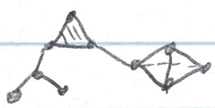
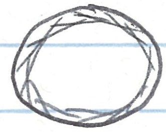
Metric reconstruction via optimal transport

Joint with Michał Adamaszek and Florian Frick

X metric space, $r>0$

Def The Vietoris-Rips simplicial complex $VR(X;r)$ has
- vertex set X
- finite simplex $\sigma \subseteq X$ when $\mathrm{diam}(\sigma) \leq r$

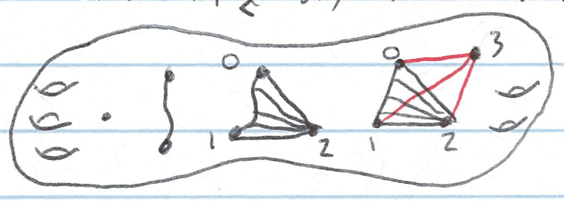
(Clique or flag simplicial complex)
($<$ subscript)

History Leopold Vietoris
- (co)homology theory for metric spaces
- recovers Čech cohomology if X compact
- Vietoris homology: counterpart to Alexander-Spanier cohomology

Ilya Rips
- Geometric group theory
- $VR\left(\substack{\delta\text{-hyperbolic group}\\ \text{word metric}};r\right) \simeq * \text{ for } r \geq 4\delta$

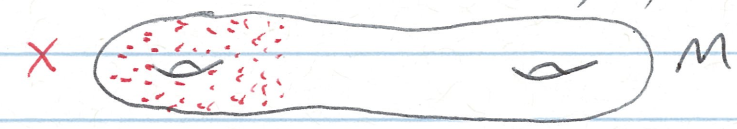
Thm (Hausmann '95) M compact Riemannian manifold.
Then $\exists\, r_0>0$ s.t. $VR_<(M;r) \simeq M$ $\forall r<r_0$.

Sketch $VR_<(M;r)$
$\downarrow$
M



Not canonical
$M \hookrightarrow VR_<(M;r)$ not continuous

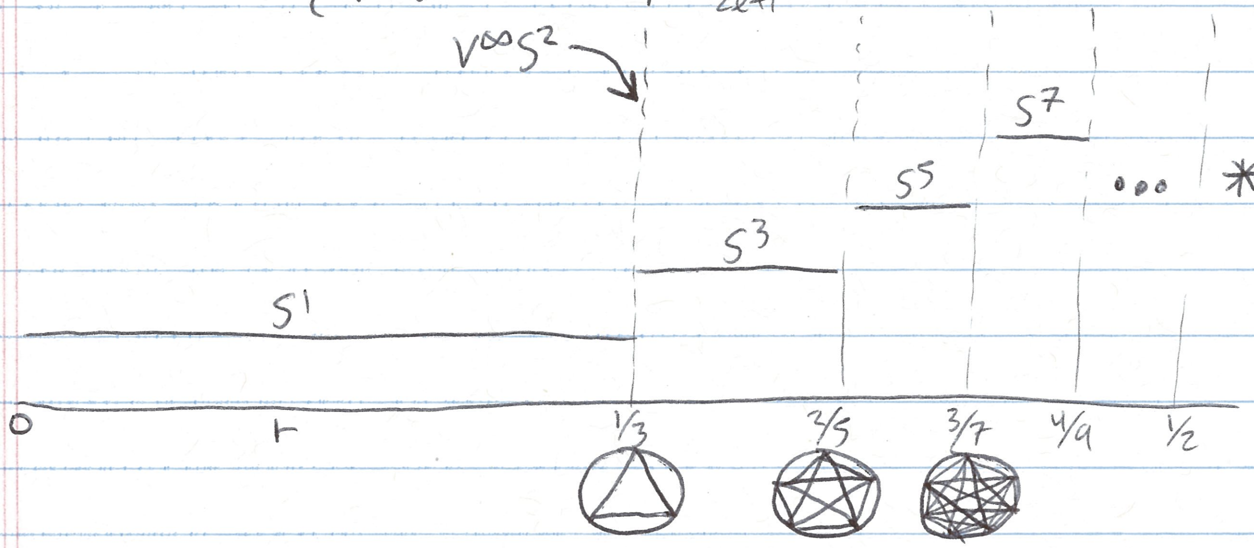
Thm (Latschev '01) M, r_0 as above.
$\forall r<r_0$ $\exists\, \delta>0$ s.t. if $d_{GH}(X;M)<\delta$, then $VR_<(X;r) \simeq M$.



Ex Cyclo-octane molecule C_8H_{16}
Martin, Thompson, Coutsias, Watson '10

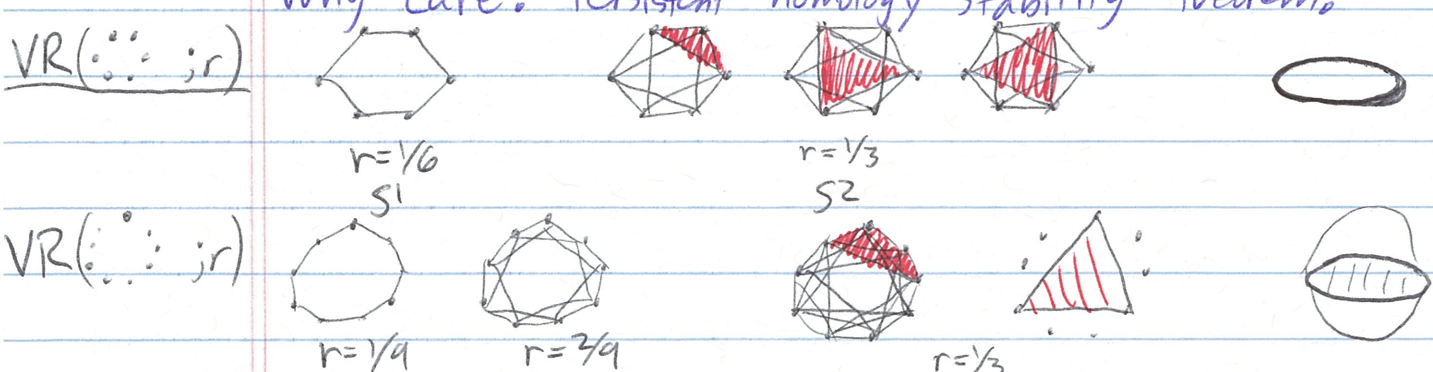
S^1 is circle w/ geodesic metric, unit circumference.

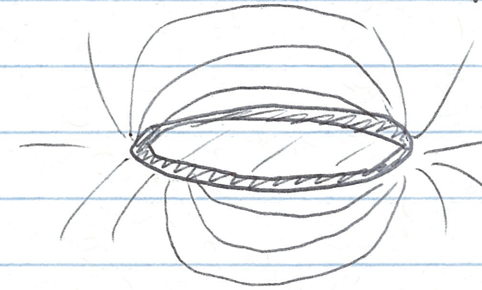
Thm $VR(S^1;r) \simeq \begin{cases} S^{2\ell+1} & \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ \bigvee^{\infty} S^{2\ell} & r = \frac{\ell}{2\ell+1} \end{cases}$ for $\ell \in \mathbb{N}$



Only connected non-contractible manifold w/ all VR homotopy types known

Why care? Persistent homology stability theorem.





$\check{C}(S^1;r)$ regime when Nerve lemma fails

Metric reconstruction

 $\rightsquigarrow$  $\rightsquigarrow \quad VR(X;r) \overset{?}{\simeq} M$

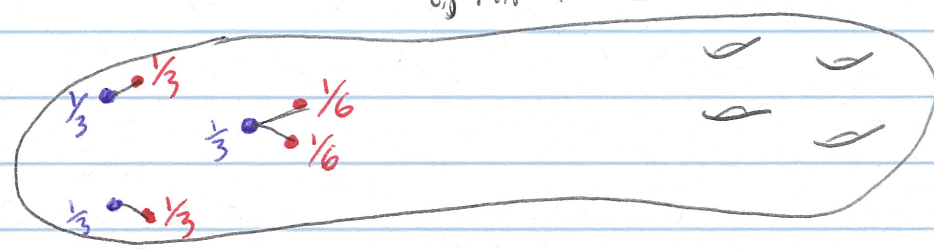
metric space M $\qquad$ $X \subseteq M$ $\qquad$ Metrizable $\Longleftrightarrow$ locally finite

Def X metric space, $r>0$. The <u>Vietoris–Rips thickening</u> $VR^m(X;r)$ is $|VR(X;r)|$ as a set, equipped w/ the 1-Wasserstein metric.

Explicitly $VR^m(X;r) = \left\{ \sum_{i=0}^{k} a_i x_i \;\middle|\; \begin{array}{ll} x_i \in X & \operatorname{diam}[x_0,\dots,x_k] \le r \\ a_i \ge 0 & \sum a_i = 1 \end{array} \right\}$

Think of x_i as Dirac δ-measures

$$d\left(\sum_{i=0}^{k} a_i x_i, \sum_{j=0}^{k'} a'_j x'_j\right) = \inf_{\substack{\{p_{i,j} \ge 0\} \\ \sum_j p_{i,j} = a_i \\ \sum_i p_{i,j} = a'_j \\ \sum_{i,j} p_{i,j} = 1}} \sum_{i,j} p_{i,j}\, d(x_i, x'_j)$$



A matching or transport plan is a joint pdf w/ given marginals

Prop $VR^m(X;r)$ is an r-thickening of X

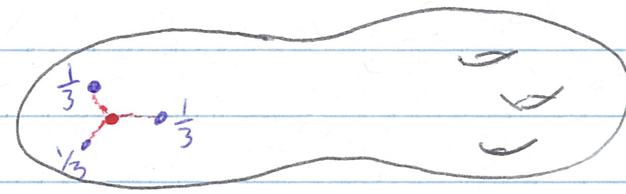
Extends metric, and $d(X, VR^m(X;r)) \le r$

Gromov studied in case X discrete

Thm M complete Riemannian manifold, $r_0 > 0$ satisfies
- balls radius r_0 geodesically convex
- $r_0 < \frac{\pi}{4} K^{-1/2}$ (K sectional curvatures)

Then $VR^m(M;r) \simeq M$ for $r < r_0$

Sketch $VR^m(M;r) \rightleftarrows M$ $\qquad \sum a_i x_i \mapsto$ Karcher or Frechet mean



Linear homotopies (dual space of continuous functions)

Rmk $VR^m(S^1; \frac{1}{3}) \simeq S^3$

Thm $VR^m(S^n; r) \simeq \begin{cases} S^n & r < r_n \\ \Sigma^{n+1} \frac{SO(n+1)}{A_{n+2}} & r = r_n \end{cases}$

r_n = diameter of inscribed regular Δ^{n+1}

A_{n+2} = alternating group (rotational symmetries of Δ^{n+1})

Sketch $VR^m(S^n; r_n) = VR^m(S^n; r_n) \setminus$ interiors of regular Δ^{n+1} $\cup \; \Delta^{n+1} \times \frac{SO(n+1)}{A_{n+2}}$

$\simeq S^n \times C\left(\frac{SO(n+1)}{A_{n+2}}\right) \cup C(S^n) \times \frac{SO(n+1)}{A_{n+2}}$

$= S^n * \frac{SO(n+1)}{A_{n+2}}$

$= \Sigma^{n+1} \frac{SO(n+1)}{A_{n+2}}$

Questions
- Larger r? (Strongly self-dual polytopes)
- Other manifolds $VR(L^\infty \text{ tori})$, flat metric, $VR(\text{ellipse})$ $<$ S^1 S^2 S^3; $\leq$ S^1 $\bigvee 5 S^2$ S^3
- Čech
- Hausmann: Homotopy connectivity a non-decreasing function of r
- Trigonometric moment curves, cyclic polytopes, orbitopes
- Joshua Mirth: Euclidean submanifold
- $VR_< \simeq VR^m$?