Updated April 23, 2018

Homework problems for AMAT 840 (Topics in Topology (geometric group theory)), Spring 2018. Over the course of the semester I'll add problems to this list, with each problem's due date specified.

Problem 1 (due Fri 2/16): Prove that none of the Platonic solids (I mean their 1-skeleta, with any orientation/labeling) can be the Cayley graph of any group for any generating set, except the octahedron. Which group/generating set gives us the octahedron? [Hint: the first question is supposed to be easy. The second might require lots of trial and error.]

Solution: The others have odd-degree vertices, so can't be Cayley graphs. The octahedron is a Cayley graph for $\mathbb{Z}/6\mathbb{Z}$, easy to work out the details.

Problem 2 (due Fri 2/16): Let Q_8 be the quaternion group of order 8, with generating set $S = \{i, j\}$. Draw $\Gamma(Q_8, S)$. [Hopefully it's clear what I mean by Q_8, i and j; if not, just ask.]

Solution: A google-able picture.

Problem 3 (due Fri 2/16): Let $G = \mathbb{Z}$ with generating set $\{1\}$ and let $H = \mathbb{Z}^2$ with generating set $\{(1, 0), (0, 1)\}$. Prove that G and H are not quasi-isometric. [Don't just prove that the inclusion $G \to H$ fails to be a QI; prove that no QI exists.] [Update: this was too hard before we discussed ends of groups. Now it's obviously true based on number of ends.]

Problem 4 (due Fri 3/9): Prove that every subgroup of $\mathbb{Z}^n$ is finitely generated.
Solution: If you use stuff from ring theory this is immediate, since $\mathbb{Z}^n$ is a finitely generated module over the noetherian ring $\mathbb{Z}$, and hence is a noetherian module. Let's do a direct proof though. Let $G \leq \mathbb{Z}^n$ (and use additive notation). Induct on n. If $n = 1$ then G is a subgroup of a cyclic group, hence cyclic, hence finitely generated. Now assume $n \geq 2$. Let $\pi\colon \mathbb{Z}^n \to \mathbb{Z}$ be the projection onto the last factor, and consider the restriction $\pi|_G\colon G \to \mathbb{Z}$. The kernel of $\pi|_G$ lies in $\mathbb{Z}^{n-1}$, so is finitely generated by induction. Say it is generated by $g_1, \dots, g_k$. Now, the image $\pi(G)$ is cyclic (being a subgroup of $\mathbb{Z}$), so we can pick $h \in G$ with $\langle \pi(h) \rangle = \pi(G)$. We claim G is generated by $g_1, \dots, g_k, h$. Let $g \in G$, so $\pi(g) = \pi(mh)$ for some $m \in \mathbb{Z}$. Hence $g - mh \in \ker(\pi|_G)$ and so $g - mh = a_1 g_1 + \cdots + a_k g_k$ for some $a_1, \dots, a_k \in \mathbb{Z}$. Hence $g = mh + a_1 g_1 + \cdots + a_k g_k$ as desired.

Problem 5 (due Fri 3/9): Let $G = F_2 \times F_2$, say the first copy of F_2 is generated by $\{a, b\}$

and the second by $\{y, z\}$, so G is generated by $\{a, b, y, z\}$. Let $\phi\colon G \to \mathbb{Z}$ be defined by sending each of a, b, y, z to 1, and set $H = \ker(\phi)$. Prove that H is generated by the finite set $\{ay^{-1}, az^{-1}, ab^{-1}\}$. [Bonus: Prove H is not finitely presented (I can't think of a proof that doesn't use discrete Morse theory and homological algebra....).]

Solution: Let's toss in $by^{-1} = (ab^{-1})^{-1}ay^{-1}$ and $bz^{-1} = (ab^{-1})^{-1}az^{-1}$ to the set for convenience. We claim this generates H; to show this we will take an arbitrary element and multiply it by elements of this set until we get the identity. Let $x_1 \cdots x_k$ be a word in $\{a^{\pm}, b^{\pm}, y^{\pm}, z^{\pm}\}^*$ representing an element of H. If for all i we have $x_i \in \{y^{\pm}, z^{\pm}\}$ then every x_i commutes with a, and so right multiplying by appropriate $x_k^{-1}a^{\pm}$ and then $x_{k-1}^{-1}a^{\pm}$ and so forth, we reach a point where every x_i is a or a^{-1}. The only power of a in H is the identity, so we are done. Now suppose some x_i is not in $\{y^{\pm}, z^{\pm}\}$. Without loss of generality there is a j such that $x_1, \dots, x_j \in \{y^{\pm}, z^{\pm}\}$ and $x_{j+1}, \dots, x_k \in \{a^{\pm}, b^{\pm}\}$. Now right multiplying by appropriate $x_k^{-1}y^{\pm}$ and then $x_{k-1}^{-1}y^{\pm}$ and so forth, we reduce to the previous case and we are done.

Problem 6 (due Fri 3/9): Since Problem 3 was too hard, here's a related, more reasonable thing: Prove that for $m \neq n$ the groups $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are not commensurable. (Prove it directly, don't just say, "well I know they're not QI so they can't be commensurable!") [Hint: How much variety is possible for finite index subgroups of $\mathbb{Z}^n$?]

Solution: WLOG $m < n$. First we claim $\mathbb{Z}^m$ cannot embed with finite index into $\mathbb{Z}^n$. Indeed, any copy of $\mathbb{Z}^m$ in $\mathbb{Z}^n \subseteq \mathbb{Q}^n$ admits an m-element generating set, and hence its $\mathbb{Q}$-span in $\mathbb{Q}^n$ is a proper subspace. But any finite index subgroup of $\mathbb{Z}^n$ spans all of $\mathbb{Q}^n$ since it contains a power of each standard generator of $\mathbb{Z}^n$. Next we claim that any finite index subgroup G of $\mathbb{Z}^m$ contains a finite index copy of $\mathbb{Z}^m$. Say $[\mathbb{Z}^m : G] = q$. Then the qth powers of the standard generators generate a finite index copy of $\mathbb{Z}^m$ that lies in G. Hence if G embeds with finite index into both $\mathbb{Z}^m$ and $\mathbb{Z}^n$, then $\mathbb{Z}^m$ embeds with finite index into $\mathbb{Z}^n$, which we ruled out. This finishes the proof, and actually shows the stronger statement that every finite index subgroup of $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^n$.

Problem 7 (due Fri April 13): Let A and B be groups and let $G = A * B$ be the free product. Prove that G is of type F_n if and only if A and B are of type F_n.

Solution: The inclusion $A \to G$ and the quotient $G \to A$ given by sending B to 1 provide a retract of G onto A, so if G is of type F_n then so is A, and B analogously. Now suppose A and B are of type F_n. Let X be a $K(A, 1)$ with compact n-skeleton and Y a $K(B, 1)$ with compact n-skeleton. Then Van Kampen says $X \vee Y$ has fundamental group G, and it has finite n-skeleton, so it suffices to see it's aspherical. We didn't really show that a wedge of aspherical spaces is aspherical, but it is true, and you can google around to find reasons.

Problem 8 (due Fri April 13): Let $G = A *_U B$ be an amalgamated product of A and B over U. Suppose that A and B are of type F_n and U is of type F_{n-1}. Prove that G is of type F_n.

Solution: Let T be the canonical tree on which G acts with fundamental domain an edge. Trees are contractible and this action is cocompact, so it suffices by Brown's Criterion (with the trivial filtration) to show that every vertex stabilizer is of type F_n and every edge stabilizer is of type F_{n-1}. The stabilizer of the vertex gA is $gAg^{-1} \cong A$, of gB is $gBg^{-1} \cong B$, and of gU is $gUg^{-1} \cong U$, so this follows from our assumptions.

Problem 9: (due Fri April 13): Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of groups. Prove that if K and Q are of type F_n then so is G.

Solution: Let X be an $(n-1)$-connected complex on which Q acts freely and cocompactly (for example the n-skeleton of the universal cover of a $K(Q,1)$ with finite n-skeleton). The action of G on X via the quotient $G \to Q$ is cocompact since every G-orbit is a Q-orbit. The action of Q on X is free, so $g \in G$ fixes a point if and only if it lies in the kernel of $G \to Q$; hence every stabilizer in G is isomorphic to K. Since K is of type F_n, Brown's Criterion (with the trivial filtration) now implies G is of type F_n.

(Big) Problem 10: (due Mon May 7): For $n \in \mathbb{N}$ let X_n be the simplicial complex with vertex set $\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq x \leq n\}$ such that a collection of vertices $\{(x_0,y_0),\dots,(x_k,y_k)\}$ spans a k-simplex if and only if $x_0,\dots,x_k$ are pairwise distinct and $y_0,\dots,y_k$ are pairwise distinct. (So you can think of the vertices as the entries of an ∞-by-n matrix and a collection of vertices spans a simplex if and only if none of them share a row or a column.) Prove that X_n is $(n-2)$-connected.

Hint: You can define a "Morse function" on X_n by lexicographically ordering $\mathbb{N} \times \mathbb{N}$, and then inspect descending links. If you're feeling ambitious explain why this is as good as an actual Morse function, i.e., why you can embed $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ taking the lexicographic order to the usual order in such a way that, even though the result is not discrete on vertices, the steps of Morse theory still work. If you're not feeling ambitious just trust that Morse theory still works with this "Morse function" even though it's not really discrete on vertices.