Problem 1: Answer the following “write down an example” questions. In each case, JUSTIFY YOUR ANSWER.

1a (3 points): Find an example of non-trivial groups $A$ and $B$ such that the direct product $G = A \times B$ has order 57. Solution: $\mathbb{Z}_3 \times \mathbb{Z}_{19}$ works.

1b (3 points): Find an example of a group $G$ of order 6 and a surjective homomorphism $\phi: G \rightarrow \mathbb{Z}_3$. Solution: Can do $G = \mathbb{Z}_6$ and $\phi: \mathbb{Z}_6 \rightarrow \mathbb{Z}_3$ given by $\phi([a]_6) = [a]_3$.

1c (3 points): An element $x$ of a ring $R$ is called nilpotent if $x^k = 0$ for some $k \in \mathbb{N}$. Find an example of a non-zero nilpotent element of $\mathbb{Z}_{76}$. [Hint: Use the fact that $76 = 4 \cdot 19$.] Solution: Can do $x = [2 \cdot 19]_{76}$. This is not zero, but when we square it we get $[2^2 \cdot 19^2]_{76} = [4 \cdot 19^2]_{76} = [76 \cdot 19]_{76} = [0]_{76}$.

1d (3 points): Find an ideal $I$ of the ring $\mathbb{Z}[x]$ such that the quotient ring $R/I$ has 39 elements. [Hint: Think about Problem #30 on the homework.] Solution: Can use the ideal generated by $x$ and 39. This works by the same reasoning as HW #30.
Problem 2: Say whether the statement is true or false. You don’t need to formally prove anything but JUSTIFY YOUR ANSWER.

2a (2 points): True or false: The principal ideal of $\mathbb{Z}[x]$ generated by $x^2 - 19^2$ is prime. [Hint: Think about Problem #31 on the homework.] Solution: False. We have $(x - 19)(x + 19) = x^2 - 19^2$ is in the ideal, but neither $x - 19$ nor $x + 19$ is.

2b (2 points): True or false: A group of order 32 can contain an element of order 21. Solution: False. 21 does not divide 32, so Lagrange says false.

Problem 3 (4 points): Prove that if $\phi: \mathbb{Z}_9 \to \mathbb{Z}$ is a function that is a group homomorphism then it must send every element to 0. Solution: For any $[a]_9 \in \mathbb{Z}_9$ we have $0 = \phi([0]_9) = \phi([9a]_9) = \phi(9[a]_9) = 9\phi([a]_9)$. Dividing both sides by 9 we get $0 = \phi([a]_9)$.

□
Problem 4 (5 points): Let $R = \{ f : \mathbb{R} \to \mathbb{R} \}$ be the ring of functions from $\mathbb{R}$ to $\mathbb{R}$. Let $I = \{ f \in R \mid f(-14) = 0 \}$. Prove that $I$ is an ideal in $R$. Solution: Clearly $0 \in I$. Now let $f, g \in I$, so $f(-14) = g(-14) = 0$. Then $(f - g)(-14) = f(-14) - g(-14) = 0 - 0 = 0$ so $f - g \in I$. Also, for any $f \in I$ and $h \in R$ we have $(fh)(-14) = f(-14)h(-14) = 0h(-14) = 0$ so $fh \in I$. Since $R$ is commutative, this proves $I$ is an ideal. □

Problem 5 (5 points): Let $R$ be a ring and $x \in R$ a fixed element. The centralizer $C_R(x)$ of $x$ in $R$ is $C_R(x) := \{ y \in R \mid xy = yx \}$. Prove that $C_R(x)$ is a subring of $R$. Solution: First note $0x = 0 = x0$ so $0 \in C_R(x)$. Now let $y, z \in C_R(x)$, so $xy = yx$ and $xz = zx$. Then $x(y - z) = xy - xz = yx - zx = (y - z)x$ so $y - z \in C_R(x)$, and $x(yz) = (xy)z = (yx)z = y(xz) = y(zx) = (yz)x$ so $yz \in C_R(x)$. □

BONUS (+2 points): A group is torsion-free if every non-trivial element has infinite order. Prove that if the group ring $\mathbb{Z}[G]$ of a group $G$ is an integral domain then $G$ is torsion-free. [Hint: See the lecture notes from 10/26/20 for the definition of group ring.] Solution: We will prove the contrapositive. Let $1 \neq g \in G$ be a non-trivial element of finite order, say order $n$. Then in $\mathbb{Z}[G]$ we have $g^n - 1 = 0$, so $(g - 1)(g^{n-1} + \cdots + g + 1) = 0$. We know $g - 1 \neq 0$ since $g \neq 1$. Also, since the $g^i$ for $i = 0, \ldots, n - 1$ are distinct, they are linearly independent over $\mathbb{Z}$, so $g^{n-1} + \cdots + g + 1 \neq 0$. This shows $g - 1$ is a zero divisor so $\mathbb{Z}[G]$ is not an integral domain. □