Updated December 5, 2022

Homework problems for AMAT 540A (Topology I), Fall 2022. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points.

Problem 1 (due Thurs 9/1): Call a relation $R \subseteq A \times A$ on a set A *reverse-transitive* if whenever aRa' and $a'Ra''$ then $a''Ra$. Prove that if R is reflexive and reverse-transitive then it is symmetric.

Solution: Assume R is reflexive and reverse-transitive. Suppose aRa'. By reflexivity, $a'Ra'$, so by reverse-transitivity $a'Ra$. Hence R is symmetric. $\square$

Problem 2 (due Thurs 9/1): Prove that an equivalence relation is antisymmetric if and only if every equivalence class consists of one element. (I more or less said this in class, but didn't write it down or prove it.)

Solution: Let R be an equivalence relation on a set A. First suppose R is antisymmetric, let $[a]$ be an equivalence class, and let $a' \in [a]$. Then $a'Ra$ and aRa', which by antisymmetry implies $a = a'$. Thus $[a]$ consists of one element. Now conversely suppose every equivalence class consists of one element, so $[a] = \{a\}$ for all $a \in A$. Let $a, a' \in A$ with aRa' and $a'Ra$. Then $a' \in [a]$, so $a' = a$. This shows R is antisymmetric. $\square$

Problem 3 (due Thurs 9/1): Let $f\colon A \to B$ be a function. Suppose that $f^{-1}(f(A_0)) = A_0$ for all $A_0 \subseteq A$. Prove that f is injective.

Solution: Let $a, a' \in A$ such that $f(a) = f(a')$. Then $a' \in f^{-1}(f(\{a\})) = \{a\}$, so $a' = a$. Thus f is injective. $\square$

Problem 4 (due Thurs 9/8): Let X be a set with power set $\mathcal{P}(X)$, and view $\mathcal{P}(X)$ as a partially ordered set with partial order $\subseteq$. Prove that every subset A of $\mathcal{P}(X)$ has a least upper bound. [Just to preemptively alleviate confusion, let me emphasize that A is a subset of $\mathcal{P}(X)$, not a subset of X.]

Solution: Let $A \subseteq \mathcal{P}(X)$. Let $\ell := \bigcup_{a \in A} a$, so $L \in \mathcal{P}(X)$. Clearly $a \subseteq \ell$ for all $a \in A$, so ℓ is an upper bound of a. Now let b be an arbitrary upper bound of A, so $a \subseteq b$ for all $a \in A$. Then $\bigcup_{a \in A} a \subseteq b$, i.e., $\ell \subseteq b$. Thus, ℓ is the least upper bound. $\square$

Problem 5 (due Thurs 9/8): Prove that $\mathbb{Q}$ is countable by finding an explicit surjection $\mathbb{N} \twoheadrightarrow \mathbb{Q}$. [Don't just blindly copy something you google!]

Every element of $\mathbb{N}$ can be written uniquely in the form $2^a 3^b 5^c d$ for some $a, b, c \in \mathbb{N} \cup \{0\}$ and some $d \in \mathbb{N}$ not divisible by 2, 3, or 5. Let $\phi\colon \mathbb{N} \to \mathbb{Q}$ send such an element to $(-1)^a \frac{b}{c}$ if $c \neq 0$, and to 0 if $c = 0$. This is well defined, and is surjective since every element of $\mathbb{Q}$ is either of the form b/c for $b, c \in \mathbb{N} \cup \{0\}$ with $c \neq 0$ or the form $-b/c$ for $b, c \in \mathbb{N} \cup \{0\}$ with $c \neq 0$. □

Problem 6 (due Thurs 9/8): A *grayscale image* is an element of the Cartesian product $\prod_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \{0, 1\}$. Prove that there exist uncountably many grayscale images. [Hint: Feel free to use that $\prod_{i\in\mathbb{N}} \{0, 1\}$ is uncountable.]

Since $\prod_{i\in\mathbb{N}} \{0, 1\}$ is uncountable, it suffices to exhibit an injection $\phi\colon \prod_{i\in\mathbb{N}} \{0, 1\} \hookrightarrow \prod_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \{0, 1\}$. Given $f \in \prod_{i\in\mathbb{N}} \{0, 1\}$, define $\phi(f) \in \prod_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \{0, 1\}$ via $\phi(f)(n, m) := f(n)$ if $(n, m) \in \mathbb{N} \times \{0\}$, and $\phi(f)(n, m) := 0$ otherwise. To see this is injective, suppose $\phi(f) = \phi(f')$. Then for any $n \in \mathbb{N}$ we have $f(n) = \phi(f)(n, 0) = \phi(f')(n, 0) = f'(n)$. Thus $f = f'$ and so ϕ is injective. □

Problem 7 (due Thurs 9/15): Let $X = \{a, b, c\}$. Find an example of a subset of $\mathcal{P}$ that contains $\emptyset$ and X but is *not* a topology.

Solution: The subset $\mathcal{A} = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}\}$ is not a topology because $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{A}$. □

Problem 8 (due Thurs 9/15): Let $X = \mathbb{Z}$ and $\mathcal{T} = \{U \subseteq \mathbb{Z} \mid 0 \in U\} \cup \{\emptyset\}$. Prove that $\mathcal{T}$ is a topology.

Solution: By construction $\emptyset \in \mathcal{T}$. Since $0 \in X$ we have $X \in \mathcal{T}$. Now let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a family of open sets. If all U_α are empty, their union is empty, hence open. Assume some U_α is non-empty, so $0 \in U_\alpha$. then 0 is in the union of all of them, hence their union is in $\mathcal{T}$. Finally, let $U, V \in \mathcal{T}$. If U or V is empty then $U \cap V$ is empty, hence open. If neither is empty, then $0 \in U$ and $0 \in V$. Thus $0 \in U \cap V$, so $U \cap V \in \mathcal{T}$. □

Problem 9 (due Thurs 9/15): Let $X = \mathbb{Z}$ and $\mathcal{B} = \{B_n \mid n \in \mathbb{Z}\}$ where $B_n = \{0, n\}$. Prove that $\mathcal{B}$ is a basis for a topology, and prove that the topology it generates is $\mathcal{T}$ from Problem 8.

Solution: For any $x \in X$ we have $x \in \{0, x\} = B_x$, so $\mathcal{B}$ covers X. Now suppose $x \in B_n \cap B_m$, for $n \neq m$. Then $B_n \cap B_m = \{0, n\} \cap \{0, m\} = \{0\}$, so $x = 0$, and we get $x \in B_0 \subseteq B_n \cap B_m$. This shows $\mathcal{B}$ is a basis. Now to show it generates $\mathcal{T}$, let $x \in X$ and $U \in \mathcal{T}$ with $x \in U$. Since $x \in U$, U is non-empty, so $0 \in U$. Hence $B_x = \{0, x\} \subseteq U$, and so $x \in B_x \subseteq U$ reveals $\mathcal{B}$ generates $\mathcal{T}$. □

Problem 10 (due Thurs 9/22): Let $X = \{1/n \mid n \in \mathbb{N}\}$. Prove that the subspace topology on X coming from $X \subseteq \mathbb{R}$ equals the discrete topology.

Solution: It suffices to prove that $\{x\}$ is open in X for all $x \in X$. Note that for any $n \in \mathbb{N} \setminus \{1\}$ we have $X \cap (1/(n+1), 1/(n-1)) = \{1/n\}$, so $\{1/n\}$ is open in X. Also, $\{1\} = X \cap (1/2, \infty)$, so $\{1\}$ is open in X. $\square$

Problem 11 (due Thurs 9/22): Let Y be a finite set with a total order $\leq$. Prove that the order topology on Y equals the discrete topology.

Solution: Say $Y = \{y_1, \ldots, y_n\}$ with $y_1 < \cdots < y_n$. Then $\{y_1\} = [y_1, y_2)$, $\{y_i\} = (y_{i-1}, y_{i+1})$ for all $1 < i < n$, and $\{y_n\} = (y_{n-1}, y_n]$, so every singleton set is open. Hence the topology is discrete. $\square$

Problem 12 (due Thurs 9/22): Let X and Y be spaces with the finite complement topology. Prove that the product topology on $X \times Y$ does not equal the finite complement topology on $X \times Y$. [Hint: Find an example with a non-empty subset that is open in the product topology but has infinite complement.]

Solution: Let $X = Y = \mathbb{Z}$ with the finite complement topology. Let $U = V = \mathbb{Z} \setminus \{0\}$, so U and V are open which means $U \times V$ is open in the product topology on $X \times Y$. But the complement of $U \times V$ is $(\{0\} \times Y) \cup (X \times \{0\})$, which is infinite and not all of $X \times Y$, so $U \times V$ is not open in the finite complement topology. $\square$

Problem 13 (due Thurs 9/29): Let S be a subset of a topological space X. Define the *boundary* of S, denoted ∂S, to be $\partial S := \overline{S} \setminus \mathring{S}$ (so the closure minus the interior). Prove that $\partial S \subseteq S$ if and only if S is closed.

Solution: First note that since $\mathring{S} \subseteq \overline{S}$, we have $\partial S \cup \mathring{S} = \overline{S}$. Now we have that $\partial S \subseteq S$ if and only if $\overline{S} \setminus \mathring{S} \subseteq S$ if and only if (unioning both sides with $\mathring{S}$) $\overline{S} \subseteq S$ if and only if S is closed. $\square$

Problem 14 (due Thurs 9/29): Let $X = \mathbb{Z}$ with the finite complement topology. Prove that $\partial S = X$ for any infinite subset S of X. Oops I meant to say, "for any infinite subset S of X with infinite complement $X \setminus S$."

Solution: Let $S \subseteq X$ with S infinite and $X \setminus S$ infinite. Since S is infinite, the only closed subset containing it is X, i.e., $\overline{S} = X$. Since $X \setminus S$ is infinite, the only open set contained in S is $\emptyset$, i.e., $\mathring{S} = \emptyset$. We conclude that $\partial S = X \setminus \emptyset = X$. $\square$

Problem 15 (due Thurs 9/29): Let X be a topological space. Suppose that for all $x \in X$ the singleton set $\{x\}$ is closed, and suppose that every infinite subset of X is dense. Prove that the topology on X equals the finite complement topology. [Hint: Think in terms of closed sets rather than open sets.]

Solution: We must show that a subset is open if and only if it has finite complement or is empty, or equivalently that a subset is closed if and only if it is finite or all of X. First note that since singletons are closed, finite sets are also closed, so that handles one direction. Now suppose C is closed in X. In particular, $\overline{C} = C$. If $C \neq X$ then this means C is not dense, and hence must be finite. $\square$

Problem 16 (due Thurs 10/6): Let $f\colon X \to Y$ be a bijection between topological spaces. Suppose that X and Y both have the finite complement topology. Prove that f is necessarily a homeomorphism.

Solution: First we claim f is continuous. Let $D \subseteq Y$ be closed. Then D is either finite or Y. Since f is a bijection, this implies $f^{-1}(D)$ is either finite or X. Hence $f^{-1}(D)$ is closed. Now we claim f^{-1} is continuous. For any closed C in X, C is either finite or X, so $(f^{-1})^{-1}(C)$ is either finite or Y, hence closed. $\square$

Problem 17 (due Thurs 10/6): Let X be a totally ordered set with the order topology. Prove that X is Hausdorff.

Solution: Let $a \neq b$ be in X, say without loss of generality $a < b$. First suppose there exists $z \in (a, b)$. If a is a global minimum and b is a global maximum, then $[a, z)$ and $(z, b]$ are disjoint open neighborhoods of a and b. If a is a global minimum but b is not a global maximum, say $y > b$, then we can use $[a, z)$ and (z, y). If a is not a global minimum then there exists $x < a$, and we can use (x, z) and either $(z, b]$ (if b is a global max) or (z, y) (if there exists $y > b$). Now suppose $(a, b) = \emptyset$. Then a parallel argument works, using either $[a, b)$ and $(a, b]$, or $[a, b)$ and (a, y), or (x, b) and $(a, b]$, or (x, b) and (a, y). $\square$

Problem 18 (due Thurs 10/6): Let X be Hausdorff and let $x_1, \dots, x_n \in X$ be distinct elements. Prove that there exist open sets $U_1, \dots, U_n \subseteq X$ such that $x_i \in U_i$ for all i and $U_i \cap U_j = \emptyset$ for all $i \neq j$.

Solution: For each i and j, choose disjoint open neighborhoods $U_j(i) \ni x_i$ and $U_i(j) \ni x_j$. For each i set U_i equal to the intersection of all $U_j(i)$ for $j \neq i$. Since there are finitely many such j, U_i is an open neighborhood of x_i. Now we claim that for any $i \neq j$, U_i and U_j are disjoint. Indeed, U_i is contained in $U_j(i)$ and x_j is contained in $U_i(j)$, and these are disjoint. $\square$

No homework due Thurs 10/13, that's the midterm.

Problem 19 (due Thurs 10/20): Let $(X_\alpha)_{\alpha\in\Lambda}$ be a family of discrete topological spaces. Prove that $\prod_{\alpha\in\Lambda} X_\alpha$ with the box topology is discrete.

Solution: We need to show that every singleton $\{f\}$ is open, for $f \in \prod_{\alpha\in\Lambda} X_\alpha$. Since each X_α is discrete, we have that $\{f(\alpha)\}$ is open in X_α, for each α. Thus (in the box topology) we have that $\prod_{\alpha\in\Lambda}\{f(\alpha)\}$ is open. But the only element of this set is f, so we conclude that $\{f\}$ is open. □

Problem 20 (due Thurs 10/20): Let $(X_\alpha)_{\alpha\in\Lambda}$ be a family of Hausdorff topological spaces. Prove that $\prod_{\alpha\in\Lambda} X_\alpha$ is Hausdorff. (This is true in either the box or product topologies, but let's say you should use the product topology here.)

Solution: Let $f \neq g$ be elements of $\prod_{\alpha\in\Lambda} X_\alpha$. Since $f \neq g$ we can choose some $\alpha_0 \in \Lambda$ such that $f(\alpha_0) \neq g(\alpha_0)$. Since X_{α_0} is Hausdorff we can choose disjoint open neighborhoods U_{α_0} of $f(\alpha_0)$ and V_{α_0} of $g(\alpha_0)$. For all $\alpha \neq \alpha_0$, set $U_\alpha = V_\alpha = X_\alpha$. Now $\prod_{\alpha\in\Lambda} U_\alpha$ is an open neighborhood of f and $\prod_{\alpha\in\Lambda} V_\alpha$ is an open neighborhood of g. Moreover, they are disjoint, since if h were in their intersection then we would have $h(\alpha_0) \in U_{\alpha_0} \cap V_{\alpha_0} = \emptyset$, a contradiction. □

Problem 21 (due Thurs 10/20): Let $(X_\alpha)_{\alpha\in\Lambda}$ be a family of discrete topological spaces. Prove that every basic open subset of $\prod_{\alpha\in\Lambda} X_\alpha$ with the product topology is closed (hence clopen). [Hint: Prove that with this setup, the complement of a basic open set is a union of basic open sets, hence is open.]

Solution: Let $\prod_{\alpha\in\Lambda} U_\alpha$ be a basic open set. This is the set of all f such that $f(\alpha) \in U_\alpha$ for all α. Thus, its complement is the set of all f such that $f(\alpha) \in X_\alpha \setminus U_\alpha$ for some α. This is the union over all α of the set of f such that $f(\alpha) \in X_\alpha \setminus U_\alpha$, so to show this union is open it suffices to show each such set is open. Indeed, it equals $\prod_{\beta\in\Lambda} V_\beta$, where $V_\alpha = X_\alpha \setminus U_\alpha$ (which is open since X_α is discrete) and $V_\beta = X_\beta$ for all $\beta \neq \alpha$, hence is open. □

Problem 22 (due Thurs 10/27): Let $f\colon X \twoheadrightarrow Y$ and $g\colon Y \twoheadrightarrow Z$ be continuous surjections. Prove that if $g \circ f$ is a quotient map then g is a quotient map.

Solution: Suppose $g \circ f$ is a quotient map. Since g is a continuous surjection, we just need to prove that for $N \subseteq Z$ non-open, $g^{-1}(N) \subseteq Y$ is non-open. Suppose $g^{-1}(N)$ is open. Since f is continuous, this implies $f^{-1}(g^{-1}(N)) \subseteq X$ is open. But $g \circ f$ is a quotient map, so this is a contradiction. □

Problem 23 (due Thurs 10/27): Let X be a metric space with metric d and let $x \in X$. Let $f_x \colon X \to [0, \infty)$ be the function $f_x(y) := d(x, y)$. Prove that f_x is continuous. [Hint: Clearly $f_x^{-1}([0, a)) = B_a(x)$ is open for all $a > 0$, so you just need to prove that sets of the form $f_x^{-1}((a, b))$ are open for all $a < b$.]

Solution: We have $f_x^{-1}((a, b)) = B_b(x) \cap \{y \in X \mid d(x, y) > a\}$, so it suffices to prove that $\{y \in X \mid d(x, y) > a\}$ is open. Let y be an element of this set. By the triangle inequality, the ball of radius $\frac{d(x,y)-a}{2}$ centered at y lies entirely in this set. Thus the set is open. □

Problem 24 (due Thurs 10/27): Let X be a metric space with metric d and let $x \in X$. Suppose that there exist $y, z \in X$ with $d(x, y) = 1$ and $d(x, z) = 2$ but there exists no $w \in X$ with $d(x, w) = 3/2$. Prove that X is not connected.

Solution: By the previous problem, the sets $f_x^{-1}([0, 3/2))$ and $f_x^{-1}((3/2, \infty))$ are open in X. They are clearly disjoint, they are both non-empty since y is in the first one and z is in the second one, and their union is all of X since $3/2$ is not in the range of f_x. Thus they form a separation of X. □

Problem 25 (due Thurs 11/3): Prove that $\mathbb{Q} \times \mathbb{Q}$ is totally disconnected (meaning its only connected non-empty subsets are the singletons).

Solution: Let $(x, y), (x', y')$ be distinct points in $\mathbb{Q} \times \mathbb{Q}$. First suppose $x \neq x'$, say $x < x'$. Choose an irrational y with $x < y < x'$. Now $(-\infty, y) \times \mathbb{Q}$ and $(y, \infty) \times \mathbb{Q}$ are a separation of $\mathbb{Q} \times \mathbb{Q}$, and no connected subspace of $\mathbb{Q} \times \mathbb{Q}$ can contain both (x, y) and (x', y'). If $x = x'$ then necessarily $y \neq y'$, and an analogous argument works. □

Problem 26 (due Thurs 11/3): Let $X = \mathbb{Z}$ with the finite complement topology. Take as given the following fact: for $a < b$, the closed interval $[a, b]$ cannot be written as a disjoint union of countably infinitely many closed subsets. Prove that X is not path connected.

Solution: Let $p \colon [a, b] \to X$ be any function with $p(a) = 0$ and $p(b) = 1$. We claim that p is not continuous, so X cannot be path connected. Indeed, if p were continuous then since singletons in X are closed we would have that every $p^{-1}(\{x\})$ is closed in $[a, b]$, so $[a, b]$ would be the disjoint union of countably many closed sets. If there are infinitely many, then this contradicts the fact we are taking as given. If there are finitely many, say $p^{-1}(\{0\})$, $p^{-1}(\{1\})$, $p^{-1}(\{x_1\})$,…,$p^{-1}(\{x_n\})$, then since finite unions of closed sets are closed all of these sets are clopen, and $p^{-1}(\{0\})$, $p^{-1}(\{1\})$ are both non-empty, so this contradicts $[a, b]$ being connected. □

Problem 27 (due Thurs 11/3): Let $Y = \mathbb{Z}$ with the particular point topology (so the open sets are those containing 0, and the empty set). Let $p \colon [a, b] \to Y$ be a function with $p(a) = 1$, $p(b) = 2$, and $0 \notin p([a, b])$. Decide whether it is possible for p to be continuous; if yes then give an example, and if no then prove it's impossible. [This will be fun, I haven't actually thought about what the right answer is! (If it turns out to be too hard, just see what you can do....)]

Solution: Not possible. Suppose p is continuous. Then $p^{-1}(\{x\})$ is closed for all $x \neq 0$, and since 0 is not in the range of p, we get that $[a,b]$ is a disjoint union of countably many closed sets. The same argument works as in the previous proof thanks to Y being connected. □

Problem 28 (due Thurs 11/10): Let (X,d) be a metric space such that for all $x,y \in X$ there exists a path $p\colon [a,b] \to X$ from x to y such that $d(x,p(t)) \leq d(x,y)$ for all $t \in [a,b]$. Prove that X is locally path connected.

Solution: Let $x \in X$ and let U be an open neighborhood of x. Choose $\varepsilon > 0$ such that $x \in B_\varepsilon(x) \subseteq U$. It suffices to show that $B_\varepsilon(x)$ is path connected, and for this it suffices to show that for any $y \in B_\varepsilon(x)$ there is a path from x to y lying in $B_\varepsilon(x)$ (since then we can get from any y to y' by passing through x). Let p be a path from x to y as in the hypothesis, and we claim it lies in $B_\varepsilon(x)$. Indeed, for any z on the path, $d(x,z) \leq d(x,y) < \varepsilon$. □

Problem 29 (due Thurs 11/10): Let X be a space in which every proper closed subspace is compact (a *proper* subset of a set A is just a subset not equal to all of A). Prove that X is compact (so in fact every closed subspace is compact).

Solution: Let $\mathcal{U}$ be an open cover of X. Pick some $\emptyset \neq U \in \mathcal{U}$ and let $C = X \setminus U$. Now C is a proper closed subspace of X and $\mathcal{U} \setminus \{U\}$ is an open cover of C. By hypothesis C is compact, so we can choose a finite subcover $\mathcal{V}$ of $\mathcal{U} \setminus \{U\}$ (covering C). Now $\mathcal{V} \cup \{U\}$ covers X, and is a finite subcover of $\mathcal{U}$. □

Problem 30 (due Thurs 11/10): Let $X = (\{0\} \times [-1,1]) \cup \{(x,y) \in (0,1] \times [-1,1] \mid y = \sin(1/x)\}$ be the topologist's sine curve restricted to the $[0,1] \times [-1,1]$ region of the plane. Prove that X is compact. [Hint: Don't try and prove compactness directly, use some theorem.]

Solution: Since $[0,1] \times [-1,1]$ is compact, it suffices to prove X is closed (since then it will be compact). We claim the complement of X in $[0,1] \times [-1,1]$ is open. Let (x,y) be in this complement, so $y \neq \sin(1/x)$. We can choose $\varepsilon > 0$ such that for all $(x',y') \in B_\varepsilon(x,y)$ we have $y' \neq \sin(1/x')$. This shows the complement is open. □

Problem 31 (due Thurs 11/17): Describe the one-point compactification of $X = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$ (where X has the subspace topology).

Solution: It's a figure 8, not much else to say really.

Problem 32 (due Thurs 11/17): Prove that a discrete space is second countable (as a space) if and only if it is countable (as a set).

Solution: Suppose X is discrete and second countable. Let $\mathcal{B}$ be a countable basis. Since X is discrete, every singleton $\{x\}$ is open, hence a union of basic open sets, hence $\mathcal{B}$ contains every singleton. This gives an injective function $X \to \mathcal{B}$ via $x \mapsto \{x\}$, and $\mathcal{B}$ is countable, so we conclude X is countable. Now suppose X is countable and discrete. Then the standard basis $\{\{x\} \mid x \in X\}$ is countable, so X is second countable. $\square$

Problem 33 (due Thurs 11/17): Prove that $\mathbb{Q}$ is not locally compact.

Solution: Suppose it is, so in particular there exists open $V \subseteq \mathbb{Q}$ with $0 \in V$ and $\overline{V}$ compact. Since closed subspaces of compact spaces are compact, we can replace V with any open subset, so without loss of generality $V = \mathbb{Q} \cap (-a, a)$ for irrational a. But then the closure of V in $\mathbb{Q}$ is just V itself, and this is not compact because it is not closed in the Hausdorff space $\mathbb{R}$, so we have a contradiction. $\square$

Problem 34 (due Thurs 12/1): Let X be a Hausdorff space with a countable basis $\mathcal{B}$ such that every basic open set $B \in \mathcal{B}$ is closed. Prove that X is metrizable.

Solution: By the Urysohn Metrization Theorem, it suffices to prove X is regular. Let $x \in X$ and U an open neighborhood of x. Choose a basic open set B with $x \in B \subseteq U$. Since B is closed, we have $x \in B \subseteq \overline{B} \subseteq U$, and since X is Hausdorff this suffices for proving X is regular. $\square$

Problem 35 (due Thurs 12/1): View S^1 as the unit circle $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $f, g\colon S^1 \to S^1$ be the maps $f(x, y) = (x, y)$ and $g(x, y) = (-x, -y)$. Prove that $f \simeq g$ (that is, that f is homotopic to g).

Solution: Use polar coordinates, so points in S^1 are $e^{i\theta}$ with $0 \le \theta < 2\pi$. Let $F(e^{i\theta}, t) := e^{i(\theta + t\pi)}$. This is clearly continuous, and has codomain S^1. Since $F(e^{i\theta}, 0) = e^{i\theta}$ and $F(e^{i\theta}, 1) = -e^{i\theta}$, this is a homotopy from f to g. $\square$

Problem 36 (due Thurs 12/1): Let X be any space and let $Y = \mathbb{Z}$ with the particular point topology $\mathcal{T} = \{U \subseteq Y \mid 0 \in U\} \cup \{\emptyset\}$. Let $c_0\colon X \to Y$ be the constant map $c_0(x) = 0$ for all $x \in X$. Prove that every map $f\colon X \to Y$ is homotopic to c_0.

Solution: Let $F(x, t)\colon X \times [0, 1] \to Y$ send (x, t) to 0 for all $t > 0$ and send $(x, 0)$ to $f(x)$. By construction $F(x, 0) = f(x)$ and $F(x, 1) = c_0(x)$, so we just need to prove F is continuous. Let $\emptyset \neq U$ be open in Y, and consider $F^{-1}(U)$. Since $0 \in U$ we know that $F^{-1}(U)$ contains all of $X \times (0, 1]$ and all of $f^{-1}(Y) \times [0, 1]$. In fact it equals their union, and they are both open, so $F^{-1}(U)$ is open. $\square$

End of homework