2-SEGAL SPACES AND ALGEBRAIC K-THEORY

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1. Introduction

The first thing to introduce in this talk is the notion of a 2-Segal space, or if we consider the discrete case (which we mostly do in this talk), a 2-Segal set.

Informally, a *1-Segal set* behaves like a *category*: it has objects, morphisms, and a unique composition law which is associative. A *2-Segal set* encodes something weaker: its composition may or may not exist, or be unique when it does, but it is nonetheless associative.

Our goal today is to look more closely at the precise structure of a 2-Segal set, and then see how such structures arise in algebraic K-theory. In particular, we will give some conjectured relationships with some of the structures which appeared in Inna's talk.

2. Simplicial sets and 1-Segal sets

To get to a precise definition of 2-Segal sets, we first need to review the notion of a simplicial set. Recall that the category Δ has objects the finite ordered sets

$$[n] = \{0 \leq 1 \leq \cdots \leq n\}$$

for each $n \geq 0$, and morphisms the order-preserving maps. Its *opposite category* Δ^{op} has the same objects but all the arrows reversed.

Definition 2.1. A *simplicial set* is a functor $K \colon \Delta^{\text{op}} \to \mathcal{S}ets$.

The "downward" arrows $[n] \to [n-1]$ are called *face maps* and can be thought of like the maps of the same name in a simplicial complex. The "upward" maps $[n] \to [n+1]$ are called *degeneracy maps* and can be thought of as a means of thinking of a simplex of a given dimension as a degenerate simplex of a higher dimension.

Example 2.2. Let $\mathcal{C}$ be a small category. Its *nerve* is a simplicial set with 0-simplices the objects of $\mathcal{C}$, 1-simplices the morphisms of $\mathcal{C}$, 2-simplices given by composable pairs of morphisms, namely elements of

$$\text{mor}(\mathcal{C}) \times_{\text{ob}(\mathcal{C})} \text{mor}(\mathcal{C}),$$

and similarly n-simplices are given by chains of n composable morphisms of $\mathcal{C}$.

Definition 2.3. A *1-Segal set* is a simplicial set K such that, for any $n \geq 2$, there is an isomorphism

$$K_n \cong K_1 \times_{K_0} \cdots \times_{K_0} K_1.$$

The maps giving these isomorphisms are defined as follows. Consider the inclusion

$$G(n) := (\bullet \to \bullet \to \cdots \to \bullet) \hookrightarrow \Delta[n].$$

Mapping into a fixed simplicial set K, we get

$$K_n = \mathrm{Hom}(\Delta[n], K) \to \mathrm{Hom}(G(n), K) = K_1 \times_{K_0} \cdots \times_{K_0} K_1.$$

Observe that a 1-Segal set coincides exactly with the nerve of a category. Composition can be defined via

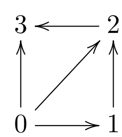
$$K_1 \times_{K_0} K_1 \overset{\cong}{\leftarrow} K_2 \to K_1.$$

3. 2-Segal sets

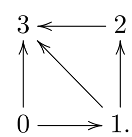
The generalization to 2-Segal sets is due to Dyckerhoff and Kapranov, and independently by Gálvez-Carrillo, Kock, and Tonks, under the name of (discrete) decomposition space.

If we think of $G(n)$ as a "triangulation" of a line segment, then we could move up a dimension and look at triangulations of regular polygons. There are two triangulations of a square:

$\mathcal{T}_1$:



$\mathcal{T}_2$:



Each gives two faces of the boundary of a 3-simplex $\Delta[3]$.

Take these two inclusions of simplicial sets and map into a simplicial set K:

$$K_3 \cong \mathrm{Hom}(\Delta[3], K) \xrightarrow{(d_1,d_3)} \mathrm{Hom}(\mathcal{T}_1, K) \cong K_2 \times_{K_1} K_2$$

$$K_3 \cong \mathrm{Hom}(\Delta[3], K) \xrightarrow{(d_0,d_2)} \mathrm{Hom}(\mathcal{T}_2, K) \cong K_2 \times_{K_1} K_2.$$

Definition 3.1. A *2-Segal set* is a simplicial set K such that for every $n \geq 3$ and every triangulation of a regular $(n+1)$-gon, the induced map

$$K_n \to \underbrace{K_2 \times_{K_1} \cdots \times_{K_1} K_2}_{n-1}$$

is an isomorphism.

A 2-Segal set still has objects K_0 and morphisms K_1 but no guaranteed composition, since the left-going map in

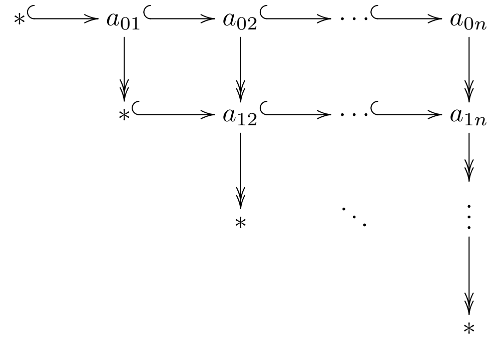
$$K_1 \times_{K_0} K_1 \leftarrow K_2 \to K_1$$

need not be an isomorphism. We can define a composition by taking a pre-image and then pushing forward. Inspecting the maps coming from the two triangulations of a square above, we can see that this composition must still be associative, however.

4. The discrete $S_\bullet$ construction

A natural question is then whether there are natural examples of 2-Segal sets. Dyckerhoff and Kapranov, as well as Gálvez-Carrillo, Kock, and Tonks, showed that 2-Segal *spaces* arise from applying Waldhausen's $S_\bullet$-construction to an exact category. Here, we'll look at a discrete version.

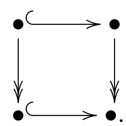
The idea is to look at diagrams



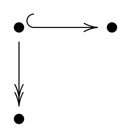
with distinguished horizontal and vertical morphisms and "bicartesian" squares. What is the minimum necessary input for such a diagram to make sense?

We claim that the desired structure is that of a *pointed stable double category.*

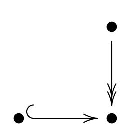
- A *double category* consists of objects, horizontal morphisms ($\hookrightarrow$), vertical morphisms ($\twoheadrightarrow$), and squares



- Being *pointed* means that there is an object $*$ which is initial in the horizontal category and terminal in the vertical category.
- Being *stable* means that any span



  or any cospan



  uniquely determines a square.

Theorem 4.1. *(B-Osorno-Ozornova-Rovelli-Scheimbauer) The discrete $S_\bullet$-construction defines an equivalence of categories between pointed stable double categories and reduced 2-Segal sets.*

Here, "reduced" means that $K_0 = *$. This result has two generalizations:

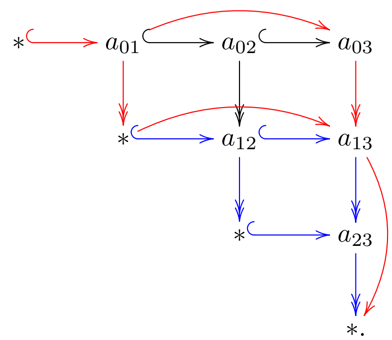
- We can replace "pointed" with "augmented" to get all 2-Segal sets, not just reduced ones.

- (Reduced) 2-Segal *spaces* correspond to pointed stable double Segal spaces, via a Quillen equivalence of model categories.

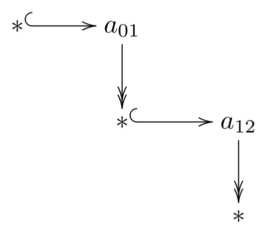
Let us look at why we would expect $S_\bullet(\mathcal{D})$ to be 2-Segal, for $\mathcal{D}$ a pointed stable double category. The map

$$(d_0, d_2)\colon \mathcal{S}_3(\mathcal{D}) \to \mathcal{S}_2(\mathcal{D}) \times_{\mathcal{S}_1(\mathcal{D})} \mathcal{S}_2(\mathcal{D})$$

is associated to:



However, it is not 1-Segal, since the diagram



need not have a completion to a square in the upper-right corner.

5. Connections with CGW categories

Pointed stable double categories seem very similar to CGW categories, as defined in Inna's talk.

- They are pointed double categories.
- There, stablility holds up to unique isomorphism (with the reverse direction of "vertical" arrows).

In joint work with Zakharevich, we conjecture that CGW categories correspond to pointed stable double Segal groupoids. A natural question we wish to answer is what *abelian* CGW categories correspond to. We could then ask whether we can go back and forth between the two contexts with results and examples.

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