Problem 1: Define what it means for a topological space to be second countable.
   Solution: Admits a countable basis.

Problem 2: Give an example of a space that is connected but not path connected (you don’t have to rigorously prove it, but give a rough explanation of why it’s connected and not path connected).
   Solution: Topologist’s sine curve (can google).

Problem 3: Let $X$ be a locally compact Hausdorff space and $Y = X \cup \{\infty\}$ its one-point compactification. Let $C$ be a closed subspace of $X$. Explain why $C \cup \{\infty\}$ is compact.
   Solution: Since $C$ is closed in $X$, $X \setminus C$ is open in $X$. The topology on $Y$ is such that $X \setminus C$ is also open in $Y$, hence $Y \setminus (X \setminus C) = C \cup \{\infty\}$ is closed in $Y$. Since $Y$ is compact this implies $C \cup \{\infty\}$ is compact. □

**Do one of 4a or 4b, you don’t have to do both.** If you do both for some reason, indicate which one you actually want me to grade and I’ll ignore the other one.

Problem 4a: Sketch a proof that every compact Hausdorff space is normal.
   Solution: Given disjoint closed $A$ and $B$, fix $a \in A$. Separate $a$ from each element of $B$ with disjoint open neighborhoods. These cover $B$, which is compact, so we only need finitely many of them, and intersecting the finitely many corresponding open neighborhoods of $a$ separates $a$ from $B$ (so now the space is regular). Now do a similar trick letting $a$ vary, intersect finitely many open nbds of $B$, and end up with disjoint open nbds of $A$ and $B$. □

Problem 4b: Sketch a proof that every metric space is normal.
   Solution: Given disjoint closed $A$ and $B$, around each $a \in A$ fit a small ball disjoint from $B$. Around each $b \in B$ fit a small ball disjoint from $A$. Replace each ball with one half the radius (same center). Now the first batch of balls covers $A$ and the second covers $B$, and the triangle inequality says each ball of the first kind is disjoint from each ball of the second kind. So the unions of these balls separate $A$ and $B$. □

**Do one of 5a or 5b, you don’t have to do both.** If you do both for some reason, indicate which one you actually want me to grade and I’ll ignore the other one.

Problem 5a: Let $f, g: S^1 \to S^1$ be $f(x, y) = (x, y)$ and $g(x, y) = (x, -y)$. Show that $f$ and $g$ are not homotopic.
   Solution: Note that $f_*$ is the identity on $Z = \pi_1(S^1)$, and $g_*$ is not since, e.g., $g_*([\omega_1]) = [\omega_{-1}]$, so $f$ and $g$ are not homotopic. □
Problem 5b: Let \( f, g : [0, 1] \to S^1 \times S^1 \) be loops in the torus with the same basepoint. Show that \( f \ast g \ast \tilde{f} \ast \tilde{g} \) is nullhomotopic.

Solution: Since \( \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2 \) is abelian, \([f \ast g \ast \tilde{f} \ast \tilde{g}] = [f] \ast [g] \ast [\tilde{f}] \ast [\tilde{g}] = [f] \ast [\tilde{f}] \ast [g] \ast [\tilde{g}] = 1.\)

Bonus: The Klein bottle \( K \) is the quotient of \([0, 1] \times [0, 1]\) by the identifications \((x, 0) \sim (x, 1)\) for all \(x\) and \((0, y) \sim (1, 1 - y)\) for all \(y\). Prove that \( K \) is not homotopy equivalent to the torus \( S^1 \times S^1 \).

Solution: It suffices to show \( \pi_1(K) \) is non-abelian. The construction shows that \( \pi_1(K) = \langle a, b \mid aba^{-1} = b^{-1} \rangle \), which is non-abelian.