Problem 1: Define what it means for \( T \subseteq \mathcal{P}(X) \) to be a topology on \( X \).

Problem 2: Let \( \{X_\alpha\}_{\alpha \in \Lambda} \) be a family of topological spaces. Define the box topology and product topology on \( \prod_{\alpha \in \Lambda} X_\alpha \).

Problem 3: Give a (concrete) example of a topological space that is not metrizable (and prove it’s not).
Solution: For example take \( \{0,1\} \) with the trivial topology. This is not Hausdorff, hence not metrizable.

Do one of 4a or 4b, you don’t have to do both. If you do both for some reason, indicate which one you actually want me to grade and I’ll ignore the other one.

Problem 4a: Let \( X = \mathbb{Z} \) with the finite complement topology and let \( A \) be an arbitrary non-empty open subset of \( X \). Prove that every \( x \in X \) is a limit point of \( A \).
Solution: Note that \( X \setminus A \) is finite. Let \( U \) be an open neighborhood of \( x \), so \( X \setminus U \) is finite. Now \( (X \setminus A) \cup (X \setminus U) \) is finite, and equals \( X \setminus (A \cap U) \), hence \( A \cap U \neq \emptyset \). Since \( U \) was arbitrary, \( x \) is a limit point of \( A \).

Problem 4b: True or False: Given any topological space \( X \) and any subset \( A \), the closure \( \overline{A} \) of \( A \) equals the union of the interior \( \mathring{A} \) with the set of limit points \( A' \). If it’s true, prove it. If it’s false, give a counterexample.
Solution: False. Counterexample \( X = \mathbb{R} \) and \( A = \{0\} \). Then \( \overline{A} = \{0\} \) but \( \mathring{A} \) and \( A' \) are both empty. (Any \( A \) with “isolated” points works, since they’ll neither be in the interior nor be limit points.)

Do one of 5a or 5b, you don’t have to do both. If you do both for some reason, indicate which one you actually want me to grade and I’ll ignore the other one.

Setup: Let \( X \) be a topological space and \( A \) a subspace. If \( f : X \to A \) is a continuous map such that \( f(a) = a \) for all \( a \in A \) we call \( f \) a retraction, and say that \( A \) is a retract of \( X \).

Problem 5a: Prove that any retraction \( f : X \to A \) is a quotient map.
Solution: By construction \( f \) is continuous and surjective. Now suppose \( U \subseteq A \) such that \( f^{-1}(U) \) is open in \( X \), and we need to show \( U \) is open in \( A \). We claim \( f^{-1}(U) \cap A = U \) and hence \( U \) is open in \( A \). Indeed, \( U \subseteq f^{-1}(U) \cap A \) since \( U \subseteq A \) by assumption and \( f(u) = u \) for all \( u \in U \). Conversely, if \( a \in f^{-1}(U) \) then \( a = f(a) \in U \) so \( f^{-1}(U) \cap A \subseteq U \).

Problem 5b: Let \( X = \mathbb{Z} \) with the finite complement topology and \( A = 2\mathbb{Z} \) the subspace of even numbers. Explain why \( A \) is a retract of \( X \).
Solution: Define $f : X \to A$ via $f(2n) = 2n$ and $f(2n - 1) = 2n$ for all $n \in \mathbb{Z}$. To see $f$ is a retraction we just need to check it’s continuous. It suffices to show for any closed $C$ in $X$, $f^{-1}(C)$ is closed. This is trivial for $C = X$ so assume $C$ is finite. Since $f^{-1}(\{2n\}) = \{2n - 1, 2n\}$ for each $n$, $|f^{-1}(C)| = 2|C| < \infty$, so $f^{-1}(C)$ is closed.

□

Bonus: Prove that every infinite subspace of $\mathbb{Z}$ with the finite complement topology is a retract.

Solution: Let $A$ be an infinite subspace. Since $\mathbb{Z} \setminus A$ is countable it embeds in $A$, say $g : \mathbb{Z} \setminus A \to A$ is some injection. Define $f : \mathbb{Z} \to A$ via $f(a) = a$ for $a \in A$ and $f(x) = g(x)$ for $x \in \mathbb{Z} \setminus A$. Then for each $a \in A$, $|f^{-1}(\{a\})|$ is either 1 or 2. In particular for any finite $C$ in $A$, $f^{-1}(C)$ is finite, so preimages of closed sets are closed and $f$ is continuous. □