The Holiday Puzzle Solution
GROUNDHOG DAY, 2024

THIS YEAR: THE DERIVATIVE GRINCH

The derivative grinch, being the mean one, wants you to find

$$\frac{d^{2023}}{dx^{2023}}(x^{12}e^{25x}).$$

If this derivative is not found, all the holiday presents in Mathville will be stuffed up chimneys. If this derivative is found, the derivative grinch's heart will grow 3 sizes, and he will join all the folks in Mathville in welcoming the holidays.

In finding this derivative, you may leave calculations involving large numbers, but you should not end up with any derivative left to find. You should also do it so that the derivative grinch says "It came without computers!"

Solution: The following proposition allows us to take multiple derivatives of a product.

Proposition 1 *If $f(x) = g(x)h(x)$, then*

$$\frac{d^n(f(x))}{dx^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j(g(x))}{dx^j} \frac{d^{n-j}(h(x))}{dx^{n-j}}.$$

Note that the 0-th derivative of a function is just the function. This proposition can be proved by induction. The case $n = 1$ is just the product rule for derivatives. Now suppose

$$\frac{d^n(f(x))}{dx^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j(g(x))}{dx^j} \frac{d^{n-j}(h(x))}{dx^{n-j}}.$$

Then

$$\begin{aligned}
\frac{d^{n+1}(f(x))}{dx^{n+1}} &= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{d^j(g(x))}{dx^j} \frac{d^{n+1-j}(h(x))}{dx^{n+1-j}} + \frac{d^{j+1}(g(x))}{dx^{j+1}} \frac{d^{n-j}(h(x))}{dx^{n-j}} \right) \\
&= \frac{d^0(g(x))}{dx^0} \frac{d^{n+1}(h(x))}{dx^{n+1}} \\
&\quad + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) \frac{d^k(g(x))}{dx^k} \frac{d^{n+1-k}(h(x))}{dx^{n+1-k}} \\
&\quad + \frac{d^{n+1}(g(x))}{dx^{n+1}} \frac{d^0(h(x))}{dx^0} \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{d^j(g(x))}{dx^j} \frac{d^{n+1-j}(h(x))}{dx^{n+1-j}}
\end{aligned}$$

where we used the equality

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $k = 1, ..., n$. Thus

$$\begin{aligned}
\frac{d^{2023}}{dx^{2023}}(x^{12}e^{25x}) &= \sum_{j=0}^{2023} \binom{2023}{j} \frac{d^j(x^{12})}{dx^j} \frac{d^{2023-j}(e^{25x})}{dx^{2023-j}} \\
&= \sum_{j=0}^{12} \binom{2023}{j} \frac{12!}{(12-j)!} x^{12-j} 25^{2023-j} e^{25x}.
\end{aligned}$$

This problem was inspired by a discussion with Jake Cordes and others about the Proposition. However, no solutions were received, and so the derivative grinch went looking for holiday presents to stuff up the chimneys in Mathville. None were to be found on February 2. So the derivative grinch then decided to stuff the groundhog up the chimney. The groundhog objected, bit the derivative grinch, and ran off. The people in Mathville were unable to hold the usual Groundhog Day ceremony and do not know whether the groundhog saw his shadow. The people in Mathville also do not know whether the derivative grinch got rabies shots after being bit by the groundhog. The derivative grinch's dog was found running loose and shivering, and so the folks in Mathville wrapped him in a blanket decorated with the phrase "$f'(a) = 0$ and $f''(a) < 0$". Henceforth, that dog was known as the local Max.

I hope to have a Holiday Puzzle next year. Look for it on the Web at `http://www.albany.edu/~martinhi/puzzle.html`

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