The Holiday Puzzle Solution: Groundhog Day, 2022

THIS YEAR: 2021

Find all ordered pairs of integers (m, n) such that

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2021}.$$

Note that we do not restrict ourselves to cases where m and n are both positive.

Solution: There are 17 such ordered pairs:

(2022,4086462)	(4086462,2022)
(2020,-4082420)	(-4082420,2020)
(2064,97008)	(97008,2064)
(1978,-92966)	(-92966,1978)
(1974,-84882)	(-84882,1974)
(3870,4230)	(4230,3870)
(172,-188)	(-188,172)
(4042,4042)	
(2068,88924)	(88924,2068)

To see this, start with

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2021}.$$

Multiply both sides by mn to get

$$n + m = \frac{mn}{2021}$$

and hence $mn = 2021(n + m)$. So $mn - 2021m - 2021n = 0$.

Thus $mn - 2021m - 2021n + 2021^2 = 2021^2$.

So $(m - 2021)(n - 2021) = 2021^2$.

Thus $m - 2021$ is an integer dividing 2021^2. Observe the prime factorization of 2021 is 43×47. So the possible values are given by the following table:

$m-2021$	m	$n-2021$	n
1	2022	$43^2 \times 47^2 = 4084441$	4086462
-1	2020	-4084441	-4082420
43	2064	$43 \times 47^2 = 94987$	97008
-43	1978	-94987	-92966
47	2068	$43^2 \times 47 = 86903$	88924
-47	1974	-86903	-84882
$43^2 = 1849$	3870	$47^2 = 2209$	4230
-1849	172	-2209	-188
$43 \times 47 = 2021$	4042	$43 \times 47 = 2021$	4042
-2021	0 invalid	-2021	0 invalid
$47^2 = 2209$	4230	$43^2 = 1849$	3870
-2209	-188	-1849	172
$43^2 \times 47 = 86903$	88924	47	2068
-86903	-84882	-47	1974
$43 \times 47^2 = 94987$	97008	43	2064
-94987	-92966	-43	1978
$43^2 \times 47^2 = 4084441$	4086462	1	2022
-4084441	-4082420	-1	2020

This solution is based upon one submitted by Nick Harrison. Solutions were also submitted by Henry Ashley, Dick Collier (who also explored some generalizations), and Anupam Srivastav. A partial solution also was received.

This question was inspired by a similar question on the 2018 Putnam Competition. That question involved finding all ordered pairs (a, b) of positive integers such that

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}$$

I hope to have a holiday puzzle next year. Look for it on the Web at `http://www.albany.edu/~martinhi/puzzle.html`

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