A combinatorial Chevalley formula for semi-infinite flag manifolds and its applications

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Geometric Aspects of Algebraic Combinatorics
AMS special session, University of Massachusetts Amherst
October 1-2, 2022

C. Lenart was supported by the NSF grant DMS-1855592.

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where $s_{(1)} = s_\square = x_1 + x_2 + \ldots$.

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Geometric interpretation (Schubert calculus on flag manifolds): $s_\lambda(x_1, \ldots, x_k)$ represent Schubert classes σ_λ (i.e., cohomology classes of Schubert varieties) in the cohomology of Grassmannians $Gr_k(\mathbb{C}^n) = SL_n/P_k$:

$$H^*(Gr_k(\mathbb{C}^n)) \simeq Sym(x_1, \ldots, x_k)/I \,.$$

Chevalley formulas (cont.)

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Chevalley (Monk) formula:

$$\mathfrak{S}_w \cdot \mathfrak{S}_{s_k} = \sum_{\substack{i \le k < j \\ \ell(wt_{ij}) = \ell(w)+1}} \mathfrak{S}_{wt_{ij}}\,,$$

where $s_k = t_{k,k+1}$ and $\mathfrak{S}_{s_k} = x_1 + \ldots + x_k$.

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- generalized flag varieties G/B or partial flag varieties G/P (G semisimple Lie group over $\mathbb{C}$, B Borel subgroup, P parabolic subgroup);
- affine versions: affine flag manifold, semi-infinite flag manifold $\mathbf{Q}_G$.

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- $QK_T(G/B)$ is closely related to $K_T(\mathbf{Q}_G)$ (breakthrough of Syu Kato);
- The semi-infinite flag manifolds have applications to the representation theory of affine Lie algebras (level 0 extremal weight modules, Kato-Naito-Sagaki).

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- Chevalley formulas for $QK_T(G/B)$ and $QK_T(G/P)$ (for G of arbitrary Lie type).
- Applications: more explicit computations and results in type A, for $QK(Fl_n)$.

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Type A_{n-1}:
$P = \{\sum_{i=1}^n \lambda_i \varepsilon_i = (\lambda_1, \ldots, \lambda_{n-1}) : \lambda_i \in \mathbb{Z}_{\geq 0}\}$ – compositions;
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Previous work

$K_T(G/B)$, as module over $K_T(\mathrm{pt}) = \mathbb{Z}[P]$, has a basis of Schubert classes $[\mathcal{O}_{X_w}]$, $w \in W$ (classes of the structure sheaves of Schubert varieties X_w).

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Chevalley formula for $K_T(G/B)$:

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[L.-Postnikov, 2003]: combinatorial Chevalley formula in terms of the alcove model.

Quantum alcove model: quantum Bruhat graph on the finite Weyl group

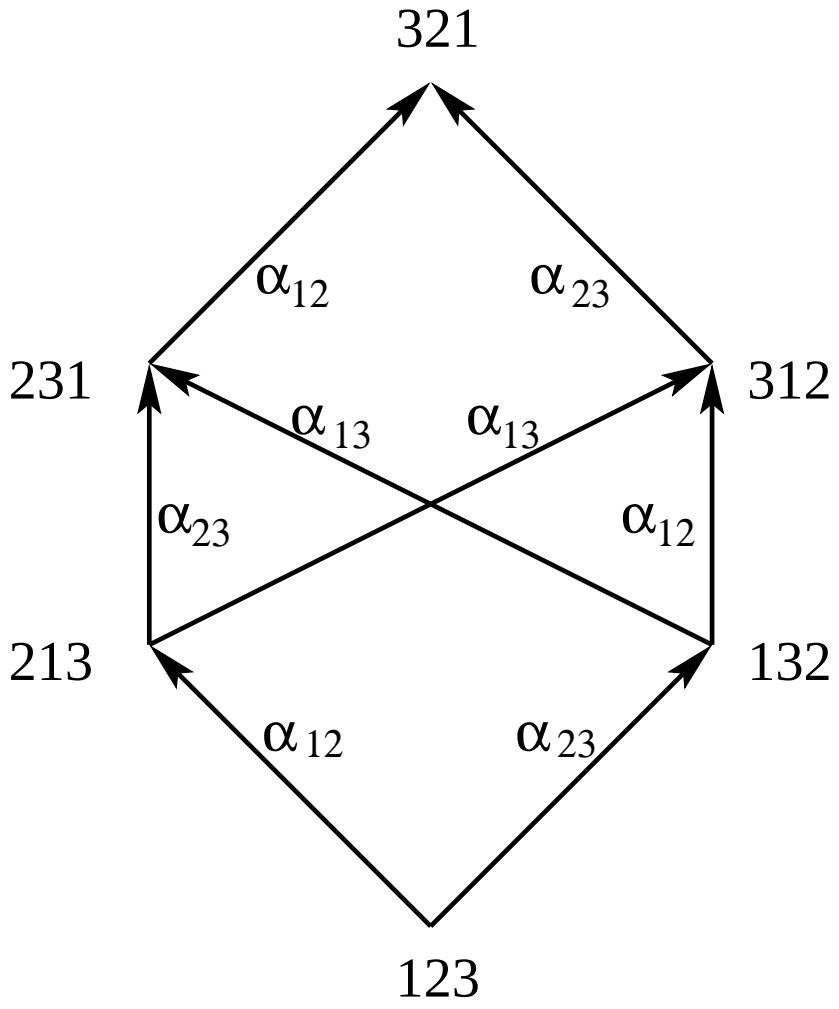
The quantum Bruhat graph on W, denoted QBG(W), is the directed graph with labeled edges

$$w \xrightarrow{\alpha} ws_\alpha\,, \qquad \text{where}$$

$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(covers of Bruhat order)}\,, \qquad \text{or}$$
$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1\,.$$

(If $\alpha^\vee = \sum_i c_i \alpha_i^\vee$, then $\mathrm{ht}(\alpha^\vee) := \sum_i c_i$.)

Hasse diagram of the Bruhat order for S_3:



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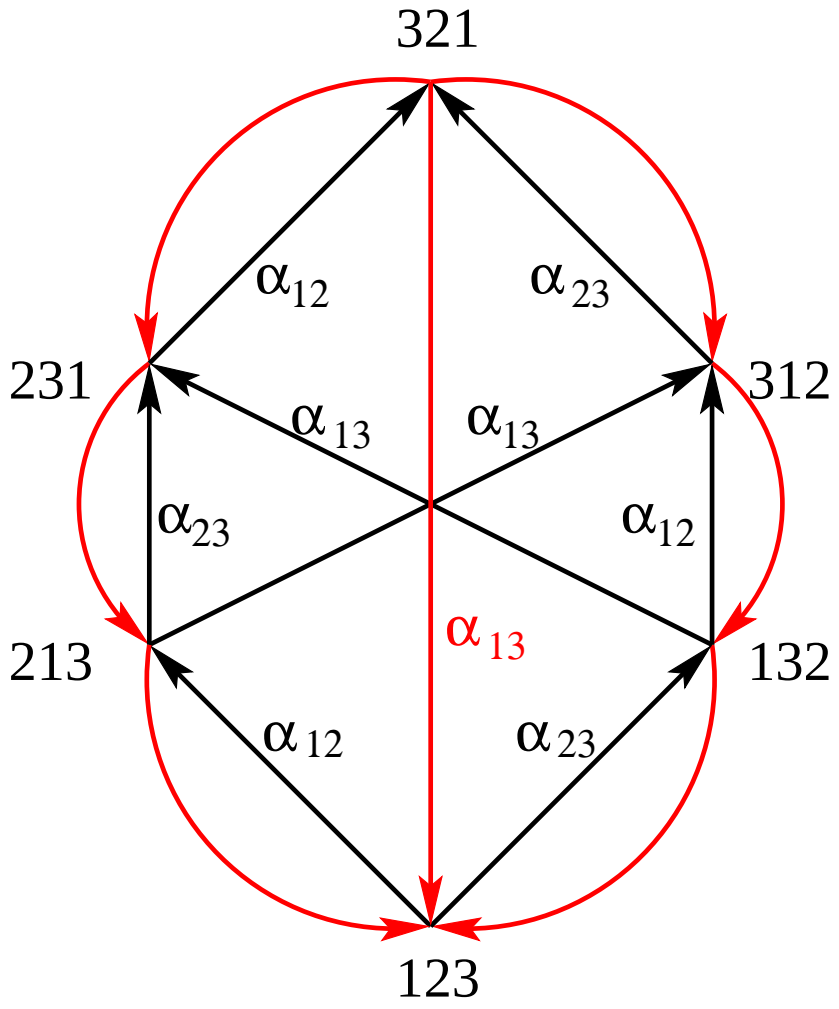

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Given **any** weight λ, we associate with it a sequence of roots, called a λ-chain:

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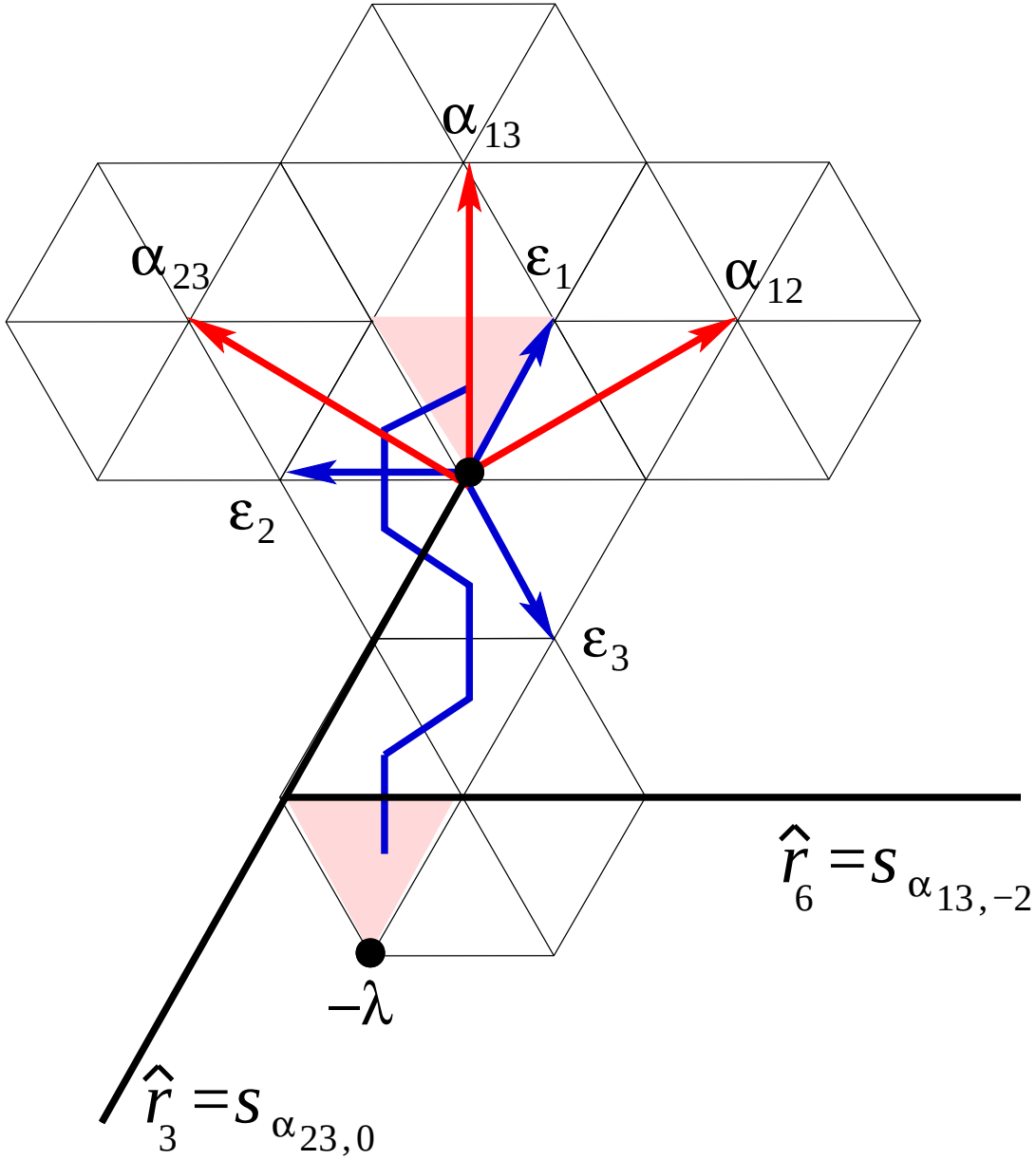
The latter gives a shortest sequence of adjacent alcoves from $A_\circ$ to $A_\circ - \lambda$.

Example. Type A_2, $\lambda = (3,1,0) = 3\varepsilon_1 + \varepsilon_2$,
$\Gamma = (\ (1,2),\ (1,3),\ (2,3),\ (1,3),\ (1,2),\ (1,3)\ )$.



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Given $\Gamma = (\beta_1, \ldots, \beta_m)$, let $r_i := s_{\beta_i}$ and $\widehat{r_i} := s_{\beta_i, -l_i}$.

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The objects of the model: subsets of positions in Γ

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For $w \in W$ and A, construct the chain $\pi(w, A)$ of elements in W:

$$w_0 = w, \quad \ldots, \quad w_i := w r_{j_1} \ldots r_{j_i}, \quad \ldots, \quad w_s = \mathrm{end}(w, A)\,.$$

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The main structure structure: w-admissible subsets

$$\mathcal{A}(w, \Gamma) := \{A \,:\, \pi(w, A) \text{ path in } \mathrm{QBG}(W)\}\,.$$

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In addition to $\mathrm{end}(w, A)$, we associate the following statistics with a pair (w, A), for $A = \{j_1 < \ldots < j_s\} \in \mathcal{A}(w, \Gamma)$:

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- $\mathrm{down}(w, A) := \sum_{j \in A^-} |\beta_j|^\vee \in Q^{\vee,+}$.

Independence of the quantum alcove model from the λ-chain

Theorem. [Kouno-L.-Naito] Given λ-chains Γ, Γ', there is a sijection [Fisher-Konvalinka] between $\mathcal{A}(w, \Gamma)$ and $\mathcal{A}(w, \Gamma')$ which preserves the relevant statistics.

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Based on quantum Yang-Baxter moves, which are root system analogues of jeu de taquin slides for semistandard Young tableaux (in type A).

The Chevalley formula for semi-infinite flag manifolds $\mathbf{Q}_G$

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- translate the Chevalley formulas for $\lambda \in P^+$ [Kato-Naito-Sagaki] and $\lambda \in P^-$ [Naito-Orr-Sagaki] from quantum LS paths to the quantum alcove model;
- generalize the new formulas to arbitrary $\lambda \in P$, via combinatorics of the quantum alcove model.

Quantum K-theory

Consider variables Q_i for $i \in I$, and let

$$\mathbb{Z}[Q] := \mathbb{Z}[Q_1, \ldots, Q_r]\,, \quad \mathbb{Z}[Q][P] := \mathbb{Z}[Q] \otimes_{\mathbb{Z}} \mathbb{Z}[P]\,.$$

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Given $\xi = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$ in $Q^{\vee,+}$, let $Q^\xi := Q_1^{d_1} \cdots Q_r^{d_r}$.

The Chevalley formula in $QK_T(G/B)$

Theorem. [L.-Naito-Sagaki, conjecture by L.-Postnikov 2003] Let $k \in I$, and fix a $(-\omega_k)$-chain of roots $\Gamma(-\omega_k)$. Then, in $QK_T(G/B)$, we have the cancellation-free formula:

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$$[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^w] = (1 - \mathbf{e}^{w(\omega_k)-\omega_k})\,[\mathcal{O}^w] +$$

$$\sum_{A \in \mathcal{A}(w,\Gamma(-\omega_k)) \setminus \{\emptyset\}} (-1)^{|A|-1}\, Q^{\mathrm{down}(w,A)}\, \mathbf{e}^{-\omega_k - \mathrm{wt}(w,A)}\, [\mathcal{O}^{\mathrm{end}(w,A)}]\,.$$

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Proof: Translate the corresponding Chevalley formula for the semi-infinite flag manifold via Kato's isomorphism:

$$QK_T(G/B) \xrightarrow{\simeq} K'_T(\mathbf{Q}_G) \subset K_T(\mathbf{Q}_G)\,.$$

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Definition. [L.-Maeno 2006, cf. Fomin-Gelfand-Postnikov] The quantum Grothendieck polynomial $\mathfrak{G}_w^Q$ is

$$\mathfrak{G}_w^Q := Q(\mathfrak{G}_w) \in \mathbb{Z}[Q, x] \quad \text{for } w \in S_n\,.$$

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Theorem. [L.-Naito-Sagaki] In the expansion of $[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^w]$, all (non-zero) coefficients are ± 1 (explicitly determined).