Problem 1: Complete the following definitions:
1a (2 points):
A set $X$ is said to be a subset of a set $Y$ if for all $x \in X$ we have $x \in Y$.

1b (2 points):
The Cartesian product of sets $A$ and $B$ is the set of ordered pairs $(a, b)$ with $a \in A$ and $b \in B$.

1c (2 points):
The contrapositive of the statement $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$.

1d (2 points):
Two statements are said to be logically equivalent if they have the same truth tables.

Problem 2: Say whether the statement is true or false. You don’t need to formally prove anything but justify your answer.
2a (3 points): True or false: For any statements $P$ and $Q$ we have $\sim (P \lor Q) = (\sim P) \lor (\sim Q)$.
False - the $\lor$ should change to $\land$.

2b (3 points): True or false: For a finite set $X$, $|P(X)| = |X| + 1$ if and only if $X = \emptyset$.
(Think carefully about this one.) False - If $|X| = 1$ then $|P(X)| = |X| + 1$ is true but $X = \emptyset$ is false, so these are not logically equivalent.

Problem 3 (6 points): Let $A \subseteq X$ and $B \subseteq Y$. Prove that $A \times B \subseteq X \times Y$. Let $(a, b) \in A \times B$, so $a \in A$ and $b \in B$. Since $A \subseteq X$ and $B \subseteq Y$ we have $a \in X$ and $b \in Y$. Hence $(a, b) \in X \times Y$. □
Problem 4 (6 points): Prove that every odd integer is a difference of two squares. Let \( n \) be an odd integer, say \( n = 2k+1 \) for some \( k \in \mathbb{Z} \). Then \( n = 2k+1 = k^2 + 2k + 1 - k^2 = (k+1)^2 - k^2 \) is a difference of two squares.

Problem 5 (4 points): For \( n \in \mathbb{N} \) let \( n\mathbb{Z} = \{ x \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \text{ with } x = nk \} \). Prove that if \( m|n \) then \( n\mathbb{Z} \subseteq m\mathbb{Z} \). Suppose \( m|n \). Let \( x \in n\mathbb{Z} \), so \( x = nk \) for some \( k \in \mathbb{Z} \). Since \( m|n \) there exists \( \ell \) such that \( n = m\ell \). Then \( x = m\ell k \), so \( x \in m\mathbb{Z} \).

BONUS (+2 points):
For \( x \in \mathbb{R} \), prove that if \( 2^{\sin(x)} + 2^{\cos(x)} \leq \sin(x) \cos(x) \) then \( \sqrt{x^2 + 1} + \ln(x^4 + 1) \geq \int_{t=0}^{t=x} \frac{\sin(t)}{e^t} \, dt \).
For any \( x \) we have \( 2^{\sin(x)} + 2^{\cos(x)} > 1 \) and \( 1 > \sin(x) \cos(x) \), so the “if” part never happens and the whole thing is vacuously true.