Asymptotic Decay Rates of Electromagnetic and Spin-2 Fields in Minkowski Space

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Abstract

This paper considers the decay rates of electromagnetic and spin-2 fields in Minkowski space. Each of the three main methods used for studying hyperbolic equations in Minkowski space — namely, the stationary phase method, the conformal compactification method, and the commuting vector fields method — is examined in turn and applied to the two particular equations under consideration.

The main part of the paper consists in rederiving the decay rates of solutions to the field equations (in particular to the Maxwell equations), by means of the stationary phase method and of the conformal compactification method.
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Chapter 1

Introduction

We shall use the notation $a \lesssim b$ whenever there exists a constant $C$ such that $|a| \leq C|b|$.

1.1 General Notions

Consider an orientable Riemannian manifold $M$ of dimension $m$, on which we select a volume form $\epsilon$. The inner product $\langle \cdot, \cdot \rangle$ on $T_P M$ can be extended to an inner product on the space of tensors defined at the point $P$, denoted with the same symbol.

**Definition 1.1.** The Hodge star operator is the linear operator that assigns to each tensor $\alpha \in \Lambda^p_P(M)$ another tensor $\ast \alpha \in \Lambda^{m-p}_P(M)$ such that (see [11, p. 87])

$$\langle \ast \alpha, \beta \rangle = \langle \alpha \wedge \beta, \epsilon_P \rangle. \quad (1.1)$$

In local coordinates one has, for

$$\alpha = \alpha_{\mu_1...\mu_p} dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p},$$

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that
\[
\ast \alpha_{\mu_1 \ldots \mu_m} = \frac{(-1)^{p(m-p)}}{p!} \epsilon_{\mu_1 \ldots \mu_m} \alpha^{\mu_{m-p+1} \ldots \mu_m}.
\] (1.2)

**Property 1.2.** For a pseudo-Riemannian manifold \( M \) of dimension \( m \) whose metric has \( s \) minus signs and for \( \omega \in \Lambda^p(M) \)
\[
\ast \ast \alpha = (-1)^{p(m-p)+s} \alpha.
\] (1.3)

Consider the exterior differential \( d \) and let us also define (see [11, p. 88]), for the pseudo-Riemannian manifold \( M \) of dimension \( m \) whose metric has \( s \) minus signs and for \( \omega \in \Lambda^p(M) \),

**Definition 1.3.**
\[
\delta \omega = (-1)^{mp+m+s+1} \ast d \ast \omega.
\] (1.4)

The most important property of this operator is that

**Property 1.4.** Denote by \( \langle \alpha, \beta \rangle \) the inner product on \( \Lambda^p(M) \)
\[
\langle \alpha, \beta \rangle = \int \alpha \wedge \ast \beta = \int \langle \alpha, \beta \rangle \sigma.
\] (1.5)

Then (assuming the integrals are finite)
\[
\langle d\alpha, \beta \rangle = \langle \alpha, \delta \beta \rangle.
\] (1.6)

We define the Laplacian on a pseudo-Riemannian manifold with a positive definite metric, such as \( \mathbb{R}^d \), by
\[
\Delta = d\delta + \delta d.
\] (1.7)

On Minkowski space we shall use \( \Box \) in order to denote the same notion.
Definition 1.5. The space $\mathcal{D}(\mathbb{R}^d)$ is the space of smooth functions of compact support in $\mathbb{R}^d$.

Let us introduce the weighted Sobolev spaces, of which we will make use throughout the remainder of this paper, as follows:

Definition 1.6. $H_{n,m}(\mathbb{R}^d)$ is the completion of $\mathcal{D}(\mathbb{R}^d)$ under the norm

$$
\|f\|^2_{H_{n,m}} = \int_{\mathbb{R}^d} \sum_{k=1}^n (1 + |x|^{2(k+m)}) |\nabla^k f(x)|^2 dx.
$$

Here $n$ is an integer and $m$ is any real number.

1.2 Notions Specific to Minkowski Space

Definition 1.7. The Minkowski space, denoted $\mathbb{R}^{3+1}$, is the set

$$
\mathbb{R}^{3+1} = \{x = (x_0, \vec{x})| x_0 \in \mathbb{R}, \vec{x} \in \mathbb{R}^3\}.
$$

It is a pseudo-Riemannian manifold, endowed with the inner product of signature $(1, -1, -1, -1)$ on $T_p(\mathbb{R}^{3+1})$

$$
\langle \xi, \eta \rangle = \xi_0\eta_0 - \langle \vec{\xi}, \vec{\eta} \rangle_{\mathbb{R}^3}.
$$

The first coordinate $x_0$ is also denoted by $t$ (meaning time) and another usual notation is $|\vec{x}| = r$. It is also customary to use Greek letters for naming any of the four coordinates and to use Roman letters in order to denote the spatial coordinates only.

The vector fields defined as $T_\mu = \partial_{x^\mu} = f_\mu$ (the coordinate unit vectors) at each point form a frame for $T(\mathbb{R}^{3+1})$, which can therefore be identified, linearly, with the trivial vector bundle $\mathbb{R}^{3+1} \times \mathbb{R}^4$. 


Definition 1.8.

\[ S_{t,r} = \{ x \in \mathbb{R}^{3+1} | x_0 = t, |\vec{x}| = r \}. \] (1.11)

Theorem 1.9. Consider a tensor \( \alpha \in \Lambda^p(\mathbb{R}^{3+1}) \). Then \( \Box \alpha = 0 \) if and only if \( d\alpha = 0 \) and \( \delta \alpha = 0 \).

Proof of Theorem 1.9. We note that

\[ \langle \Box \alpha, \beta \rangle = \langle \delta d\alpha, \beta \rangle + \langle d\delta \alpha, \beta \rangle = \langle d\alpha, d\beta \rangle + \langle \delta \alpha, \delta \beta \rangle \] (1.12)

Thus, if \( d\alpha \) and \( \delta \alpha \) are 0, it follows that \( \Box \alpha = 0 \). Conversely, if \( \Box \alpha = 0 \), we see that \( \langle d\alpha, d\beta \rangle + \langle \delta \alpha, \delta \beta \rangle = 0 \) for any \( \beta \in \Lambda^p_c(\mathbb{R}^{3+1}) \). However, by Poincaré’s Lemma, for any \( \gamma_1 \in \Lambda^p_c(\mathbb{R}^{3+1}) \) and \( \gamma_2 \in \Lambda^{p-1}_c(\mathbb{R}^{3+1}) \) there exists a unique \( \beta \) of compact support such that \( \delta \beta = \gamma_1 \) and \( d\beta = \gamma_2 \). It follows that \( d\alpha = 0 \), \( \delta \alpha = 0 \).

1.3 The Newman-Penrose Formalism

The Newman-Penrose formalism, such as it was introduced in [7], consists in defining a spin structure on the space-time (see [9], whose exposition I am going to follow) and expressing the gravitational tensor as a 2-spinor.

On the Minkowski space \( \mathbb{R}^{3+1} \), consider a 2-dimensional complex vector bundle \( V \), on which a symplectic product \((\cdot, \cdot)\) is defined. In local coordinates, for any given basis \((v_1, v_2)\), the symplectic product is expressed as an antisymmetric, nondegenerate \(2 \times 2\) matrix: \((v_A, v_B) = \epsilon_{AB}\). One can also choose a basis for which \(\epsilon\) takes the form

\[ \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \] (1.13)
on a neighborhood of any given point in $\mathbb{R}^{3+1}$, the base of $V$.

The conjugate bundle $\overline{V}$ is a symplectic bundle with the product $\overline{\epsilon}$ (in local coordinates). The symplectic product on $V$ also induces one on the dual vector bundle $V^*$, namely $\epsilon^{AB}$ (the inverse of $\epsilon$), and one on $\overline{V}^*$, $\overline{\epsilon}^{AB}$.

For a basis $(v_A)$ on $V$, let us denote by $(v_A')$, $(v^A)$, and $(v^{A'})$ the bases corresponding to it on $\overline{V}$, $V^*$, and $\overline{V}^*$ respectively.

**Definition 1.10.** The tensor product of any number of copies of $V$, $\overline{V}$, $V^*$, and $\overline{V}^*$ is called a spinor bundle and its sections are named spinors.

The symplectic product on $V$ induces on any spinor bundle either a symmetric or an antisymmetric product, depending on the bundle’s dimension.

In particular, let us consider the spinor bundle $W \subset V \otimes \overline{V}$ of Hermitian spinors, that is of those spinors $w$ for which $w_{AB'} = \overline{w}_{BA'}$.

**Theorem 1.11.** Assume that the spinor bundle $V$ over $\mathbb{R}^{3+1}$ is the trivial vector bundle $\mathbb{R}^{3+1} \times \mathbb{C}^2$ with the symplectic product given at every point by $\epsilon$ (see above). Then, there exists an isometry $\sigma$ between the tangent space $T(\mathbb{R}^{3+1})$ and $W$ given at each point by

\[
\sigma(f_0) = \frac{1}{\sqrt{2}}(v_0 \otimes v_0' + v_1 \otimes v_1'), \quad \sigma(f_1) = \frac{1}{\sqrt{2}}(v_0 \otimes v_0' - v_1 \otimes v_1'),
\]

\[
\sigma(f_2) = \frac{1}{\sqrt{2}}(v_0 \otimes v_1' + v_1 \otimes v_0'), \quad \sigma(f_3) = \frac{1}{\sqrt{2}}(iv_0 \otimes v_1' - iv_1 \otimes e_0').
\] (1.14)

**Proof.** Clearly $(f_\mu)$ form a basis for $T_p(\mathbb{R}^{3+1})$ and the spinors enumerated above form a basis for $W_p$, so this is a well-defined linear isomorphism. It can also be checked that it preserves the dot product, for example

\[
\left( \frac{1}{\sqrt{2}}(v_0 \otimes v_0' + v_1 \otimes v_1'), \frac{1}{\sqrt{2}}(v_0 \otimes v_0' + v_1 \otimes v_1') \right) = 1 = \langle e_0, e_0 \rangle.
\] (1.15)
On the basis of this isomorphism, one obtains a new frame for the complexification of the tangent space of $\mathbb{R}^{3+1}$, $T(\mathbb{R}^{3+1}) \otimes \mathbb{C}$, given by $X_\mu = e_\mu = \sigma^{-1}(v_A \otimes v_B')$ at each point. Since all the vectors $e_\mu = \sigma^{-1}(v_A \otimes v_B')$ are null (they have the property that $\langle e_\mu, e_\mu \rangle = 0$), this frame is called the null frame. However, we want to avoid the presence of vectors with complex coordinates, so let us redefine the notion as follows:

**Definition 1.12.** A null frame is one consisting of four vectors $(E_+, E_-, e_3, e_4)$ at each point where it is defined, such that $(E_+, E_-)$ is an orthonormal basis for the tangent space to the 2-sphere $S_{r,t}$ going through that point, while $e_3 = \frac{\partial}{\partial t} - \frac{\partial}{\partial r}$ and $e_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial r}$.

We also require that $\epsilon(E_+, E_-, e_3, e_4) = 2$, where $\epsilon$ is the volume form of the Minkowski space.

One can extend the isomorphism between $T(\mathbb{R}^{3+1})$ and $W$ to tensors of higher rank, both contravariant and covariant. For example, we see that $\Lambda^1(\mathbb{R}^{3+1}) \cong W^*$.

**Definition 1.13.** A spin-2 tensor field is one that can be represented as $W = \sigma^{-1}(\psi \otimes \bar{\psi})$, where $\psi$ is a 4-spinor, $\psi \in V^* \otimes V^* \otimes V^* \otimes V^*$, and is symmetric in all indices.

**Theorem 1.14.** A 4-covariant tensor field is a spin-2 tensor field if and only
if it possesses the following symmetries:

\[ W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma} \]
\[ W_{[\alpha\beta\gamma]}\delta = 0 \ (the \ Bianchi \ identities) \]
\[ W_{\alpha\beta\gamma\delta} = W_{\gamma\delta\alpha\beta} \]
\[ W^{\alpha}_{\quad\beta\alpha\delta} = 0. \] (1.16)

The interior derivative of an antisymmetric 2-covariant tensor field is the 1-form given by

\[ i_X F(Y) = F(Y, X). \] (1.17)

The corresponding notion for spin-2 tensors is

\[ i_{(X_1, X_2)} W(Y_1, Y_2) = W(Y_1, X_1, Y_2, X_2). \] (1.18)

**Definition 1.15.** The null decomposition of a tensor field is its decomposition into components using the null coordinate frame. More precisely, when \( F \) is a 2-covariant antisymmetric form, \( F \in \Lambda^2(\mathbb{R}^{3+1}) \), we obtain from its decomposition the 1-forms \( \alpha \) and \( \alpha^\perp \) tangent to the 2-spheres \( S_{r,t} \) and the scalar quantities \( \rho \) and \( \sigma \), where

\[ \alpha(X) = i_{e_3} F(X) = F(X, e_3) \]
\[ \alpha^\perp(X) = i_{e_4} F(X) = F(X, e_4) \]
\[ \rho = \frac{1}{2} F(e_3, e_4) \]
\[ \sigma = F(e_A, e_B). \] (1.19)

Since \( e_A \wedge e_B \) is the area element of the 2-sphere \( S_{r,t} \), the definition of \( \rho \) is coordinate-independent. Similarly, we define for a spin-2 tensor \( W \) the
following quantities:

\[ \Omega(X, Y) = i_{(e_3, e_3)} W(X, Y) = W(X, e_3, Y, e_3) \]
\[ \alpha(X, Y) = i_{(e_4, e_4)} W(X, Y) = W(X, e_4, Y, e_4) \]
\[ \beta(X) = \frac{1}{2} W(X, e_3, e_3, e_4) \]
\[ \rho = \frac{1}{4} W(e_3, e_4, e_3, e_4) \]
\[ \sigma = \frac{1}{4} W(e_A, e_B, e_3, e_4) \].

(1.20)

One can easily show that the totality of these components uniquely determines the tensor from which they were obtained.

Another useful way of writing the tensors is by means of their electric and magnetic parts, \( E \) and \( H \). Namely, consider the vector field \( T_0 = e_0 \). We can define, for an antisymmetric 2-covariant tensor field \( F \),

\[ E = i_{T_0} F = F(\cdot, T_0), \quad H = i_{T_0} F. \] (1.21)

Here \( E \) and \( H \) are 1-forms tangent to the hyperplanes (\( t \) constant). Similarly, for a spin-2 field \( W \), one can define the symmetric 2-forms \( E \) and \( H \)

\[ E = i_{(T_0, T_0)} W, \quad H = i_{(T_0, T_0)} \ast W. \] (1.22)

**Observation** 1.16. We consider \( W \) to be a 2-covariant antisymmetric tensor with values in the set of alternating 2-forms and accordingly we define \( \ast W \) in the above expression by

\[ \ast W_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\mu\nu} W^{\mu\nu}_{\gamma\delta}. \] (1.23)

Under this interpretation, \( E \) and \( H \) can also be seen as 1-forms with values in the space of 1-forms.
Observation 1.17. We can express the null components of the tensors $F$ and $W$ as a function of $E$ and $H$ as follows (see [3, p. 152, 171]): for $F$

\[
\alpha_A = E_A - \epsilon^B_A H_B \\
\rho = -E_N, \quad \sigma = -H_N,
\]

where $\epsilon_{AB}$ is the area form of the 2-sphere $S_{t,r}$ and $N = \frac{x^i}{r} \partial_{x^i}$. For the spin-2 field $W$ we have

\[
\alpha_{AB} = 2(E_{AB} - \epsilon^C_A H_{CB}) + \rho \delta_{AB} - \sigma \epsilon_{AB} \\
\rho = E_{NN}, \quad \sigma = H_{NN}.
\]

1.4 The Field Equations

The equations that we intend to study are

\[
\Box F = 0 \tag{1.26}
\]

where $F$ is an antisymmetric 2-covariant tensor field, and

\[
\nabla_{[\alpha} W_{\beta\gamma]\delta} = 0 \tag{1.27}
\]

for a spin-2 tensor field, with initial values given on the hyperplane $t = 0$ ($F \mid_{t=0}$ or $W \mid_{t=0}$ is known).

By Theorem 1.9, (1.26) is equivalent to saying that $dF = 0$ and $d*F = 0$. Furthermore, instead of considering $W$ to be a 4-tensor, let us take it to be
a 2-covariant antisymmetric tensor with values in the set of alternating 2-forms. Then, equation (1.27) is just $dW = 0$ and we also have that $\delta W = 0$ (see [3] for the proof).

**Theorem 1.18.** Equation $dF = 0$ can be rewritten as either

$$\nabla_{[\mu} F_{\nu\lambda]} = 0 \text{ or } \nabla^\mu * F_{\mu\nu} = 0.$$  \hspace{1cm} (1.28)

Similarly, equation $d* F = 0$ is the same as either

$$\nabla_{[\mu} * F_{\nu\lambda]} = 0 \text{ or } \nabla^\mu F_{\mu\nu} = 0.$$  \hspace{1cm} (1.29)

**Proof of Theorem 1.18.** The first statement is trivial, the second follows from the fact that

$$0 = \nabla^\mu * F_{\mu\nu} = \nabla_{\mu} (e^\mu_\nu \alpha \beta F_{\alpha\beta}) = e^\mu_\nu \alpha \beta \nabla_{\mu} F_{\alpha\beta}.$$  \hspace{1cm} (1.30)

Now, the last two statements are a consequence of the fact that $* * F = -F$. \hspace{1cm} □

**Theorem 1.19.** Equation $dW = 0$ is equivalent to either of the following four systems of equations:

$$\nabla_{[\alpha} W_{\beta\gamma]\delta\epsilon} = 0$$  \hspace{1cm} (1.31)

$$\nabla^\alpha * W_{\alpha\beta\gamma\delta} = 0$$  \hspace{1cm} (1.32)

$$\nabla_{[\alpha} * W_{\beta\gamma]\delta\epsilon} = 0$$  \hspace{1cm} (1.33)

$$\nabla^\alpha W_{\alpha\beta\gamma\delta} = 0.$$  \hspace{1cm} (1.34)
Proof of theorem 1.19. The equivalence of the first two statements to \(dW = 0\) follows from Theorem 1.18. However, the tensors \(W\) and \(*W\), as \(\text{spin-2}\) tensors, have additional symmetries that allow us to prove the last two statements as well. Contracting \(\alpha\) and \(\delta\) in (1.31) we obtain (1.34), which is equivalent to (1.33) by the previous theorem.

Theorem 1.20. In terms of the electric and magnetic components, equations (1.26), (1.27) can be written as

\[
\nabla . E = 0, \quad \nabla . H = 0
\]

\[
\partial_t E = \nabla \times H, \quad \partial_t H = -\nabla \times E,
\]

where

\[
\nabla . \alpha = \nabla^j \alpha_j, \quad (\nabla \times \alpha)_i = \epsilon^{jk}_i \nabla^j \alpha_k
\]

for 1-forms and

\[
(\nabla . \alpha)_i = \nabla^j \alpha_{ji}, \quad (\nabla \times \alpha)_d = \epsilon^{jk}_i \nabla^j \alpha_{kd}
\]

in the case of 2-forms (which is the same as the previous definition, if we consider \(\alpha\) to be a 1-form with values in the space of 1-forms).

Observation 1.21. Both in the case of antisymmetric 2-covariant tensors and in that of spin-2 tensors, the time derivatives of the tensor at \(t=0\) need not be given explicitly, since they can be computed by knowing the value of the tensor at the origin.

Indeed, in the first case the equation can be rewritten in local coordinates as

\[
\nabla_\lambda F_{\mu\nu} = 0, \quad \nabla^{\mu} F_{\mu\nu} = 0,
\]
where $[\lambda \mu \nu]$ stands for the sum of all cyclical permutations $(\lambda, \mu, \nu)$, $(\mu, \nu, \lambda)$, and $(\nu, \lambda, \mu)$. Thus, all the time derivatives of the tensor can be expressed in terms of the spatial derivatives, for example $\partial_t F_{10} = \nabla^0 F_{01} = -(\nabla_2 F_{21} + \nabla_3 F_{31} + \nabla_4 F_{41})$.

In the second case we can rewrite the equation as

$$\nabla_{[\alpha} W_{\beta\gamma]\delta\epsilon} = 0 \quad (1.39)$$

and we can express the time derivatives of $W$ as a function of the other derivatives, in a similar manner.
Chapter 2

Survey of Known Results

The decay rate of solutions to the two equations (1.26) and (1.27) has been studied extensively, the reference works being [3] and [1].

2.1 Stationary Phase Method

The classical way of treating hyperbolic equations is to apply the stationary phase method to the solution written explicitly with the help of the fundamental solution. It is represented by papers such as [10]. There, the author proves that for the scalar equation

\[ \Box u = 0 \]  \hspace{1cm} (2.1)

with initial data \( u \mid_{t=0} \in W^{[n/2]+1,1}, \partial_t u \mid_{t=0} \in W^{[n/2],1} \), the solution has a decay rate

\[ u \lesssim (1 + t + r)^{(n-1)/2}. \]  \hspace{1cm} (2.2)

However, as far as I know, this method has not been applied in order to obtain the improved estimates that are attainable for tensors.
2.2 The Conformal Compactification Method

In their paper [1], the authors took a new approach to the study of this problem. Namely, they applied the conformal compactification method introduced by Penrose [8] in order to prove the existence of global solutions to field equations in Minkowski space and related spaces, obtaining their asymptotic decay rates as a side result. More precisely, for the Yang-Mills equation in the Minkowski space $\mathbb{R}^{3+1}$,

$$\hat{\nabla} \lambda F^{\lambda \alpha \nu} = J^{\mu, \alpha} \quad (2.3)$$

$$J^{\mu, \alpha} = i \bar{\Psi} \gamma^\mu S^a \Psi + (\bar{\Phi} T^a \hat{\nabla}^\mu \Phi + \hat{\nabla}^\mu \Phi T^a \Phi) \quad (2.4)$$

$$\bar{\nabla} \Psi = H(\Phi, \Psi), \quad \hat{\Box} \Phi = K(\Phi, \Psi) \quad (2.5)$$

the authors obtained the following decay rates (see [1, p. 501]):

$$(\Phi \Phi)^{1/2} \lesssim (1 + (t + r)^2)^{-1/2} (1 + (t - r)^2)^{-1/2},$$

$$(\bar{\Psi} \gamma^\lambda h_{\lambda \alpha} \Psi)^{1/2} \lesssim (1 + (t + r)^2)^{-3/4} (1 + (t - r)^2)^{-5/4},$$

$$F_{\lambda A} \lesssim (1 + (t + r)^2)^{-3/2} (1 + (t - r)^2)^{-1/2},$$

$$(F_{AB}, F_{43}) \lesssim (1 + (t + r)^2)^{-1} (1 + (t - r)^2)^{-1},$$

$$F_{3A} \lesssim (1 + (t + r)^2)^{-1/2} (1 + (t - r)^2)^{-3/2}. \quad (2.6)$$

Christodoulou used again the same method in [2], in order to prove the existence of solutions to the quasilinear system of hyperbolic equations

$$\Box u = f(u, \partial u, \partial^2 u), \quad (2.7)$$

where $u = (u_1, \ldots, u_N)$, $f^A(u, v, w) = \alpha^{\mu \nu}(u, v) u^A_{\mu \nu} + \beta^A(u, v)$, and in dimension 3 $f$ is also required to satisfy the null condition. Under these assumptions, together with conditions on the initial data

$$u \big|_{t=0} \in H_{(d+1)/2+2, (d+1)/2+1}(\mathbb{R}^d), \quad \partial_t u \big|_{t=0} \in H_{(d+1)/2+1, (d+1)/2+2}(\mathbb{R}^d), \quad (2.8)$$
u was shown to have the property that

\[ u(t, x) \lesssim (1 + (t + r)^2)^{-(d-1)/4} (1 + (t - r)^2)^{-(d-1)/4}. \]  \hspace{1cm} (2.9)

### 2.3 The Commuting Vector Fields Method

In [5] the *commuting vector fields method* is introduced for the first time. This method allows one to obtain results similar to those in [2] under fewer assumptions for the initial data (and can be applied to a more general category of spaces as well). Namely, in his papers [5] and [6], Professor Klainerman proved that, if \( u \) is a solution of the homogeneous hyperbolic equation

\[ \Box u = 0 \]  \hspace{1cm} (2.10)

with

\[ u|_{t=0} \in H_{s,1}(\mathbb{R}^d), \quad \partial_t u|_{t=0} \in H_{s-1,1}(\mathbb{R}^d), \]  \hspace{1cm} (2.11)

where \( s > n/2 \) (in particular \( s \geq \lceil n/2 \rceil \), since \( s \) is an integer), then (see Corollary 1, p. 133, in [6])

\[ u(t, x) \lesssim (1 + (t + r)^2)^{-(d-1)/4} (1 + (t - r)^2)^{-1/4}. \]  \hspace{1cm} (2.12)

In their subsequent paper [3], Professors Klainerman and Christodoulou proved a better result concerning electromagnetic and spin-2 fields. Under the condition that

\[ F|_{t=0} \in H_{k,1}, \]  \hspace{1cm} (2.13)

where \( F \) is a 2-covariant antisymmetric tensor field satisfying the electromagnetic field equation

\[ dF = 0, \quad d^* F = 0, \]  \hspace{1cm} (2.14)
they proved that its components in the null frame satisfy the estimates

\[ \nabla_3^m \nabla_4^n \nabla^l \alpha(t, x) \lesssim r^{-1-n-l} \tau_\tau^{-3/2-m} \| F(0) \|_{H_{m+n+l+2,2}(\mathbb{R}^3)}, \quad (2.15) \]

\[ \nabla_3^m \nabla_4^n \nabla^l (\rho, \sigma)(t, x) \lesssim r^{-2-n-l} \tau_\tau^{-1/2-m} \| F(0) \|_{H_{m+n+l+2,1}(\mathbb{R}^3)}, \quad (2.16) \]

\[ \nabla_4^n \nabla^l \beta(t, x) \lesssim r^{-5/2-n-l} \| F(0) \|_{H_{m+n+l+2,1}(\mathbb{R}^3)}, \quad (2.17) \]

\[ \nabla_3^m \nabla_4^n \nabla^l \alpha(t, x) \lesssim r^{-3-n-l} \tau_\tau^{-1/2-m} \| F(0) \|_{H_{m+n+l+3,3}(\mathbb{R}^3)}, \quad (2.18) \]

on the set where \( r > 1 + t/2 \), where \( \tau_\tau = (1 + (t - r)^2)^{1/2} \), and the interior decay estimate

\[ \nabla^l F(t, x) \lesssim t^{5/2-l} \| F(0) \|_{H_{l+2,1}(\mathbb{R}^3)} \quad (2.19) \]

in the region \( r \leq 1 + 1/2 \). The rate of decay for \( \alpha \) is no better than the one that can be obtained simply by knowing the fact the components of \( F \) satisfy the scalar hyperbolic equation \( \square F_{\mu\nu} = 0 \), but the other results represent a new phenomenon, different from the scalar case.

Similarly, for the 2-spin tensor \( W \),

\[ \nabla_3^m \nabla_4^n \nabla^l \alpha(t, x) \lesssim r^{-1-n-l} \tau_\tau^{-5/2-m} \| W(0) \|_{H_{m+n+l+2,2}(\mathbb{R}^3)}, \quad (2.20) \]

\[ \nabla_3^m \nabla_4^n \nabla^l \beta(t, x) \lesssim r^{-2-n-l} \tau_\tau^{-3/2-m} \| W(0) \|_{H_{m+n+l+2,2}(\mathbb{R}^3)}, \quad (2.21) \]

\[ \nabla_3^m \nabla_4^n \nabla^l (\rho, \sigma)(t, x) \lesssim r^{-3-n-l} \tau_\tau^{-1/2-m} \| W(0) \|_{H_{m+n+l+2,2}(\mathbb{R}^3)}, \quad (2.22) \]

\[ \nabla_4^n \nabla^l (\beta, \alpha)(t, x) \lesssim r^{-7/2-n-l} \| F(0) \|_{H_{m+n+l+2,2}(\mathbb{R}^3)}, \quad (2.23) \]

\[ \nabla_3^m \nabla_4^n \nabla^l \beta(t, x) \lesssim r^{-4-n-l} \tau_\tau^{-1/2-m} \| W(0) \|_{H_{m+n+l+3,3}(\mathbb{R}^3)}, \quad (2.24) \]

\[ \nabla_3 \nabla_4^n \nabla^l \alpha(t, x) \lesssim r^{-9/2-n-l} \| W(0) \|_{H_{m+n+l+3,2}(\mathbb{R}^3)}, \quad (2.25) \]

\[ \nabla_3^m \nabla_4^n \nabla^l \alpha(t, x) \lesssim r^{-5-n-l} \tau_\tau^{-1/2-m} \| W(0) \|_{H_{m+n+l+4,2}(\mathbb{R}^3)}, \quad (2.26) \]

One can also obtain an interior decay estimate of the form

\[ \nabla^l W(t, x) \lesssim t^{7/2-l} \| W(0) \|_{H_{l+2,1}(\mathbb{R}^3)}. \quad (2.27) \]
In the course of this paper all the three methods will be employed in order to obtain various results concerning the decay rates of electromagnetic and spin-2 fields (equations (1.26) and (1.27)).
Chapter 3

Main Results

We are going to apply all the three methods separately, in order to obtain the corresponding decay results.

3.1 Stationary Phase Method

We are interested, to begin with, in the asymptotic behavior of the scalar solutions of hyperbolic equations in $\mathbb{R}^{d+1}$.

Lemma 3.1. The Fourier transform of the space $H_{n,m}$, when both $n$ and $m$ are non-negative integers, is

$$\hat{H}_{n,m}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \mid \|f\|_{\hat{H}_{n,m}(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \sum_{k=0}^{n+m} |\nabla^k f|^2 (|x|^{2\min(0,k-m)} + |x|^{2n}) dx \right)^{1/2} \right\}$$ (3.1)

Lemma 3.2. Consider a solution of the hyperbolic equation $\Box F = 0$, repre-
sented in the form
\[ F(t, x) = \int_{\mathbb{R}^3} e^{2\pi i(x \cdot \xi + t |\xi|)} \hat{F}_+(\xi) d\xi + \int_{\mathbb{R}^3} e^{2\pi i(x \cdot \xi - t |\xi|)} \hat{F}_-(\xi) d\xi. \quad (3.2) \]

Then this representation is unique.

Proof. Indeed, consider two different representations of \( F \) given by the pairs of functions \((\hat{F}_{1+}, \hat{F}_{1-})\) and \((\hat{F}_{2+}, \hat{F}_{2-})\). By considering equation (3.2) and its derivative with respect to \( t \) at \( t = 0 \), we obtain that
\[
\hat{F}_{1+}(\xi) + \hat{F}_{1-}(\xi) = \hat{F}_{2+}(\xi) + \hat{F}_{2-}(\xi)
\]
and
\[
|\xi|(\hat{F}_{1+}(\xi) - \hat{F}_{1-}(\xi)) = |\xi|(\hat{F}_{2+}(\xi) - \hat{F}_{2-}(\xi))
\]
for any \( \xi \), whence the conclusion follows (for almost every \( \xi \)). \( \square \)

Consider a covariant antisymmetric 2-tensor \( F \), which is a solution of the wave equation (1.26), and whose electric and magnetic parts are \( E \), respectively \( H \).

**Lemma 3.3.** Assume the the electric and the magnetic parts of \( F \) are represented by
\[ E(t, x) = \int_{\mathbb{R}^3} e^{2\pi i(x \cdot \xi + t |\xi|)} \hat{E}_+(\xi) d\xi + \int_{\mathbb{R}^3} e^{2\pi i(x \cdot \xi - t |\xi|)} \hat{E}_-(\xi) d\xi \quad (3.3) \]
and similarly for \( H \). Then we have
\[
\hat{E}_\pm(\xi) \cdot \xi = 0, \quad \hat{H}_\pm(\xi) \cdot \xi = 0,
\]
\[
\hat{E}_+(\xi) = \frac{\xi}{|\xi|} \times \hat{H}_+(\xi), \quad \hat{E}_-(\xi) = -\frac{\xi}{|\xi|} \times \hat{H}_-(\xi).
\]
and if \( E \mid_{t=0} \) and \( H \mid_{t=0} \) both belong to \( H_{n,m}(\mathbb{R}^3) \), then \( \hat{E}_\pm, \hat{H}_\pm \in \hat{H}_{n,m}(\mathbb{R}^3) \).
Proof of Lemma 3.3. Let us replace $E$ and $H$ with their explicit representations (3.3) in the Maxwell equations (1.35). We obtain that the pairs $(\xi, \hat{E}_+, \xi, \hat{E}_-)$ and $(0, 0)$ represent the same functions (the left and the right hand of the equation $\nabla \cdot E = 0$), so one of the conclusions of the lemma follows (and the same for $H_\pm$). Then, the pairs $(\xi \hat{E}_+, \xi \hat{E}_-)$ and $(0, 0)$ also represent the same function ($\partial_t E$ and $\nabla \times H$, the left and right side of the other Maxwell’s equation), so they coincide.

Finally, when $E \mid_{t=0}$ and $H \mid_{t=0}$ are in $H_{n,m}(\mathbb{R}^3)$, we see that $\partial_t E \mid_{t=0} = \nabla \times H \mid_{t=0}$ and $\partial_t H \mid_{t=0} = \nabla \times E \mid_{t=0}$. Hence we infer that (due to the unicity of the representation)

$$\hat{E}_+ = \frac{1}{2}(E(0, \cdot) + \frac{1}{|\xi|} \partial_t E(0, \cdot)) = \frac{1}{2}(E(0, \cdot) + \frac{\xi}{|\xi|} \times H(0, \cdot)) \in \hat{H}_{n,m}(\mathbb{R}^3) \quad (3.6)$$

The same applies to $E_-$ and to $H_\pm$, so the proof is complete. \hfill \square

Observation 3.4. The same representation can be obtained for the components of a spin-2 field $W$, if we define the dot product and the cross product as follows:

$$(v \cdot E)_j = v^i E_{ij}, \quad (v \times E)_{ij} = \epsilon_{ik} v^k E_{ij}. \quad (3.7)$$

Let us denote $|x| = r$, $\hat{x} = x/r$ and $|\xi| = \rho$, $\hat{\xi} = \xi/\rho$. Following a change of variables, we can assume that $\hat{x} = e_1$.

Theorem 3.5 (Interior Decay). Assume $f \in H_{d,1}(\mathbb{R}^d)$. Then, for $r = |x| < \frac{1}{2} t$,

$$F(t, x) = \int_{\mathbb{R}^d} e^{2\pi i(x \cdot \xi + t|\xi|)} \hat{f}(\xi) d\xi \lesssim t^{-d}. \quad (3.8)$$
Proof of Theorem 3.5. Rewriting the integral in polar coordinates, we get

\[ F(t, x) = \int_0^\infty \int_{S^{d-1}} e^{2\pi i p(r \xi_1 + t)} \hat{f}(\rho \xi) \rho^{d-1} d\xi d\rho. \] (3.9)

Integrating by parts \( d \) times, we obtain

\[
F(t, x) = (-1)^{d-1} \int_{S^{d-1}} \frac{1}{(2\pi i (r \xi_1 + t))^{d}} \hat{f}(0) d\xi + \\
+ (-1)^{d} \int_0^\infty \int_{S^{d-1}} \frac{e^{2\pi i p(r \xi_1 + t)}}{(2\pi i (r \xi_1 + t))^{d}} \partial^d_p(\hat{f}(\rho \xi) \rho^{d-1}) d\xi d\rho \lesssim \\
\lesssim \frac{1}{t^d} \|f\|_{L^1(\mathbb{R}^d)} + \sum_{k < d/2} \int_{B(0,1)} |\nabla^{d-k} \hat{f}| |\xi|^{-k} d\xi + \\
+ \sum_{k \geq d/2} \|\nabla^{d-k} \hat{f}\|_{L^\infty(\mathbb{R}^d)} + \sum_{k = 0}^{d-1} \int_1^\infty \int_{S^{d-1}} |\nabla^{d-k} \hat{f}(\rho \xi)| \rho^{d-1-k} d\xi d\rho \lesssim \\
\lesssim \frac{1}{t^d} \left( \left\| (1 + |x|^{(d+1)/2}) f \right\|_{L^2(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} \frac{1}{(1 + |x|^{(d+1)/2})^2} dx \right)^{1/2} + \\
+ \sum_{k < d/2} \|\nabla^{d-k} \hat{f}\|_{L^2(\mathbb{R}^d)} \left( \int_{B(0,1)} |\xi|^{-2k} d\xi \right)^{1/2} + \\
+ \sum_{k \geq d/2} \left( \int_{\mathbb{R}^d} \left| x \right|^{2(d-k)} \left( \int_{\mathbb{R}^d} \frac{|x|^{2(d-k)}}{(1 + |x|^{d+1})^2} dx \right)^{1/2} + \\
+ \sum_{k = 0}^{d-1} \left( \int_{\mathbb{R}^d} \left| \nabla^{n-k} \hat{f} \right|^2 |\xi|^{2d} d\xi \right)^{1/2} \left( \int_{|\xi| > 1} |\xi|^{-2(d+k)} d\xi \right)^{1/2} \right) \lesssim \\
\lesssim \frac{1}{t^d} \|f\|_{H^{s,m}(\mathbb{R}^d)}. \tag{3.10}
\]

Observation 3.6. The condition on \( f \) can be reduced to \( f \in H_{n,1}(\mathbb{R}^d) \), with

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\( n > d/2 \), in which case
\[
F \lesssim t^{-n}. \tag{3.11}
\]

**Theorem 3.7.** If \( \hat{E} \) is a vector field on \( \mathbb{R}^3 \), such that \( \hat{E}(\xi) \cdot \xi = 0 \) for any \( \xi \) and \( \hat{E} \in \dot{H}_{2,1}(\mathbb{R}^3) \), then the following is true:
\[
F(t, x) = N \int_{\mathbb{R}^3} e^{2\pi i (x \cdot \xi + t|\xi|)} \hat{E}(\xi) d\xi \lesssim r^{-2}, \tag{3.12}
\]
where \( N \) is the unit vector normal to the 2-spheres \( S_{t,r} \) in the hyperplane of constant \( t \), \( N = \frac{x^i}{r} \partial_{x^i} = \frac{\xi}{r} \).

**Proof of Theorem 3.7.** We note that \( \nabla_{\xi} e^{2\pi i (x \cdot \xi + t|\xi|)} = 2\pi i e^{2\pi i (x \cdot \xi + t|\xi|)} (x + t \frac{\xi}{|\xi|}) \).

Thus,
\[
e^{2\pi i (x \cdot \xi + t|\xi|)} N \hat{E}(\xi) = \frac{1}{2\pi i r} \left( \nabla_{\xi} (e^{2\pi i (x \cdot \xi + t|\xi|)}) \cdot \hat{E}(\xi) - 2\pi i e^{2\pi i (x \cdot \xi + t|\xi|)} \frac{\xi}{|\xi|} \cdot \hat{E}(\xi) \right) = \frac{1}{2\pi i r} \nabla_{\xi} (e^{2\pi i (x \cdot \xi + t|\xi|)}) \cdot \hat{E}(\xi),
\]
using the fact that \( \hat{E}(\xi) \cdot \xi = 0 \). Integrating by parts, we obtain that
\[
F(t, x) = -\frac{1}{2\pi i r} \int_{\mathbb{R}^3} e^{2\pi i (x \cdot \xi + t|\xi|)} \nabla \cdot \hat{E}(\xi) d\xi. \tag{3.13}
\]

What follows is a standard argument. We can assume that \( x \) is parallel to \( e_1 \), one of the coordinate vectors. Let us reddenote \( -\frac{1}{2\pi i} \nabla \cdot \hat{E}(\xi) \) by \( G \). We have that \( G \in \dot{H}_{2,0}(\mathbb{R}^3) \), because \( G \) is the Fourier transform of \( x \cdot E(x) \). Let us make the changes of variable (to polar coordinates) \( \xi \mapsto (\rho, \hat{\xi}) \), where \( \rho = |\xi|, \hat{\xi} = \frac{\xi}{|\xi|} \) and \( \xi \mapsto (\xi_1, \omega) \), where \( \xi_1 = \xi \cdot e_1 \) and \( \omega = \frac{\hat{\xi} - \xi_1 e_1}{(1 - \xi_1^2)^{1/2}} \). We obtain
\[
F(t, x) = \frac{1}{r} \int_{-1}^{1} \int_{0}^{\infty} \int_{S^1} e^{2\pi i \rho (\xi_1 + t)} G \rho^2 d\omega d\rho d\xi_1. \tag{3.14}
\]
Integrating again by parts in $\xi_1$ we get
\[ F(t, x) = \frac{1}{r} \left( \int_0^\infty \int_{S^1} e^{2\pi i \rho (r \xi_1 + t)} G d\omega d\rho \right) \bigg|_{\xi_1=-1} - \int_{-1}^1 \int_0^\infty \int_{S^1} e^{2\pi i \rho (r \xi_1 + t)} \partial_\xi_1 G d\omega d\rho d\xi_1 \bigg). \] (3.15)

Hence we have obtained two powers of $r$ and the remaining quantities under the integral sign are bounded. Indeed, let us deal with each of them separately. The boundary term for $\xi_1 = 1$ becomes (since $G(\rho, \omega, 1) = G(\rho, e_1) = G(\rho e_1)$)
\[ \int_0^\infty \int_{S^1} \frac{e^{2\pi i \rho (r \xi_1 + t)}}{2\pi i \rho r} G(\rho e_1) \rho d\omega d\rho \lesssim \int_0^\infty \| G(\rho, \cdot) \|_{L^\infty(S^2)} \rho d\rho \lesssim \int_0^\infty \| G(\rho, \cdot) \|_{H^{1+\epsilon}(S^2)} \rho d\rho \lesssim \left( \int_0^\infty \| G(\rho, \cdot) \|^2_{H^{1+\epsilon}(S^2)} (\rho^2 + \rho^{1-2\epsilon}) \rho d\rho \right)^{1/2} \left( \int_0^\infty \frac{\rho^2 + \rho^{1-2\epsilon}}{\rho} d\rho \right)^{1/2} \lesssim \| G \|_{\dot{H}^{2,0}(\mathbb{R}^3)}. \] (3.16)

The other boundary term, for $\xi_1 = -1$, is entirely similar and the last term in (3.15) is
\[ \int_{-1}^1 \int_0^\infty \int_{S^1} \frac{e^{2\pi i \rho (r \xi_1 + t)}}{2\pi i} \partial_\xi_1 G(\rho) d\omega d\rho d\xi_1 \lesssim \int_0^\infty \| \nabla G \|_{L^{1+\epsilon}(S^2)} \rho d\rho \lesssim \int_0^\infty \| G(\rho, \cdot) \|_{L^{2+\epsilon}(S^2)} \rho d\rho \lesssim \int_0^\infty \| G(\rho, \cdot) \|_{H^{1+\epsilon}(S^2)} \rho d\rho \lesssim \| G \|_{\dot{H}^{2,0}(\mathbb{R}^3)}. \] (3.17)

Here, by $\epsilon$ we have denoted various small positive numbers. The last step of (3.17) is exactly the same as the last step of (3.16). Thus, the proof is complete. \hfill \Box
Corollary 3.8. If the initial data for equation (1.26) satisfies $F_{\mid t=0} \in H_{2,1}(\mathbb{R}^3)$, then $\rho \lesssim r^{-2}$ and $\sigma \lesssim r^{-2}$ (where $\rho$ and $\sigma$ are the null components of $F$ defined by (1.24)).

Proof of Corollary 3.8. Indeed, we know by Lemma 3.3 that

$$\rho = -N.E = -N. \int_{\mathbb{R}^3} e^{2\pi i (x, \xi + t|\xi|)} \hat{E}_+(\xi) d\xi - N. \int_{\mathbb{R}^3} e^{2\pi i (x, \xi - t|\xi|)} \hat{E}_-(\xi) d\xi,$$

where $\xi. \hat{E}_\pm(\xi) = 0$. The result that $\rho \lesssim r^{-2}$ follows directly from the previous theorem and from the corresponding estimate for $E_-$. By replacing $E$ with $H$, one obtains the result for $\sigma$.

Lemma 3.9. If $\alpha_1$ is a coordinate of $\alpha$ (one of the null components of an electromagnetic tensor $F$, satisfying equation (1.26)), then there exists a vector $v \perp x$ such that

$$\alpha_1(x,t) = \int_{\mathbb{R}^3} e^{2\pi i (x, \xi + t|\xi|)} \left( \left( \frac{x}{|x|} + \frac{\xi}{|\xi|} \right), \hat{H}_+(\xi), v \right) d\xi + \int_{\mathbb{R}^3} e^{2\pi i (x, \xi - t|\xi|)} \left( \left( \frac{x}{|x|} - \frac{\xi}{|\xi|} \right), \hat{H}_-(\xi), v \right) d\xi. \quad (3.18)$$

Proof of Lemma 3.9. Consider a negatively oriented orthonormal basis $(e_1, e_2)$ for $T_x(S_{t,r})$, such that $\epsilon_{12} = 1$ (where $\epsilon$ is the area form of $S_{t,r}$). Indeed, here the area form $\epsilon$ is the same one that appears in the formula (1.24), given by $\epsilon(e_A, e_B) = \frac{1}{2} \epsilon(e_A, e_B, e_3, e_4)$, with $e_3 = \partial_t - \partial_r$, $e_4 = \partial_t + \partial_r$ (see [3, p. 152]). Thus, it defines the negative orientation on the 2-sphere $S_{t,r}$. Clearly $e_1 \perp x$ and, because of the negative orientation, we have that $e_2 = -\frac{x}{|x|} \times e_1$. Then,
by formula (1.24),
\[ \alpha_1 = E_1 + H_2 = e_1 \cdot E - \left( \frac{x}{|x|} \times e_1 \right) \cdot H. \]

Using the representation given by Lemma 3.3 and the fact that \( \hat{E}_\pm = \pm \frac{\xi}{|\xi|} \times \hat{H}_\pm \) (proved in the same Lemma), we obtain that
\[
\alpha_1 = \int_{\mathbb{R}^3} e^{2\pi i (x \cdot \xi + |\xi|)} \left( e_1 \cdot \left( \frac{\xi}{|\xi|} \times \hat{H}_+ (\xi) \right) - \left( \frac{x}{|x|} \times e_1 \right) \cdot \hat{H}_+ (\xi) \right) d\xi + \]
\[
+ \int_{\mathbb{R}^3} e^{2\pi i (x \cdot \xi - |\xi|)} \left( - e_1 \cdot \left( \frac{\xi}{|\xi|} \times \hat{H}_- (\xi) \right) - \left( \frac{x}{|x|} \times e_1 \right) \cdot \hat{H}_- (\xi) \right) d\xi. \tag{3.19}
\]

By rearranging the terms and denoting \( e_1 \) by \( v \), we see that we can rewrite these expressions in the form in which they appear in (3.18). Thus, for example,
\[
e_1 \cdot \left( \frac{\xi}{|\xi|} \times \hat{H}_+ (\xi) \right) - \left( \frac{x}{|x|} \times e_1 \right) \cdot \hat{H}_+ (\xi) = \left( \frac{x}{|x|} + \frac{\xi}{|\xi|} \right) \cdot \hat{H}_+ (\xi), v. \]

This completes the proof of the lemma.

**Theorem 3.10.** For a solution \( F \) of Maxwell’s field equations corresponding to sufficiently smooth initial data,
\[ \alpha \lesssim r^{-5/2}. \tag{3.20} \]

**Proof of Theorem 3.10.** In the neighborhood of any point there exists a local coordinate system in which \( \alpha_1 = |\alpha| \); thus, we actually have to prove the decay estimate for an expression of the form (3.18). Since both the plus and
the minus term that add up to $\alpha_1$ behave similarly, we shall only compute
the former,

$$
\alpha_+(t, x) = \int_{\mathbb{R}^3} e^{2\pi i (x \xi + t |\xi|)} \left( \left( \frac{x}{|x|} + \frac{\xi}{|\xi|} \right), \hat{H}_+ (\xi), v \right) d\xi.
$$

As in the previous proof, let us rewrite the integral in polar coordinates and
integrate by parts. Let us write $|x| = r$, $\frac{x}{|x|} = e_1$ and make the change of
variables $\xi \mapsto (\rho, \hat{\xi})$, $\hat{\xi} \mapsto (\xi_1, \omega)$, where $\rho = |\xi| \in [0, \infty)$, $\hat{\xi} = \frac{\xi}{\rho} \in S^2$,
$\xi_1 = \xi.e_1 \in [-1, 1]$, and $\omega = \frac{\xi - \xi_1 e_1}{(1 + \xi_1^2)^{1/2}} \in S^1$ (the same as in the proof of
Theorem 3.7). In polar coordinates, the integral becomes

$$
\alpha_+(t, x) = \int_0^\infty \int_{-1}^1 \int_{S^1} e^{2\pi i \rho (r \xi_1 + t)} \left( \left( e_1 + \hat{\xi} \right), \hat{H}_+ (\xi), v \right) \rho^2 d\omega d\xi_1 d\rho. \tag{3.21}
$$

Also, let us denote $\left( \left( e_1 + \hat{\xi} \right), \hat{H}_+ (\xi), v \right) = G(\xi)$. Due to the fact that
$\hat{H}_+(\xi) \perp \hat{\xi}$ and $v \perp e_1$, this function has the property that $G(\xi) = O(1 + \xi_1)$,
where $\xi_1 = \hat{\xi}.e_1$. To make this statement more precise, let us choose a
positively oriented orthonormal basis in which

$$
\hat{\xi} = \xi_1 e_1 + (1 - \xi_1^2)^{1/2} e_2,
$$

$$
v = e_2 \cos \theta + e_3 \sin \theta \quad \text{(because $v \perp e_1$), and}
$$

$$
\hat{H}_+(\xi) = |\hat{H}_+(\xi)| \left( \cos \varphi (\xi) \left((1 - \xi_1^2)^{1/2} e_1 - \xi_1 e_2 \right) + \sin \varphi (\xi) e_3 \right)
$$

(which expresses the fact that $\hat{H}_+(\xi) \perp \xi$). We obtain by computing the func-
tion $G$ that

$$
G(\xi) = |\hat{H}_+(\xi)| \left( (1 + \xi_1) (\cos \theta \sin \varphi (\xi) + \xi_1 \sin \theta \cos \varphi (\xi)) + (1 - \xi_1^2) \sin \theta \cos \varphi (\xi) \right)
$$

$$
= (1 + \xi_1) K (\xi), \tag{3.23}
$$

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where we have denoted \( K(\xi) = |\hat{H}_+(\xi)| (\cos \theta \sin \varphi(\xi) + \sin \theta \cos \varphi(\xi)) \). Coming back to formula (3.22) and looking at the definition of \( \omega = \frac{\xi - \xi_1 e_1}{(1 - \xi_1^2)^{1/2}} \), we see that \( e_2 = \omega \). Since

\[
e_3 = e_1 \times e_2 = e_1 \times \omega
\]

(where \( e_1 \) is fixed) and

\[
|\hat{H}_+(\xi)| \sin \varphi(\xi) = \hat{H}_+(\xi).e_3, \quad |\hat{H}_+(\xi)| \cos \varphi(\xi) = \hat{H}_+(\xi).(\hat{\xi} \times e_3),
\]

we have that \( K(\xi) \) is a function only of \( \hat{H}_+(\xi) \) and \( \hat{\xi} \) (it does not depend directly on \( \rho \), the other component of \( \xi \)). Therefore, for any \( k, \ l \) we easily find that

\[
|\partial^k_\rho \nabla^l K| \lesssim \sum_{j=1}^l |\partial^k_\rho \nabla^j \hat{H}_+|,
\]

where \( \nabla \) stands for covariant differentiation on the sphere.

Integrating by parts in \( \xi_1 \) in (3.21), we obtain

\[
\alpha_+ = \int_0^\infty d\varphi \int_{\mathbb{S}^1} e^{2\pi i \rho (r_1 + t)} \frac{G \rho^2 d\varphi d\rho}{2\pi i \rho r} \left[ G_{\rho} \right]_{\xi_1 = -1} + \\
\int_{-1}^1 \int_0^\infty d\varphi \int_{\mathbb{S}^1} e^{2\pi i \rho (r_1 + t)} \frac{G \rho^2 d\varphi d\rho d\xi_1}{2\pi i \rho r}. \quad (3.25)
\]

When \( \xi_1 = -1 \) the boundary term cancels because \( G = 0 \) and we have for \( \xi_1 = 1 \) a boundary term of

\[
2\pi \int_0^\infty e^{2\pi i \rho (r + t)} \frac{G(\rho c_1) \rho^2 d\rho}{2\pi i \rho r}.
\]
Integrating by parts twice in $\rho$ we obtain that

$$
2\pi \int_0^\infty \frac{e^{2\pi i \rho (r+t)}}{2\pi i \rho r} G(\rho e_1) \rho^2 d\rho \lesssim \frac{1}{r(r+t)^2} \left( G(0) + \int_0^\infty |\partial_\rho G(\rho e_1)| + \rho |\partial_\rho^2 G(\rho e_1)| d\rho \right). \quad (3.26)
$$

Here by $G(0)$ I have denoted $\limsup_{\rho \to 0} G(\rho e_1)$ (since we do not know whether $G$ is well-defined at 0). But

$$
G(0) \lesssim \int_0^\infty |\partial_\rho G(\rho e_1)| d\rho \lesssim \int_0^\infty \rho |\partial_\rho^2 G(\rho e_1)| d\rho.
$$

Therefore, it is sufficient to evaluate the last integral, which converges if the initial data is sufficiently regular. Indeed, since $G = (1 - \xi_1) K$, using (3.24) we obtain that $\partial_\rho^2 G \lesssim \partial_\rho^2 \hat{H}_+$. Then we have the following inequalities:

$$
\int_0^\infty \rho |\nabla^2 G| d\rho \lesssim \int_0^\infty \rho |\nabla^2 \hat{H}_+| d\rho \lesssim \int_0^\infty \rho^{1+\epsilon} \|\hat{H}_+(\rho, \cdot)\|_{H^{3+\epsilon}(S(0,\rho))} d\rho \lesssim \left( \int_0^\infty \rho^{2+2\epsilon} + \rho^{3+3\epsilon} \|\hat{H}_+(\rho, \cdot)\|_{H^{3+\epsilon}(S(0,\rho))}^2 d\rho \right)^{1/2} \left( \int_0^\infty \frac{\rho^{2+2\epsilon}}{\rho^{2+2\epsilon} + \rho^{3+3\epsilon}} d\rho \right)^{1/2} \lesssim \|\hat{H}_+\|_{\dot{H}_{2,1+\epsilon}(\mathbb{R}^3)}. \quad (3.27)
$$

Thus, we have proved that the boundary term for $\xi_1 = 1$ in (3.25) decays like $r^{-3}$.

Now let us look at the last term in (3.25). Let us consider separately the cases when $\xi_1$ belongs to the intervals $[-1, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. On the second interval we can apply the same kind of reasoning as above and conclude by
partial integration in $\rho$ that, for $\xi_1 \in [\frac{1}{2}, 1]$, the integral

$$\int_0^\infty \frac{e^{2\pi i \rho (r \xi_1 + t)}}{2\pi i r} \nabla G \rho d\rho$$

(3.28)
decays like $r^{-1}(r \xi_1 + t)^{-2}$ for sufficiently regular initial data, uniformly in $\xi_1$ (see (3.26), (3.33)). Then, since $\partial_{\xi_1} G \cong (1 + \xi_1^2)^{-1/2} \nabla G$ and \( \int_{1/2}^1 (1 + \xi_1^2)^{-1/2} d\xi_1 \)
converges, we obtain a decay rate of $r^{-3}$ for the overall integral

$$\int_{1/2}^1 \int_0^\infty \int_{S^1} e^{2\pi i \rho (r \xi_1 + t)} \partial_{\xi_1} G \rho d\omega d\rho d\xi_1 \lesssim \int_{1/2}^1 \int_0^\infty (r \xi_1 + t)^{-2} d\omega d\rho.$$  

(3.29)
as well.

We know that we can write $G(\xi) = (1 + \xi_1)K(\xi)$, where $K$ has the same regularity properties as $\hat{H_+}$ (see (3.23)). In terms of $K$ the derivatives of $G$ are

$$\partial_{\xi_1} \partial_\rho^k G = \partial_\rho^k K(\xi) + (1 + \xi_1) \partial_{\xi_1} \partial_\rho^k K(\xi) \cong \nabla^k K + (1 + \xi_1) (1 - \xi_1^2)^{-1/2} \nabla^{k+1} K$$

(3.30)

and

$$\partial_{\xi_1}^2 \partial_\rho^k G = 2 \partial_{\xi_1} \partial_\rho^k K(\xi) + (1 + \xi_1) \partial_{\xi_1}^2 \partial_\rho^k K(\xi) \cong (1 - \xi_1^2)^{-1/2} \nabla^{k+1} K + (1 + \xi_1) (1 - \xi_1^2)^{-1} \nabla^{k+2} K.$$  

(3.31)

On the interval $[-1, \frac{1}{2}]$, integrating again by parts in $\xi_1$ in

$$\int_{-1}^{1/2} \int_0^\infty \int_{S^1} e^{2\pi i \rho (r \xi_1 + t)} \partial_{\xi_1} G \rho d\omega d\rho d\xi_1,$$  

(3.32)
as well.
we obtain
\[
\left. \int \int_{-1}^{1} e^{2\pi i \rho(r\xi_1 + t)} \frac{\partial_{\xi_1} G \rho d\omega d\rho d\xi_1}{2\pi i r} \right|_{\xi_1 = -1} = \int \int_{-1}^{1} e^{2\pi i \rho(r\xi_1 + t)} \frac{\partial_{\xi_1} G \rho d\omega d\rho}{(2\pi i r)^2} \left. \right|_{\xi_1 = -1} - \int \int_{-1}^{1} e^{2\pi i \rho(r\xi_1 + t)} \frac{\partial_{\xi_1}^2 G \rho d\omega d\rho d\xi_1}{(2\pi i r)^2}.
\]
\[(3.33)\]

The boundary term at 1/2 can be integrated again by parts, for a decay rate of \( r^{-3} \), while the one at -1 cancels.

We also see that
\[
\int_{0}^{\infty} |\nabla K(\rho \hat{\xi})| + |\nabla^2 K(\rho \hat{\xi})| + |\nabla^3 K(\rho \hat{\xi})| + |\nabla^4 K(\rho \hat{\xi})| d\rho
\]
has a bound independent of \( \hat{\xi} \) for sufficiently smooth initial data. For the last term (the others are similar) this can be proved as follows:
\[
\int_{0}^{\infty} |\nabla^4 K(\rho \hat{\xi})| d\rho \lesssim \int_{0}^{\infty} |\nabla^4 \hat{H}_+(\rho \hat{\xi})| d\rho \lesssim \|
\hat{H}_+\|_{L^\infty(\mathbb{R}^3)} + \int_{0}^{\infty} \|
\hat{H}_+(\rho, \cdot)\|_{H^{5+\epsilon}(S^2)} \rho^\epsilon d\rho \lesssim \|
\hat{H}_+\|_{H_{n,m}(\mathbb{R}^3)}
\]
\[(3.35)\]

for sufficiently large \( m \) and \( n \). Therefore, in the last term of (3.33) we can integrate by parts in \( \rho \) outside the interval \( \{\xi_1 : |r\xi_1 + t| \leq 1\} \) and obtain
that

\[
\int_{-1}^{\frac{1}{2}} \int \int e^{2\pi i \rho (r\xi_1 + t)} \partial^2_{\xi_1} G d\omega d\rho d\xi_1 =
\]

\[
= \int_{-1}^{\frac{1}{2}} \int \int \frac{e^{2\pi i \rho (r\xi_1 + t)}}{2\pi i (r\xi_1 + t)} \partial^2_{\xi_1} G - \frac{e^{2\pi i \rho (r\xi_1 + t)}}{(2\pi i)^2 (r\xi_1 + t)^2} \partial^2_{\xi_1} \partial_\rho G d\omega d\rho d\xi_1 +
\]

\[
+ \int_{-1}^{\frac{1}{2}} \int \int_{|r\xi_1 + t| > 1} \frac{e^{2\pi i \rho (r\xi_1 + t)}}{(2\pi i)^2 (r\xi_1 + t)^2} \partial^2_{\xi_1} \partial_\rho G d\omega d\rho d\xi_1 +
\]

\[
+ \int \int_{|r\xi_1 + t| \leq 1} e^{2\pi i \rho (r\xi_1 + t)} \partial^2_{\xi_1} G d\omega d\rho d\xi_1. \quad (3.36)
\]

However, for \(\rho = 0\) we have that \(G(\rho, \cdot) \equiv G(0, \cdot) = 0\) (even though we cannot say the same thing about \(\partial_\rho G\)). For all the other terms, the integrand in \(\xi_1\) is equal to \((1 - \xi_1^2)^{-1/2}\) times a function bounded uniformly in \(\xi_1\) (by the previous estimates involving \(K\)). Then the expression above is seen to be comparable to

\[
\int_{|r\xi_1 + t| > 1} \frac{1}{2} (1 - \xi_1^2)^{-1/2} |r\xi_1 + t|^{-2} d\xi_1 + \int_{|r\xi_1 + t| \leq 1} (1 - \xi_1^2)^{-1/2} d\xi_1. \quad (3.37)
\]

The second term is clearly at most \(r^{-1/2}\) (because we are integrating something like \(x^{-1/2}\) on an interval of length \(r^{-1}\)) and the first term can be shown by computations to have an order of \(r^{-1}\).

Therefore, I have proved that the last term in (3.33) decays like \(r^{-5/2}\), which completes the proof of the theorem.
Observation 3.11. The same method can be applied in order to derive the decay rates of the components of the spin-2 field.

Observation 3.12. The problem of interior decay is entirely similar to the scalar case. Thus, we note that there is no need for any new statement concerning interior decay, since the problem falls under the conditions of Theorem 3.5.

3.2 Conformal Compactification Method

The following statement can be found (together with its proof) in [3, p. 162] and [3, p. 183].

Theorem 3.13. Consider a conformal transformation on the manifold, $g = \Omega^2 g$, for some positive smooth function $\Omega$. If $F$ satisfies equation (1.26) in the original coordinates, then $F$ satisfies the same equation in the new coordinates. If $W$ satisfies equation (1.27) in the original coordinates, then $\Omega^{-1} W$ satisfies the equation in the new coordinates.

Lemma 3.14. ([1, p. 497], [8]) The Minkowski space $\mathbb{R}^{3+1}$ with the metric $g$ is conformal to a subset of the Einstein cylinder $\Sigma_4 = \mathbb{R} \times S^3$, of metric $\underline{g}$, under the conformal transformation

$$
\underline{g} = \Omega^2 g,
$$

where $\Omega^2 = 4 \left(1 + (r + t)^2\right)^{-1} \left(1 + (r - t)^2\right)^{-1}$. The metric $\underline{g}$ is the natural metric of the cylinder.
If we represent points in $\Sigma_4$ by the standard coordinates $(T, \alpha, \theta, \phi)$, then the image of $\mathbb{R}^{3+1}$ is represented by the set

$$
0 \leq \alpha < \pi, 0 \leq \theta < \pi, 0 \leq \phi < 2\pi
$$
$$
\alpha - \pi < T < \pi - \alpha.
$$

(3.39)

The coordinate change is given by

$$
x^1 = r \sin \theta \sin \phi, x^2 = r \sin \theta \cos \phi, x^3 = r \cos \theta
$$
$$
x_0 = \frac{1}{2} \left( \tan \frac{T+\alpha}{2} + \tan \frac{T-\alpha}{2} \right), r = \frac{1}{2} \left( \tan \frac{T+\alpha}{2} - \tan \frac{T-\alpha}{2} \right).
$$

(3.40)

In the new coordinates, $\Omega$ is

$$
\Omega^2 = (\cos \alpha + \cos T)^2.
$$

(3.41)

**Theorem 3.15.** If $\alpha$ is a $p$-covariant tensor field defined on $\mathbb{R}^3$, then $\Omega^k\alpha \in H^s(S^3)$ under the condition that $\alpha \in H_{s, s+2p-3-2k}$.

**Theorem 3.16.** A solution of the linear homogenous hyperbolic equation on $\Sigma_4$ with initial data $(u(0), u_t(0)) \in H_{s,0}(S^3) \times H_{s-1,0}(S^3)$ belongs to $E_s(\Sigma_4)$, where

$$
E_s(\Sigma_4) = \{ f \in L^2(\Sigma_4) | \sup_T \left( \sum_{k=1}^{[s]} \| \partial^k_T f \|_{H^{s-k}(S^3)}^2 \right)^{1/2} < \infty \}. \tag{3.42}
$$

Here $S_\tau = \Sigma_4 \cap \{(T, \omega) | T = \tau\} = \{\tau\} \times S^3$ and all the norms are evaluated using the metric induced on $S_\tau$ by the metric $g$ of $\Sigma_4$, which is the natural metric of the sphere.

**Theorem 3.17.** The following imbedding holds:

$$
E_{3/2+\epsilon}(\Sigma_4) \subset C_b(\Sigma_4), \tag{3.43}
$$

where $C_b$ is the set of bounded continuous functions.
Theorem 3.18. The null coordinate vectors \((E_+ , E_- , e_3 , e_4)\) are transformed by the conformal mapping in the following manner:

\[
e_A = \left(1 + (t + r)^2\right)^{-1/2} \left(1 + (t - r)^2\right)^{-1/2} e_A, \\
e_3 = \left(1 + (t - r)^2\right)^{-1} e_3, \\
e_4 = \left(1 + (t + r)^2\right)^{-1} e_4,
\]

where \(e_\mu\) form an orthonormal basis for \(T_P(\Sigma_4)\).

Theorem 3.19. Consider the equation (1.26), with initial data \(F \mid_{t=0} \in H_{2,3}(\mathbb{R}^3)\). Then the null components of \(F\) have the following decay rates:

\[
\alpha \lesssim (1 + (t + r)^2)^{-1/2} (1 + (t - r)^2)^{-3/2} \\
(\rho, \sigma) \lesssim (1 + (t + r)^2)^{-1} (1 + (t - r)^2)^{-1} \\
\alpha \lesssim (1 + (t + r)^2)^{-3/2} (1 + (t - r)^2)^{-1/2}.
\]

Also, if we consider a solution \(W\) of equation (1.26), for initial data \(W \mid_{t=0} \in H_{2,9}(\mathbb{R}^3)\), we obtain

\[
\alpha \lesssim (1 + (t + r)^2)^{-1/2} (1 + (t - r)^2)^{-5/2} \\
\beta \lesssim (1 + (t + r)^2)^{-1} (1 + (t - r)^2)^{-2} \\
(\rho, \sigma) \lesssim (1 + (t + r)^2)^{-3/2} (1 + (t - r)^2)^{-3/2} \\
\beta \lesssim (1 + (t + r)^2)^{-2} (1 + (t - r)^2)^{-1} \\
\alpha \lesssim (1 + (t + r)^2)^{-5/2} (1 + (t - r)^2)^{-1/2}.
\]

The computation is straightforward, considering the fact that \(F\) and \(\Omega^{-1}W\) are bounded in the \(\Sigma_4\) metric.

Observation 3.20. In both cases, the conditions imposed on the initial data can be lowered by almost 1/2, i.e. \(F \mid_{t=0} \in H_{3/2 + \epsilon, 3}(\mathbb{R}^3)\) and \(W \mid_{t=0} \in H_{3/2 + \epsilon, 9}(\mathbb{R}^3)\). In order to make sense of such conditions, one may use the Fourier transform.
Observation 3.21. By the same method, one can obtain estimates for the
derivatives of the tensors, because if $F |_{t=0} \in H_{2+l,3+l}(\mathbb{R}^3)$ it follows that $F \in E_{2+l}(\Sigma_4) \subset C^l_0(\Sigma_4)$, from which we can obtain estimates for the derivatives.
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Bibliography


This paper represents my own work in accordance with University regulations.

Marius Beceanu