Homework problems for AMAT 540A (Topology I), Fall 2021. Over the course of the semester I’ll add problems to this list, with each problem’s due date specified. Each problem is worth 2 points.

Problem 1 (due Weds 9/15): Let $\Lambda$ be any set. Prove that there exists a bijection between $\mathcal{P}(\Lambda)$ and $\prod_{\alpha \in \Lambda} \{0, 1\}$.

**Solution:** Let $\Phi: \mathcal{P}(\Lambda) \to \prod_{\alpha \in \Lambda} \{0, 1\}$ be the function defined by setting $\Phi(\Theta)(\alpha)$ equal to 1 if $\alpha \in \Theta$ and 0 if $\alpha \notin \Theta$. Let $\Psi: \prod_{\alpha \in \Lambda} \{0, 1\} \to \mathcal{P}(\Lambda)$ be the function defined by $\Psi(f) = f^{-1}(1)$. Now we have $\Phi(\Psi(f))(\alpha) = \Phi(f^{-1}(1))(\alpha) = f(\alpha)$ for all $\alpha$, so $\Phi(\Psi(f)) = f$, and clearly $\Psi(\Phi(\Theta)) = \Theta$, so $\Phi$ and $\Psi$ are inverses, hence bijective. \qed

Problem 2 (due Weds 9/15): Let $f: A \to B$ be a function. Suppose that $B$ is countable and that for all $b \in B$ the preimage $f^{-1}\{\{b\}\}$ in $A$ is countable. Prove that $A$ is countable.

**Solution:** We have that $A$ is the union of the $f^{-1}(\{b\})$ for all $b \in B$. Thus $A$ is a countable union of countable sets, hence countable. \qed

Problem 3 (due Weds 9/15): Construct an explicit bijection between $\mathbb{Z}$ and $\mathbb{Q}$ (don’t just google it, be creative!)

**Solution:** First fix a bijection $f: \mathbb{N} \cup \{0\} \to \mathbb{Z}$ with $f(0) = 0$ (e.g., order $\mathbb{Z}$ as $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$). Now any non-zero element of $\mathbb{N}$ is uniquely expressible via its prime decomposition $\pm 2^{a_2} 3^{a_3} \cdots (a_i \in \mathbb{N} \cup \{0\})$, and any non-zero element of $\mathbb{Q}$ is uniquely expressible via its “prime decomposition” $\pm 2^{b_2} 3^{b_3} \cdots (b_i \in \mathbb{Z})$. In each, only finitely many $a_i$ or $b_i$ can be non-zero. So, define a bijection $\mathbb{Z} \to \mathbb{Q}$ by sending 0 to 0 and otherwise sending $\pm 2^{a_2} 3^{a_3} \cdots$ to $\pm 2^{f(a_2)} 3^{f(a_3)} \cdots$. \qed

Problem 4 (due Weds 9/22): Let $X = \{a, b, c\}$. Find a non-trivial, non-discrete topology on $X$.

**Solution:** Lots of options here. I’ll go with $\{\emptyset, \{a\}, \{a, b, c\}\}$. (I should have asked for something that’s *not* a topology, that would have been more interesting.)

Problem 5 (due Weds 9/22): A topological space is Alexandrov if the intersection of any collection of open sets is open (not just finitely many). Prove that $\mathbb{Z}$ with the finite complement topology is not Alexandrov.
**Solution:** For any \( n \in \mathbb{N} \) the set \( \mathbb{Z} \setminus \{n\} \) is open, but the intersection of all of these is \( \mathbb{Z} \setminus \mathbb{N} \), which is neither \( \mathbb{Z} \) nor has a finite complement, hence is not open. \( \square \)

Problem 6 (due Weds 9/22): Let \( X \) be a set and let \( x, x' \in X \) with \( x \neq x' \). Suppose that \( T = \{ U \subseteq X \mid x \in U \text{ or } x' \in U \} \cup \{\emptyset\} \) is a topology on \( X \). Prove that \( |X| = 2 \).

**Solution:** By construction \( |X| \geq 2 \). Suppose \( |X| > 2 \), say \( x'' \in X \setminus \{x, x'\} \). Then \( \{x, x''\} \) and \( \{x', x''\} \) are both open, but their intersection is \( \{x''\} \), which is not open, contradicting that \( T \) is a topology. We conclude that \( |X| = 2 \). \( \square \)

Problem 7 (due Weds 9/29): Let \( T := \{ U \subseteq \mathbb{Z} \mid \mathbb{Z} \setminus U \text{ is infinite} \} \cup \{\mathbb{Z}\} \). Prove that \( T \) is not a topology on \( \mathbb{Z} \).

**Solution:** Let \( U = \{1, 2, 3, \ldots\} \) and \( V = \{-1, -2, -3, \ldots\} \). These both have infinite complement, hence are in \( T \), but their intersection \( \{0\} \) is neither infinite nor all of \( \mathbb{Z} \), hence is not in \( T \). Thus \( T \) is not a topology. \( \square \)

Problem 8 (due Weds 9/29): Let \( n \in \mathbb{N} \) and set \( B_n := \{ B \subseteq \mathbb{Z} \mid n \leq |\mathbb{Z} \setminus B| < \infty \} \). Prove that \( B_n \) is a basis for the finite complement topology on \( \mathbb{Z} \).

**Solution:** Let \( U \) be an open set and \( x \in U \). Then \( \mathbb{Z} \setminus U \) is finite, so in particular \( U \) is infinite. Thus we can choose a subset \( A \subseteq U \) such that \( x \notin A \) and \( |A| \geq n \). Set \( B = U \setminus A \). Now \( |\mathbb{Z} \setminus B| \geq |U \setminus B| = |A| \geq n \), so \( B \in B_n \), and moreover \( x \in B \subseteq U \), so we conclude \( B_n \) is a basis. \( \square \)

Problem 9 (due Weds 9/29): Let \( n \in \mathbb{N} \) and set \( C_n := \{ C \subseteq \mathbb{Z} \mid n \geq |\mathbb{Z} \setminus C| \} \). Prove that \( C_n \) is not a basis (for any topology).

**Solution:** Let \( B = \mathbb{Z} \setminus \{1, \ldots, n\} \) and \( C = \mathbb{Z} \setminus \{-1, \ldots, -n\} \), so \( |\mathbb{Z} \setminus B| = |\mathbb{Z} \setminus C| = n \), which means \( B, C \in C_n \). Now consider \( 0 \in B \cap C \), and let \( D \) be any set satisfying \( 0 \in D \subseteq B \cap C \). This implies \( |\mathbb{Z} \setminus D| \geq |\mathbb{Z} \setminus (B \cap C)| = 2n \), and \( n \geq 2n \), so \( D \notin C_n \). We conclude \( C_n \) is not a basis. \( \square \)

Problem 10 (due Weds 10/6): Prove that if \( X \) and \( Y \) have the discrete topology, then the product topology on \( X \times Y \) is also discrete.

**Solution:** It suffices to show that every singleton set \( \{(x, y)\} \) is open. Indeed, \( \{(x, y)\} = \{x\} \times \{y\} \), and each of \( \{x\} \) and \( \{y\} \) are open, so \( \{(x, y)\} \) is open. \( \square \)
Problem 11 (due Weds 10/6): Let $X$ be a space with the finite complement topology and let $Y$ be a finite subset of $X$. Prove that the subspace topology on $Y$ equals the discrete topology on $Y$.

**Solution:** It suffices to show that every singleton set $\{y\}$ in $Y$ is open in the subspace topology. Let $U = X \setminus (Y \setminus \{y\})$. Since $Y$ is finite, so is $Y \setminus \{y\}$, so $U$ is open. Also, $Y \cap U = Y \setminus (Y \setminus \{y\}) = \{y\}$, so we conclude that $\{y\}$ is open in $Y$. □

Problem 12 (due Weds 10/6): Let $X$ and $Y$ be topological spaces, with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively. Let $\mathcal{T}_{X \times Y}$ be the product topology on $X \times Y$. Let $A \subseteq X$ and $B \subseteq Y$ be subspaces, and let $\mathcal{T}_A$ and $\mathcal{T}_B$ be the subspace topologies. Let $\mathcal{T}_{A \times B}$ be the product topology on $A \times B$ (using $\mathcal{T}_A$ and $B$). Prove that $\mathcal{T}_{A \times B}$ equals the subspace topology on $A \times B$ as a subspace of $\mathcal{T}_{X \times Y}$. (More intuitively: Prove that the subspace and product topologies on $A \times B$ agree, either viewing $A \times B$ as a subspace of a product or as a product of subspaces.)

**Solution:** First let $U \times V$ be a basic open set in the product topology on $A \times B$. Since $U$ is open in $A$, we can choose $U'$ open in $X$ such that $U = A \cap U'$, and similarly choose $V'$ open in $Y$ with $V = B \cap V'$. Now $U' \times V'$ is a (basic) open set in $X \times Y$ and $U \times V = (A \times B) \cap (U' \times V')$, so we conclude that $U \times V$ is open in the subspace topology. This shows that the product topology on $A \times B$ is a refinement of the subspace topology, and it remains to show the converse. We know that a basis for the subspace topology can be obtained by intersecting the subspace with all the basic open sets from a basis of the ambient space. Hence $\{(A \times B) \cap (U \times V) \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ is a basis for the subspace topology on $A \times B$. Now since $(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$, each of these basic open sets in the subspace topology is also a (basic) open set in the product topology, as desired. □

Nothing due Weds 10/13 thanks to the timing of October break.

Problem 13 (due Weds 10/20): Let $X$ and $Y$ be sets, each with the finite complement topology. Prove that a function $f : X \to Y$ is continuous if and only if either $f(X) = \{y\}$ for some $y \in Y$ or else $f^{-1}(\{y\})$ is finite for all $y \in Y$.

**Solution:** Suppose $f$ is continuous. Suppose $f^{-1}(\{y\})$ is infinite for some $y \in Y$ (so $X$ is infinite), and we must show that it equals all of $X$. Since $\{y\}$ is closed in $Y$ and $f$ is continuous, $f^{-1}(\{y\})$ is closed in $X$. The only closed infinite subset of $X$ is $X$, so we are done. Now suppose that either $f(X) = \{y\}$ for some $y \in Y$ or else $f^{-1}(\{y\})$ is finite for all $y \in Y$. In the first case, $f$ is continuous because any constant map is continuous. In the second case, the preimage in $X$ of any closed subset $C$ of $Y$ is either $X$ (if $C = \{y\}$ or is a finite union of finite sets (if $C$ is finite), so in either case it is closed. Hence $f$ is continuous. □
Problem 14 (due Weds 10/20): Let $X$ be a metric space with metric $d$, viewed as a topological space with the metric topology. Fix $x_0 \in X$, and define $\phi_{x_0} : X \to \mathbb{R}$ via $\phi_{x_0}(x) := d(x_0, x)$. Prove that $\phi_{x_0}$ is continuous.

**Solution:** Let $(a, b)$ be a basic open set in $\mathbb{R}$, so $a < b$. The preimage of $(a, b)$ under $\phi_{x_0}$ is $\{x \in X \mid a < d(x_0, x) < b\}$. This equals the intersection of the open ball of radius $b$ centered at $x_0$ (which is open) with the complement of the closed ball of radius $a$ centered at $x_0$ (which is also open), hence is open. Thus, $\phi_{x_0}$ is continuous. \[\square\]

Problem 15 (due Weds 10/20): Prove that if a topological space is both Hausdorff and Alexandrov (see problem 5) then it is discrete.

**Solution:** Hausdorff implies singletons are closed, and Alexandrov implies arbitrary unions of closed sets are closed, and every set is the union of its singletons, so we conclude that every subset is closed; hence the topology is discrete. \[\square\]

Problem 16 (due Weds 11/3): Let $X = \mathbb{Z}$ with the finite complement topology, let $Y$ be a topological space in which every singleton $\{y\}$ is closed, and let $p : X \to Y$ be a surjective function. Prove that if $p$ is continuous then the topology on $Y$ is the finite complement topology.

Problem 17 (due Weds 11/3): With the same setup as Problem 16, prove that if $p$ is continuous then it is automatically a quotient map.

Problem 18 (due Weds 11/3): Let $\{0, 1\}$ have the discrete topology, and consider the product $\prod_{N} \{0, 1\}$ with the product (not box) topology. Prove that every basic open set in $\prod_{N} \{0, 1\}$ is closed.

Problem 19 (due Weds 11/10): Let $X$ be a topological space and $Y, Z \subseteq X$ open subspaces such that $X = Y \cup Z$. Suppose that $X, Y$, and $Y \cap Z$ are connected. Prove that $Z$ is connected.

Problem 20 (due Weds 11/10): Let $q : X \to Y$ be a quotient map. Suppose that $Y$ is connected, and that for each $y \in Y$ the preimage $q^{-1}(\{y\})$ is connected. Prove that $X$ is connected.

Problem 21 (due Weds 11/10): Prove that Problem #20 is *not* true with “connected” replaced by “path connected” everywhere. [Hint: Find a quotient of the topologist’s sine curve that is path connected and where every preimage of a singleton is path connected.]