 2 \cdot 1 + 3 \cdot 0$ and $3 = 2 \cdot 0 + 3 \cdot 1$, so these hold. Now assume for strong induction that $n \geq 4$ and that the result holds for all $m < n$. In particular it holds for $n - 2$, i.e., $n - 2 \in X$, say $n - 2 = 2c + 3d$ for some $c, d \in \mathbb{N}$. Then $n = 2(c+1) + 3d$, and $c + 1 \in \mathbb{N}$, so this shows that $n \in X$. □

Problem 31 (due Weds 5/4): Let R be the relation on $\mathbb{R}$ defined by: xRy iff $xy > 0$. Prove that R is symmetric and transitive, but not reflexive.

Solution: Symmetric: Let $x, y \in \mathbb{R}$ such that xRy. Then $xy > 0$, so $yx > 0$, so yRx. Transitive: Let $x, y, z \in \mathbb{R}$ such that xRy and yRz. Then $xy > 0$ and $yz > 0$. In particular

$y \neq 0$, so $y^2 > 0$. Now $xyyz > 0$ implies $y^2xz > 0$, so $xz > 0$, i.e., xRz. Non-reflexive: We note that $0^2 = 0 \not> 0$. $\square$

Problem 32 (due Weds 5/4): Let S be the relation on $\mathbb{R}$ defined by: xSy iff $xy \geq 0$. Is S an equivalence relation? Prove or disprove.

Solution: No, we claim it is not transitive. Indeed, $1 \cdot 0 = 0 \geq 0$ and $0 \cdot (-1) = 0 \geq 0$, but $1 \cdot (-1) = -1 \not\geq 0$. $\square$

Problem 33 (due Weds 5/4): Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function $f(a, b) = a^b$. Prove that f is surjective but not injective.

Solution: Let $x \in \mathbb{N}$. Then $x = x^1 = f(x, 1)$, so f is surjective. For non-injectivity, observe that $(1, 1) \neq (1, 2)$ but $f(1, 1) = 1^1 = 1 = 1^2 = f(1, 2)$. $\square$

End of homework.
