Updated November 23, 2021

Homework problems for AMAT 540A (Topology I), Fall 2021. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points.

Problem 1 (due Weds 9/15): Let Λ be any set. Prove that there exists a bijection between $\mathcal{P}(\Lambda)$ and $\prod_{\alpha\in\Lambda}\{0,1\}$.

Solution: Let $\Phi\colon \mathcal{P}(\Lambda) \to \prod_{\alpha\in\Lambda}\{0,1\}$ be the function defined by setting $\Phi(\Theta)(\alpha)$ equal to 1 if $\alpha \in \Theta$ and 0 if $\alpha \notin \Theta$. Let $\Psi\colon \prod_{\alpha\in\Lambda}\{0,1\} \to \mathcal{P}(\Lambda)$ be the function defined by $\Psi(f) = f^{-1}(1)$. Now we have $\Phi(\Psi(f))(\alpha) = \Phi(f^{-1}(1))(\alpha) = f(\alpha)$ for all α, so $\Phi(\Psi(f)) = f$, and clearly $\Psi(\Phi(\Theta)) = \Theta$, so Φ and Ψ are inverses, hence bijective. $\square$

Problem 2 (due Weds 9/15): Let $f\colon A \to B$ be a function. Suppose that B is countable and that for all $b \in B$ the preimage $f^{-1}(\{b\})$ in A is countable. Prove that A is countable.

Solution: We have that A is the union of the $f^{-1}(\{b\})$ for all $b \in B$. Thus A is a countable union of countable sets, hence countable. $\square$

Problem 3 (due Weds 9/15): Construct an explicit bijection between $\mathbb{Z}$ and $\mathbb{Q}$ (don't just google it, be creative!)

Solution: First fix a bijection $f\colon \mathbb{N}\cup\{0\} \to \mathbb{Z}$ with $f(0) = 0$ (e.g., order $\mathbb{Z}$ as $\{0, 1, -1, 2, -2, 3, -3, \dots\}$). Now any non-zero element of $\mathbb{N}$ is uniquely expressible via its prime decomposition $\pm 2^{a_2}3^{a_3}\cdots$ ($a_i \in \mathbb{N}\cup\{0\}$), and any non-zero element of $\mathbb{Q}$ is uniquely expressible via its "prime decomposition" $\pm 2^{b_2}3^{b_3}\cdots$ ($b_i \in \mathbb{Z}$). In each, only finitely many a_i or b_i can be non-zero. So, define a bijection $\mathbb{Z} \to \mathbb{Q}$ by sending 0 to 0 and otherwise sending $\pm 2^{a_2}3^{a_3}\cdots$ to $\pm 2^{f(a_2)}3^{f(a_3)}\cdots$. $\square$

Problem 4 (due Weds 9/22): Let $X = \{a, b, c\}$. Find a non-trivial, non-discrete topology on X.

Solution: Lots of options here. I'll go with $\{\emptyset, \{a\}, \{a, b, c\}\}$. (I should have asked for something that's *not* a topology, that would have been more interesting.)

Problem 5 (due Weds 9/22): A topological space is *Alexandrov* if the intersection of any collection of open sets is open (not just finitely many). Prove that $\mathbb{Z}$ with the finite complement topology is not Alexandrov.

Solution: For any $n \in \mathbb{N}$ the set $\mathbb{Z} \setminus \{n\}$ is open, but the intersection of all of these is $\mathbb{Z} \setminus \mathbb{N}$, which is neither $\mathbb{Z}$ nor has a finite complement, hence is not open. $\square$

Problem 6 (due Weds 9/22): Let X be a set and let $x, x' \in X$ with $x \neq x'$. Suppose that $\mathcal{T} = \{U \subseteq X \mid x \in U \text{ or } x' \in U\} \cup \{\emptyset\}$ is a topology on X. Prove that $|X| = 2$.

Solution: By construction $|X| \geq 2$. Suppose $|X| > 2$, say $x'' \in X \setminus \{x, x'\}$. Then $\{x, x''\}$ and $\{x', x''\}$ are both open, but their intersection is $\{x''\}$, which is not open, contradicting that $\mathcal{T}$ is a topology. We conclude that $|X| = 2$. $\square$

Problem 7 (due Weds 9/29): Let $\mathcal{T} := \{U \subseteq \mathbb{Z} \mid \mathbb{Z} \setminus U \text{ is infinite}\} \cup \{\mathbb{Z}\}$. Prove that $\mathcal{T}$ is not a topology on $\mathbb{Z}$.

Solution: Let $U = \{1, 2, 3, \dots\}$ and $V = \{-1, -2, -3, \dots\}$. These both have infinite complement, hence are in $\mathcal{T}$, but their intersection $\{0\}$ is neither infinite nor all of $\mathbb{Z}$, hence is not in $\mathcal{T}$. Thus $\mathcal{T}$ is not a topology. $\square$

Problem 8 (due Weds 9/29): Let $n \in \mathbb{N}$ and set $\mathcal{B}_n := \{B \subseteq \mathbb{Z} \mid n \leq |\mathbb{Z} \setminus B| < \infty\}$. Prove that $\mathcal{B}_n$ is a basis for the finite complement topology on $\mathbb{Z}$.

Solution: Let U be an open set and $x \in U$. Then $\mathbb{Z} \setminus U$ is finite, so in particular U is infinite. Thus we can choose a subset $A \subseteq U$ such that $x \notin A$ and $|A| \geq n$. Set $B = U \setminus A$. Now $|\mathbb{Z} \setminus B| \geq |U \setminus B| = |A| \geq n$, so $B \in \mathcal{B}_n$, and moreover $x \in B \subseteq U$, so we conclude $\mathcal{B}_n$ is a basis. $\square$

Problem 9 (due Weds 9/29): Let $n \in \mathbb{N}$ and set $\mathcal{C}_n := \{C \subseteq \mathbb{Z} \mid n \geq |\mathbb{Z} \setminus C|\}$. Prove that $\mathcal{C}_n$ is not a basis (for any topology).

Solution: Let $B = \mathbb{Z} \setminus \{1, \dots, n\}$ and $C = \mathbb{Z} \setminus \{-1, \dots, -n\}$, so $|\mathbb{Z} \setminus B| = |\mathbb{Z} \setminus C| = n$, which means $B, C \in \mathcal{C}_n$. Now consider $0 \in B \cap C$, and let D be any set satisfying $0 \in D \subseteq B \cap C$. This implies $|\mathbb{Z} \setminus D| \geq |\mathbb{Z} \setminus (B \cap C)| = 2n$, and $n \not\geq 2n$, so $D \notin \mathcal{C}_n$. We conclude $\mathcal{C}_n$ is not a basis. $\square$

Problem 10 (due Weds 10/6): Prove that if X and Y have the discrete topology, then the product topology on $X \times Y$ is also discrete.

Solution: It suffices to show that every singleton set $\{(x, y)\}$ is open. Indeed, $\{(x, y)\} = \{x\} \times \{y\}$, and each of $\{x\}$ and $\{y\}$ are open, so $\{(x, y)\}$ is open. $\square$

Problem 11 (due Weds 10/6): Let X be a space with the finite complement topology and let Y be a finite subset of X. Prove that the subspace topology on Y equals the discrete topology on Y.

Solution: It suffices to show that every singleton set $\{y\}$ in Y is open in the subspace topology. Let $U = X \setminus (Y \setminus \{y\})$. Since Y is finite, so is $Y \setminus \{y\}$, so U is open. Also, $Y \cap U = Y \setminus (Y \setminus \{y\}) = \{y\}$, so we conclude that $\{y\}$ is open in Y. $\square$

Problem 12 (due Weds 10/6): Let X and Y be topological spaces, with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively. Let $\mathcal{T}_{X\times Y}$ be the product topology on $X \times Y$. Let $A \subseteq X$ and $B \subseteq Y$ be subspaces, and let $\mathcal{T}_A$ and $\mathcal{T}_B$ be the subspace topologies. Let $\mathcal{T}_{A\times B}$ be the product topology on $A \times B$ (using $\mathcal{T}_A$ and $\mathcal{B}$). Prove that $\mathcal{T}_{A\times B}$ equals the subspace topology on $A \times B$ as a subspace of $\mathcal{T}_{X\times Y}$. (More intuitively: Prove that the subspace and product topologies on $A \times B$ agree, either viewing $A \times B$ as a subspace of a product or as a product of subspaces.)

Solution: First let $U \times V$ be a basic open set in the product topology on $A \times B$. Since U is open in A, we can choose U' open in X such that $U = A \cap U'$, and similarly choose V' open in Y with $V = B \cap V'$. Now $U' \times V'$ is a (basic) open set in $X \times Y$ and $U \times V = (A \times B) \cap (U' \times V')$, so we conclude that $U \times V$ is open in the subspace topology. This shows that the product topology on $A \times B$ is a refinement of the subspace topology, and it remains to show the converse. We know that a basis for the subspace topology can be obtained by intersecting the subspace with all the basic open sets from a basis of the ambient space. Hence $\{(A \times B) \cap (U \times V) \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ is a basis for the subspace topology on $A \times B$. Now since $(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$, each of these basic open sets in the subspace topology is also a (basic) open set in the product topology, as desired. $\square$

Nothing due Weds 10/13 thanks to the timing of October break.

Problem 13 (due Weds 10/20): Let X and Y be sets, each with the finite complement topology. Prove that a function $f\colon X \to Y$ is continuous if and only if either $f(X) = \{y\}$ for some $y \in Y$ or else $f^{-1}(\{y\})$ is finite for all $y \in Y$.

Solution: Suppose f is continuous. Suppose $f^{-1}(\{y\})$ is infinite for some $y \in Y$ (so X is infinite), and we must show that it equals all of X. Since $\{y\}$ is closed in Y and f is continuous, $f^{-1}(\{y\})$ is closed in X. The only closed infinite subset of X is X, so we are done. Now suppose that either $f(X) = \{y\}$ for some $y \in Y$ or else $f^{-1}(\{y\})$ is finite for all $y \in Y$. In the first case, f is continuous because any constant map is continuous. In the second case, the preimage in X of any closed subset C of Y is either X (if $C = Y$) or is a finite union of finite sets (if C is finite), so in either case it is closed. Hence f is continuous. $\square$

Problem 14 (due Weds 10/20): Let X be a metric space with metric d, viewed as a topological space with the metric topology. Fix $x_0 \in X$, and define $\phi_{x_0} \colon X \to \mathbb{R}$ via $\phi_{x_0}(x) := d(x_0, x)$. Prove that ϕ_{x_0} is continuous.

Solution: Let (a, b) be a basic open set in $\mathbb{R}$, so $a < b$. The preimage of (a, b) under ϕ_{x_0} is $\{x \in X \mid a < d(x_0, x) < b\}$. This equals the intersection of the open ball of radius b centered at x_0 (which is open) with the complement of the closed ball of radius a centered at x_0 (which is also open), hence is open. Thus, ϕ_{x_0} is continuous. $\square$

Problem 15 (due Weds 10/20): Prove that if a topological space is both Hausdorff and Alexandrov (see problem 5) then it is discrete.

Solution: Hausdorff implies singletons are closed, and Alexandrov implies arbitrary unions of closed sets are closed, and every set is the union of its singletons, so we conclude that every subset is closed; hence the topology is discrete. $\square$

Problem 16 (due Weds 11/3): Let $X = \mathbb{Z}$ with the finite complement topology, let Y be a topological space in which every singleton $\{y\}$ is closed, and let $p \colon X \to Y$ be a surjective function. Prove that if p is continuous then the topology on Y is the finite complement topology.

Solution: Since singletons are closed in Y, so are all finite subsets. It remains to prove that every closed subset of Y is either finite or is all of Y. Let $C \subseteq Y$ be closed. Since p is continuous, $p^{-1}(C)$ is closed in X, so either $p^{-1}(C) = X$ or else $p^{-1}(C)$ is finite. By surjectivity, $C = p(p^{-1}(C))$, so we get that either $C = p(X)$, i.e., $C = Y$, or else C is the image of a finite set, hence finite. $\square$

Problem 17 (due Weds 11/3): With the same setup as Problem 16, prove that if p is continuous then it is automatically a quotient map.

Solution: It suffices to show it is a closed map. Any closed set in X is either finite or X, so its image is either finite or Y, hence closed. $\square$

Problem 18 (due Weds 11/3): Let $\{0, 1\}$ have the discrete topology, and consider the product $\prod_{\mathbb{N}} \{0, 1\}$ with the product (not box) topology. Prove that every basic open set in $\prod_{\mathbb{N}} \{0, 1\}$ is closed.

Solution: (Easier than I thought....) Since the closure of a product is the product of the closures, and since $\{0, 1\}$ is discrete, this is just immediate. $\square$

Problem 19 (due Weds 11/10): Let X be a topological space and $Y, Z \subseteq X$ open subspaces such that $X = Y \cup Z$. Suppose that X, Y, and $Y \cap Z$ are connected. Prove that Z is connected.

Solution: If $Z = \emptyset$ then it is connected. Now assume $Z \neq \emptyset$. Let $A \neq \emptyset$ be a connected component of Z, and let $B = Z \setminus A$, so we want to show $B = \emptyset$. Since $Y \cap Z$ is connected, it must lie in either A or B, say WLOG in A. In particular Y and B are disjoint. If $Y \cap Z = \emptyset$ then since X is connected we have $Z = X$, so Z is connected. Now assume $Y \cap Z \neq \emptyset$. Then $Y \cup A$ is a union of connected spaces over a non-empty intersection, hence is connected. Now X is the disjoint union of open sets $Y \cup A$ and B, and $Y \cup A \neq \emptyset$, so connectivity of X says $B = \emptyset$ as desired. □

Problem 20 (due Weds 11/10): Let $q \colon X \to Y$ be a quotient map. Suppose that Y is connected, and that for each $y \in Y$ the preimage $q^{-1}(\{y\})$ is connected. Prove that X is connected.

Solution: Let $X = A \cup B$ for disjoint open A and B. For each $y \in Y$, the fact that $q^{-1}(\{y\})$ is connected implies that $q^{-1}(\{y\})$ must lie either in A or B. Let $C = \{y \in Y \mid q^{-1}(\{y\}) \subseteq A\}$ and $D = \{y \in Y \mid q^{-1}(\{y\}) \subseteq B\}$. Now $q^{-1}(C) = A$ and $q^{-1}(D) = B$, so thanks to q being a quotient map we conclude that C and D are both open. They are also disjoint, and their union is Y, so by connectivity of Y one of them must be empty, which implies that one of A or B must be empty. Hence X is connected. □

Problem 21 (due Weds 11/10): Prove that Problem #20 is *not* true with "connected" replaced by "path connected" everywhere. [Hint: Find a quotient of the topologist's sine curve that is path connected and where every preimage of a singleton is path connected.]

Solution: Let $\pi \colon \mathbb{R}^2 \to \mathbb{R}$ be $\pi(x, y) = x$. This restricts to a surjective map $\pi' \colon T \to [0, \infty)$ where T is the topologist's sine curve. The preimage of any $y \in [0, \infty)$ is either a single point (if $y > 0$) or the entire y-axis (if $y = 0$), hence path connected. Also $[0, \infty)$ is path connected, and T is not. So, we just need to show $\pi' \colon T \to [0, \infty)$ is a quotient map. Let $U \subseteq [0, \infty)$ and suppose $(\pi')^{-1}(U)$ is open in T. If $0 \notin U$ then this is immediate (since the "wiggly" part of T by itself is well-behaved), so assume $0 \in U$. This implies the y-axis is in $(\pi')^{-1}(U)$, so by openness some open neighborhood of the y-axis is also in $(\pi')^{-1}(U)$. This "catches" all but a well-behaved part of the wiggly part, so once again everything is well-behaved now. (Sorry, not very precise, kind of complicated to be precise, I won't care about this when grading.) □

Problem 22 (due Weds 11/17): Let $X = \mathbb{Z}$ with the particular point topology $\mathcal{T}_{pp} = \{U \subseteq \mathbb{Z} \mid 0 \in U\} \cup \{\emptyset\}$. Prove that X is not compact.

Solution: For each $n \in \mathbb{Z}$ let $U_n := \{0, n\}$. This is clearly an open cover of X. However, the union of finitely many U_n's is finite, whereas X is infinite, so there exists no finite subcover. □

Problem 23 (due Weds 11/17): Let $X = \mathbb{R}$ with the "countable complement topology" $\mathcal{T}_{cc} = \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is countable}\} \cup \{\emptyset\}$. Prove that X is not compact.

Solution: For each $n \in \mathbb{Z}$ let $U_n := X \setminus (\mathbb{Z} \setminus \{n\})$. This is clearly an open cover of X. For finitely many $n_1, \dots, n_k$ we have $U_{n_1} \cup \cdots \cup U_{n_k} = X \setminus (\mathbb{Z} \setminus \{n_1, \dots, n_k\})$, which does not equal all of X, so there is ano finite subcover. $\square$

Problem 24 (due Weds 11/17): Let $f \colon X \to Y$ be a surjective map. Suppose that Y is totally disconnected, and that for each $y \in Y$ the preimage $q^{-1}(\{y\})$ is totally disconnected. Prove that X is totally disconnected.

Solution: [Oops I called it f in one place and q in another. No one got confused though. Also, I just realized we didn't actually need f to be surjective!] Let C be a non-empty connected subset of X. Then the image $f(C)$ is non-empty and connected, and since Y is totally disconnected this implies $f(C)$ is a singleton, say $\{y\}$. Hence $C \subseteq f^{-1}(\{y\})$, which since $f^{-1}(\{y\})$ is totally disconnected implies C is a singleton. We conclude that X is totally disconnected. $\square$

Problem 25 (due Mon 12/6): Let X be a metric space such that the metric topology coincides with the discrete topology (for example, $\mathbb{Z}$ with the usual metric). Prove that X is locally compact if and only if every (finite-radius) ball in X is finite.

Problem 26 (due Mon 12/6): Call a topological space X *normal-ish* if for all disjoint closed subsets $C, D \subseteq X$ there exist disjoint open subsets $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$ (so if X is normal-ish and singletons are closed, then X is normal). Prove that $X = \mathbb{Z}$ with the "excluded point topology" $\mathcal{T}_{ep} = \{U \subseteq \mathbb{Z} \mid 0 \notin U\} \cup \{\mathbb{Z}\}$ is normal-ish (but obviously not normal).

Problem 27 (due Mon 12/6): Call a topological space X *super-Hausdorff* if for all $x \in X$ and all $A \subseteq X \setminus \{x\}$ there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $A \subseteq V$. Prove that the only super-Hausdorff spaces are discrete.