Problem 1: Complete the following definitions:

1a (2 points): Let $G$ and $H$ be groups and let $\phi: G \to H$ be a function. Then $\phi$ is a \textit{homomorphism} if $\phi(gg') = \phi(g)\phi(g')$ for all $g, g' \in G$.

1b (2 points): An element of $S_n$ is \textit{even} (respectively \textit{odd}) if it is a product of an even (respectively odd) number of 2-cycles. The \textit{alternating group} $A_n$ is defined to be \textit{the subset of all even elements of $S_n$}.

1c (2 points): Let $G$ be a group and $H$ a subgroup. A \textit{left coset} of $H$ in $G$ is a subset of $G$ of the form $\{gh \mid h \in H\}$ for some $g \in G$.

Problem 2: Say whether the statement is true or false, and justify your answer.

2a (3 points): True or false: For any group $G$ and any $g, h \in G$, we have $(gh)^{-1} = g^{-1}h^{-1}$. False: Counterexample is any $g, h$ that don’t commute, e.g., $(1\ 2)$ and $(2\ 3)$ in $S_3$.

2b (3 points): True or false: Every group of order 97 is isomorphic to $\mathbb{Z}_{97}$. True: Since 97 is prime.

Problem 3 (6 points): Prove that in any group $G$, the identity element is unique. Suppose $e$ and $e'$ are identity elements, so $eg = ge = g$ and $e'g = ge' = g$ for all $g \in G$. Then $ee' = e'$ and $ee' = e$, so $e = e'$.
Problem 4 (6 points): Let $G$ be a group with odd order. Prove that no element of $G$ has order 2. If $g \in G$ has order 2 then the subgroup $\langle g \rangle$ has order 2, but Lagrange says the order of a subgroup divides the order of the group, so this is impossible.

Problem 5 (6 points): Let $G$ be a group and $H$ a subgroup of $G$. The normalizer of $H$ is

$$N_G(H) := \{ n \in G \mid nHn^{-1} = H \}$$

and the centralizer of $H$ is

$$C_G(H) := \{ c \in G \mid chc^{-1} = h \text{ for all } h \in H \}.$$

Prove that $C_G(H)$ is a subgroup of $N_G(H)$. Since $ehe^{-1} = h$ for all $h \in H$ we know $e \in C_G(H)$, so $C_G(H) \neq \emptyset$. Now let $c, c' \in C_G(H)$, so $chc^{-1} = c'h(c')^{-1} = h$ for all $h \in H$. Then $h = c^{-1}hc$, and so we get $(c^{-1}c'h(c')^{-1} = c^{-1}c'h(c')^{-1}c = c^{-1}hc = h$ for all $h \in H$, which implies $c^{-1}c' \in C_G(H)$. We conclude $C_G(H)$ is a subgroup.

BONUS (+2 points): Prove that $C_G(H)$ is a normal subgroup of $N_G(H)$. We need to show that $nC_G(H)n^{-1} = C_G(H)$ for all $n \in N_G(H)$. Since $N_G(H)$ is closed under inverses it suffices to show that $ncn^{-1} \in C_G(H)$ for all $n \in N_G(H)$ and $c \in C_G(H)$, since the other inclusion will then follow for free. Indeed, for any $n \in N_G(H)$, $c \in C_G(H)$, and $h \in H$, we have $(ncn^{-1})h(ncn^{-1})^{-1} = ncn^{-1}hnc^{-1}n^{-1} = nc(n^{-1}hn)c^{-1}n^{-1} = n(n^{-1}hn)n^{-1} = h$, since $n^{-1}hn \in H$.  

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