DUAL HOPF ORDERS IN GROUP RINGS OF ELEMENTARY ABELIAN $p$-GROUPS

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Let $R$ be the valuation ring of $K$, a finite extension of $\mathbb{Q}_p$ containing a primitive $p$th root of unity, and let $G$ be an elementary abelian $p$-group of order $p^n$, with dual group $\hat{G}$. We construct a new family of triangular Hopf orders over $R$ in $KG$, a proper subfamily whose duals are also triangular, and a proper subfamily of that family whose construction extends the truncated exponential construction of [GC98]. In contrast to the Hopf orders of [GC98], the construction yields examples where none of the rank $p^2$ subquotients are Larson orders.

For $G$ a finite abelian group of order $p^n$, the classification of Hopf orders over $R$ in $KG$ is known only for $n = 1$ [TO70] and 2 [Gr92], [By93], [Un94]. For $n > 2$ only a few families of Hopf orders were known until recently: [Ra74] for $G$ elementary abelian of order $p^n$, [La76] for arbitrary $G$ ("Larson orders"), [Un96] for $G$ cyclic of order $p^3$, [CS98] and [GC98] for $G$ elementary abelian of order $p^n$. For $G$ of order $p^n$, Larson orders are described completely by $n$ valuation parameters (that determine the discriminant of the Hopf order); the examples of [Gr92] and [CS98], [GC98] suggest that a general Hopf order of rank $p^n$ should involve, in addition, $n(n-1)/2$ unit parameters, which can conveniently be laid out as entries of a lower triangular matrix.

There are two ways that these unit parameters can arise. In the formal group construction of [CS98], the matrix of unit parameters is used to construct an isogeny of polynomial formal groups whose kernel is represented by a Hopf order. The matrix entries then show up as coefficients of group elements in the algebra generators of the Hopf order. In the constructions of [Un96] and [CS98], the unit parameters appear directly in the algebra generators, generalizing Greither’s construction in [Gr92]. This approach was codified in [UC05] in the definition of triangular Hopf orders.

This is the fourth in a recent series of papers that construct new families of Hopf orders with the desired number of parameters. [CU03] used formal groups to construct families of Hopf orders in $KG$ when $G$ is cyclic of order $p^n$; [UC05] obtained several families of triangular Hopf

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orders in $KG$, $G$ cyclic of order $p^3$ and strengthened the formal group construction of [CU03] when $n = 3$, and [CU04] identified a subfamily of the formal group Hopf orders of [CU03] in the cyclic $p^n$ case whose duals are triangular. In this paper we obtain new families of triangular Hopf orders in the elementary abelian $p^n$ case.

The construction in [GC98] used unit parameters in the form of truncated exponentials. These parameters are particularly attractive for constructing dual pairs of Hopf orders, because the matrices of unit parameters associated to dual pairs are essentially inverses of each other. The smaller family of dual pairs of Hopf orders in this paper is constructed using truncated exponentials, similar to that in [CG98], but with less restrictive conditions on the parameters. Many of the rank two subquotients of the Hopf orders constructed in [CG98] must be Larson orders, whereas we find conditions sufficient to insure that the Hopf orders constructed here have no rank two Larson subquotients.

Here is an outline of the paper. In Section 1, we apply the strategy of utilizing Larson orders inside arbitrary Hopf orders, used in [GC98], to construct (Theorem 3) a new class of triangular Hopf orders in $KG$ for arbitrary $n$. In Section 2, we extend the construction (Theorem 5) to obtain dual pairs of triangular Hopf orders in $KG$. In Section 3 we construct (Theorem 9) dual pairs of triangular Hopf orders using the truncated exponential function introduced in [CG98]. To do the construction requires subjecting the unit parameters to an additional set of inequalities beyond those required in Section 2. Examples can be found (using the simplex algorithm) that show that the three constructions are in fact increasingly restrictive. In the final section we obtain a family of examples of dual pairs constructed by Theorem 9 that are "Larsonless", that is, have no rank two subquotients that are Larson orders.

1. Hopf Orders

Let $K$, $R$ be as noted above. Let $\pi$ be a parameter for (generator of the maximal ideal of) $R$, let $e$ be the absolute ramification index of $K$, assume $K$ contains a primitive $p$th root of unity $\zeta$, and let $e' = \text{ord}(\zeta - 1) = e/(p - 1)$. Let $G$ be an elementary abelian $p$-group of order $p^n$, $G = G_1 \times \ldots \times G_n$ with $G_r = \langle \sigma_r \rangle$ cyclic of order $p$. In $KG_r = K\langle \sigma_r \rangle$, let

$$e_j^{(r)} = \frac{1}{p} \sum_{i=1}^{p-1} \zeta^{-ij} \sigma_i^r,$$
$j = 1, \ldots, p - 1$, be the primitive idempotents. For $u$ in $K$ let

$$a_u^{(r)} = \sum_{k=0}^{p-1} u^k e_k^{(r)}.$$  

Then $a_u^{(r)}$ is a multiplicative homomorphism from $K^\times$ to $KG^\times_r$ satisfying $a_\zeta^{(r)} = \sigma_r$.

Let $i_1, \ldots, i_n$ be valuation parameters: numbers satisfying $0 \leq i_j \leq e'$ for all $j$, and denote $i'_j = e' - i_j$ as usual. Assume $i_r \leq pi_j$ for all $r$ and all $j < r$, and also that $i'_s \leq pi'_k$ for all $s$ and all $k > s$. Let $U = (u_{i,j})$ be a lower triangular matrix with diagonal entries $= \zeta$. Let $a_{i,j} = a_{u_{i,j}}^{(j)}$ in $K_{G_j}$. Then $a_{i,j} = a_{\zeta}^{(j)} = \sigma_j$. Consider the $R$-algebra

$$H_n = R[\frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{a_{2,1}\sigma_2 - 1}{\pi^{i_2}}, \ldots, \frac{a_{n,1}a_{n,2} \cdots a_{n,n-1}\sigma_n - 1}{\pi^{i_n}}].$$

In this section we find conditions on the entries of $U$ for $H_n$ to be a Hopf order over $R$ in $KG$.

Our approach uses the following result of Greither [Gr92], for which a convenient reference is [Ch00, (31.8), (31.10)]:

**Proposition 1.** Let $0 \leq i, k \leq e'$ and $k < pi$. Let $G, G'$ be cyclic of order $p$ with generators $\sigma, \sigma'$, respectively. For $v$ in $R$, let $a_v$ be in $K$ and $t = \frac{a_v - 1}{\pi^i}$. Let $H(i) = R[\frac{\sigma^i - 1}{\pi^i}]$ in $KG$. Then $E_v = H(i)[t]$ is an order in $K[G \times G']$, free of rank $p$ as an $H(i)$-module with power basis $1, t, \ldots, t^{p-1}$, if and only if

$$\text{ord}(v - 1) \geq \frac{i'}{p} + k;$$

$H$ is then a Hopf order in $K[G \times G']$ if $\Delta(a_v) \equiv a_v \otimes a_v$, iff

$$\text{ord}(v - 1) \geq \frac{i'}{p} + k.$$  

We proceed to construct the order $H_n$ in $KG$ with $G = G_1 \times \ldots \times G_n$, inductively.

For each $r$ with $1 \leq r < n$, having constructed $H_r$, the construction of $H_{r+1}$ involves finding a Larson order $H(\mu_r) = R[\frac{\sigma_{r'} - 1}{\pi^{i_{r'}}}]$ inside $H_r$. We call $\mu_1, \mu_2, \ldots, \mu_n$ a set of Larson parameters for $H_n$ if $H(\mu_r) \subseteq H_r$ for $r = 1, \ldots, n$.

Let

$$H_1 = R[\frac{\sigma_1 - 1}{\pi^{i_1}}] = R[\frac{a_{1,1} - 1}{\pi^{i_1}}],$$
a Larson order in $KG_1$, and set $\mu_1 = i_1$, the first Larson parameter. Recalling the matrix $U$, let

$$H_2 = H_1 \left[ \frac{a_{2,1}\sigma_2 - 1}{\pi^{\mu_2}} \right] = H_1 \left[ \frac{a_{2,1}a_{2,2} - 1}{\pi^{\mu_2}} \right].$$

By Proposition 1, this is a Hopf order in $K[G_1 \times G_2]$ iff

$$\text{ord}(u_{2,1} - 1) \geq \frac{i'_1}{p} + i_2$$

and

$$\text{ord}(u_{2,1} - 1) \geq \frac{i'_1 + \frac{i_2}{p}}. $$

Given that $H_2$ is a Hopf order, we have

$$H_2 \cap K[\sigma_1] = H_1 = H(i_1) = R[\frac{\sigma_1 - 1}{\pi^{i_1}}],$$

and we need to find a Larson parameter $\mu_2$ so that

$$H(\mu_2) = R[\frac{\sigma_2 - 1}{\pi^{\mu_2}}] \subseteq H_2 \cap K[\sigma_2].$$

**Proposition 2.** $H(\mu_2) \subseteq H_2 \cap K[\sigma_2]$ if $\mu_2 \leq \min\{i_2, \text{ord}(u_{2,1} - 1) - i'_1\}$.

**Proof.** We have

$$\frac{a_{2,1}\sigma_2 - 1}{\pi^{\mu_2}} = \frac{a_{2,1}(\sigma_2 - 1)}{\pi^{\mu_2}} + \frac{a_{2,1} - 1}{\pi^{\mu_2}}. $$

Assume that $\mu_2 \leq i_2$, so that $\frac{a_{2,1}\sigma_2 - 1}{\pi^{\mu_2}}$ is in $H_2$. Then $a_{2,1}$ is a unit of $H_2$, and so $\frac{a_{2,1} - 1}{\pi^{\mu_2}}$ will be in $H_2$ if and only if $\frac{a_{2,1} - 1}{\pi^{\mu_2}}$ is in $H_1$. But by [UC05, Proposition 2.1],

$$R[\frac{a_{2,1} - 1}{\pi^{\mu_2}}] \subseteq H(i_1)$$

iff

$$\text{ord}(u_{2,1} - 1) - \mu_2 \geq \epsilon' - i_1 = i'_1,$$

or $\mu_2 \leq \min\{i_2, \text{ord}(u_{2,1} - 1) - i'_1\}$. \hfill \square

We proceed by induction. Suppose we have found conditions on the entries of the first $r - 1$ rows of the matrix $U$ so that $H_{r-1}$ is an $R$-Hopf order in $K[G_1 \times \ldots \times G_{r-1}]$ in such a way that for $2 \leq j \leq r - 1$, $H_j$ is free of rank $p$ over $H_{j-1}$ on powers of the algebra generator $t_j = \frac{a_{i_1,i_2,\ldots,i_{r-1}}\sigma_j - 1}{\pi^{\mu_j}}$ of $H_j$ over $H_{j-1}$. Suppose also we have found $r - 1$ Larson parameters $\mu_1 = i_1, \mu_2, \ldots, \mu_{r-1}$ so that

$$H_j \cap K[G_j] = H_{r-1} \cap K[G_j] \supseteq H(\mu_j) = R[\frac{\sigma_j - 1}{\pi^{\mu_j}}].$$
Consider
\[ H_r = H_{r-1}\left[ \frac{a_{r,1}a_{r,2} \cdots a_{r,r-1} \sigma_r - 1}{\pi^{i_r}} \right] = H_{r-1}\left[ \frac{a_{r,1}a_{r,2} \cdots a_{r,r} - 1}{\pi^{i_r}} \right]. \]

For \( H_r \) to be free of rank \( p \) over \( H_{r-1} \) with power basis generated by
\[ t = \frac{a_{r,1}a_{r,2} \cdots a_{r,r-1} \sigma_r - 1}{\pi^{i_r}} \]
it suffices that
\[ \frac{a_{r,1}^p a_{r,2}^p \cdots a_{r,r-1}^p - 1}{\pi^{pi_r}} \in H_{r-1}, \]
which follows if
\[ \frac{a_{u_{r,j}} - 1}{\pi^{pi_r}} \in H(\mu_j) \subset H_{r-1} \cap KG_j \]
for all \( j < r \), which in turn follows from
\[ \text{ord}(w_{r,j} - 1) - pi_r \geq e' - \mu_j \]
that is,
\[ \text{ord}(u_{r,j} - 1) \geq \frac{\mu_j'}{p} + i_r \]
for \( j = 1, \ldots, r-1 \).

To show that \( H_r \) is a Hopf order, that is, the comultiplication on
\( KG \) maps \( H_r \) to \( H_r \otimes H_r \), we need
\[ \Delta(a_{r,1} \cdots a_{r,r-1}) \equiv a_{r,1} \cdots a_{r,r-1} \otimes a_{r,1} \cdots a_{r,r-1} \mod \pi^{i_r} H_{r-1} \otimes H_{r-1}. \]
Now
\[ \Delta(a_{r,1} \cdots a_{r,r-1}) = \Delta(a_{r,1}) \cdots \Delta(a_{r,r-1}) \]
and
\[ a_{r,1} \cdots a_{r,r-1} \otimes a_{r,1} \cdots a_{r,r-1} = (a_{r,1} \otimes a_{r,1}) \cdots (a_{r,r-1} \otimes a_{r,r-1}). \]
So it suffices that for each \( j \),
\[ \Delta(a_{r,j}) \equiv a_{r,j} \otimes a_{r,j} \mod \pi^{i_r} H_{r-1} \otimes H_{r-1}. \]
But \( a_{r,j} \in H_{r-1} \cap KG_j \supseteq H(\mu_j) \). So it suffices that
\[ \Delta(a_{r,j}) \equiv a_{r,j} \otimes a_{r,j} \mod \pi^{i_r} H(\mu_j) \otimes H(\mu_j)), \]
which holds if \( i_r \leq p\mu_j \) and
\[ \text{ord}(u_{r,j} - 1) \geq \mu_j' + \frac{i_r}{p}. \]
Thus we need
\[ i_r \leq p\mu_j \]
for all \( j = 1, \ldots, r - 1 \), and

\[
\text{ord}(u_{r,j} - 1) \geq \mu'_j + \frac{i_r}{p}
\]

for \( j = 1, \ldots, r - 1 \).

To complete the inductive construction, we need an \( r \)th Larson parameter \( \mu_r \) so that

\[
H_r \cap K[G_r] \supseteq H(\mu_r).
\]

We have

\[
\frac{a_{r,1}a_{r,2} \cdot \cdot \cdot a_{r,r-1}\sigma_r - 1}{\pi^{\mu_r}} = \frac{a_{r,1} - 1}{\pi^{\mu_r}} + \sum_{i=2}^{r-1} (a_{r,1}a_{r,2} \cdot \cdot \cdot a_{r,i-1}) \frac{(a_{r,i} - 1)}{\pi^{\mu_r}} + (a_{r,1}a_{r,2} \cdot \cdot \cdot a_{r,r-1}) \frac{(\sigma_r - 1)}{\pi^{\mu_r}}.
\]

Now

\[
\frac{a_{r,1}a_{r,2} \cdot \cdot \cdot a_{r,r-1}\sigma_r - 1}{\pi^{\mu_r}}
\]

is in \( H_r \) if \( \mu_r \leq i_r \). If

\[
\frac{a_{r,k} - 1}{\pi^{\mu_r}} \in H_k
\]

for \( k = 1, \ldots, r - 1 \), then the \( a_{r,k} \) will be units of \( H_r \), and so

\[
\frac{\sigma_r - 1}{\pi^{\mu_r}} \in H_r,
\]

hence

\[
R[\frac{\sigma_r - 1}{\pi^{\mu_r}}] \subseteq H_r \cap K[G_r].
\]

Now since

\[
R[\frac{a_{r,k} - 1}{\pi^{\mu_r}}] = R[\frac{\sigma_k - 1}{\pi^{\nu_k}}]
\]

with \( \text{ord}(u_{r,k} - 1) - \mu_r = \epsilon' - \nu_k \) by [UC05, Proposition 2.1], and

\[
R[\frac{\sigma_k - 1}{\pi^{\nu_k}}] \subseteq H_k
\]

if \( \nu_k \leq \mu_k \), it follows that

\[
\frac{a_{r,k} - 1}{\pi^{\mu_r}} \in H_k
\]

if

\[
\text{ord}(u_{r,k} - 1) - \mu_r \geq \epsilon' - \mu_k
\]

for \( k = 1, \ldots, r - 1 \).

Thus we require \( \mu_r \) to satisfy

\[
\mu_r \leq i_r
\]
and
\[ \text{ord}(u_{r,k} - 1) - \mu_r \geq \epsilon' - \mu_k \]
for \( k = 1, \ldots, r - 1 \).

To summarize:

**Theorem 3.** Given \( i_1, \ldots, i_n \) with \( 0 \leq i_r \leq \epsilon' \) for all \( r \) and \( i_r \leq p \sigma_s \) for \( r > s \), suppose \( U = (u_{r,s}) \) is a lower triangular \( n \times n \) matrix with entries in \( R \) and diagonal entries \( \zeta \). Set \( H_0 = R \) and for \( 1 \leq r \leq n \), define \( H_r \) by \( H_r = H_{r-1}[t_r] \), where
\[ t_r = a_{r,1} a_{r,2} \cdots a_{r,r-1} \sigma_r - 1 \]

Then \( H_r \) is free over \( H_{r-1} \) with basis \( \{ 1, t_r, \ldots, t_{p-1} \} \) and \( H(U) = H_n \) is a Hopf order with Larson parameters \( \mu_1 = i_1, \mu_2, \ldots, \mu_n \) if the following inequalities hold for all \( r, s \) with \( 1 \leq s < r \leq n \):
\[
\begin{align*}
i_r &\leq p \mu_s \\
\text{ord}(u_{r,s} - 1) &\geq \frac{\mu'_s}{p} + i_r \\
\text{ord}(u_{r,s} - 1) &\geq \mu'_s + \frac{i_r}{p} \\
\mu_r &\leq i_r \\
\mu_r &\leq \text{ord}(u_{r,s} - 1) - \mu'_s
\end{align*}
\]

Thus given \( i_1, \ldots, i_n \) satisfying the conditions of Theorem 3, the matrix \( U \) yields a Hopf order if the valuations \( \text{ord}(u_{r,s} - 1) \) of the \( n(n - 1)/2 \) off-diagonal entries of \( U \) and the \( n - 1 \) Larson parameters \( \mu_1, \mu_2, \ldots, \mu_{n-1} \) satisfy a collection of \( 2n^2 - 2n \) linear inequalities.

2. **Duality**

Here is the corresponding result for a potential dual:

**Theorem 4.** Given \( i'_n, \ldots, i'_1 \) with \( 0 \leq i'_s \leq \epsilon' \) for all \( s \) and \( i'_s \leq p \sigma'_s \) for \( s < r \), suppose \( W = (w_{s,r}) \) is an upper triangular \( n \times n \) matrix with entries in \( R \) and diagonal entries \( \zeta \). Set \( J_{n+1} = R \) and for \( 1 \leq s \leq n \), define \( J_s \) by \( J_s = H_{s+1}[q_s] \), where
\[ q_s = \frac{b_{s,n} b_{s,n-1} \cdots b_{s,s+1} \gamma_s - 1}{\pi'_s} \]

Then \( J_s \) is free over \( J_{s+1} \) with basis \( \{ 1, q_s, \ldots, q_{p-1}^s \} \) and \( J(W) = J_1 \) is a Hopf order with Larson parameters \( \delta'_n = i_n, \delta'_{n-1}, \ldots, \delta'_1 \) if the following
inequalities hold for all \( s, r \) with \( 1 \leq s < r \leq n \):

\[
i'_s \leq p\delta'_r
\]

\[
\text{ord}(w_{s,r} - 1) \geq \frac{\delta_r}{p} + i'_s
\]

\[
\text{ord}(w_{s,r} - 1) \geq \delta_r + \frac{i'_s}{p}
\]

\[
\delta'_s \leq i'_s
\]

\[
\delta'_s \leq \text{ord}(w_{s,r} - 1) - \delta_r
\]

We now find conditions on \( U \) and \( W \) so that \( H(U) \) and \( J(W) \) are dual Hopf orders.

Let \( G = \langle \sigma_1 \rangle \times \ldots \times \langle \sigma_n \rangle \) and \( \hat{G} = \langle \gamma_1 \rangle \times \ldots \times \langle \gamma_n \rangle \) with

\[
\langle \sigma_r, \gamma_s \rangle = 1 \text{ if } s \neq r, \quad = \zeta \text{ if } s = r.
\]

Then \( KG \) and \( \hat{K}\hat{G} \) are dual group rings.

For \( x \) and \( y \) units in \( R \) the quantity

\[
G(x, y) = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} x^i \zeta_1^{-ij} y^j,
\]

is the Gauss sum of \( x \) and \( y \) ([GC98]). Note that \( G(x, 1) = 1 \). Also,

\[
G(\zeta_1^k, w) = \frac{1}{p} \sum_{i,j=0}^{p-1} \zeta_1^{ki} \zeta_1^{-ij} w^j = \frac{1}{p} \sum_{j=0}^{p-1} \left( \sum_{i=0}^{p-1} \zeta_1^{(k-j)i} \right) w^j = w^k.
\]

The Gauss sum arises in connection with duality because

\[
G(x, y) = \langle a_x, a_y \rangle
\]

(where \( a_x \in KC_p, a_y \in \hat{K}\hat{C}_p \)), as is easily verified (cf. [GC98]).

Assuming that \( H(U) \), \( J(W) \) are Hopf orders in \( KG \) and \( \hat{K}\hat{G} \), respectively, then by the choice of denominators (valuation parameters), \( J(W) \) will be the dual of \( H(U) \) iff \( \langle H(U), J(W) \rangle \subset R \), by a routine discriminant argument. Since both are Hopf orders, it suffices that the duality map applied to generators maps into \( R \), that is, for all \( r, s \),

\[
\frac{\langle a_{r,1}a_{r,2} \cdots a_{r,r-1}\sigma_r - 1}{\pi^{i_r}}, b_{s,n}b_{s,n-1} \cdots b_{s,s+1}\gamma_s - 1}{\pi^{i_s}} \subset R,
\]

that is,

\[
D_{r,s} - 1 := \frac{\langle a_{r,1}a_{r,2} \cdots a_{r,r-1}\sigma_r, b_{s,n}b_{s,n-1} \cdots b_{s,s+1}\gamma_s \rangle - 1}{\pi^{i_r+i_s}} \subset R.
\]
One sees easily that $D_{r,s} = 1$ if $r < s$ and $D(r,r) = \zeta$, and so $D_{r,s} - 1$ is in $\pi^{i_r+i'_r}R$ if $r \leq s$. For $r > s$,
$$D_{r,s} = \langle a_{r,s}, \gamma_s \rangle \langle a_{r,s+1}, b_{s,s+1} \rangle \cdot \ldots \cdot \langle a_{r,r-1}, b_{s,r-1} \rangle \langle a_{r,s}, \gamma_s \rangle = u_{r,s}G(u_{r,s+1}, w_{s,s+1}) \cdot \ldots \cdot G(u_{r,r-1}, w_{r,r-1})w_{s,r}$$
(c.f. [GC98, Lemma 2.1]).

In order to construct $W$ so that $D_{r,s} \equiv 1 \pmod{\pi^{i_r+i'_r}}$ for $r > s$, we make the assumptions on the entries of $U$:
$$\text{ord}(u_{r,s} - 1) > 0$$
for all $r > s$, and
$$\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) > e' + \text{ord}(u_{r,s} - 1)$$
for all $r > k > s$. The first assumption follows from the inequalities of Theorem 3 provided that $i_r > 0$. The second assumption implies that
$$\text{ord}(G(u_{r,k}, u_{k,s}) - 1) = \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e'$$
by [UC05, Proposition 2.3].

Assuming these inequalities, we define the off-diagonal entries of $W$ inductively.

For all $r$ define $w_{r-1,r}$ by
$$D_{r,r-1} = u_{r,r-1}w_{r-1,r} = 1.$$ Then $D_{r,r-1} - 1 \in \pi^{i_r+i'_r-1}R$ and $\text{ord}(u_{r,r-1} - 1) = \text{ord}(w_{r-1,r} - 1)$.

Define $w_{r-2,r}$ by
$$D_{r,r-2} = u_{r,r-2}G(u_{r-1,r}, w_{r-2,r-1})w_{r-2,r} = 1.$$ This definition makes sense because
$$\text{ord}(u_{r,r-1} - 1) + \text{ord}(w_{r-2,r-1} - 1) - e'$$
$$= \text{ord}(u_{r,r-1} - 1) + \text{ord}(u_{r-1,r-2} - 1) - e'$$
$$> \text{ord}(u_{r,r-2} - 1) > 0$$
and so
$$\text{ord}(G(u_{r,r-1}, w_{r-2,r-1}) - 1) = \text{ord}(u_{r,r-1} - 1) + \text{ord}(w_{r-2,r-1} - 1) - e'$$
$$> \text{ord}(u_{r,r-2} - 1),$$

hence both $G(u_{r,r-1}, w_{r-2,r-1})$ and $u_{r,r-2}$ are units. Also,
$$\text{ord}(u_{r,r-2} - 1) = \text{ord}(w_{r-2,r} - 1)$$
by the isosceles triangle inequality applied to $0 = D_{r,r-2} - 1$ since
$$D_{r,r-2} - 1 = u_{r,r-2}G(u_{r,r-1}, w_{r-2,r-1})(w_{r-2,r} - 1) + u_{r,r-2}(G(u_{r,r-1}, w_{r-2,r-1}) - 1) + (u_{r,r-2} - 1).$$
Assume that \( w_{s,r} \) has been defined for all \( r, s \) with \( r - s = d > 0 \) so that \( \text{ord}(w_{s,r}) = \text{ord}(u_{r,s}) \). We have

\[
  u_{r+1,s} G(u_{r+1,s+1}, w_{s,s+1}) \cdots G(u_{r+1,r}, w_{s,r}) w_{s,r+1} = 1.
\]

Now for all \( k \) with \( r + 1 > k > s \),

\[
  \text{ord}(u_{r+1,k}) + \text{ord}(w_{s,k}) - \epsilon' = \text{ord}(u_{r+1,k}) + \text{ord}(w_{s,k}) - \epsilon' > \text{ord}(w_{s,r+1}) > 0,
\]

hence for \( r + 1 > k > s \), \( G(u_{r+1,k}, w_{s,k}) \) and \( u_{r+1,s} \) are units of \( R \). Therefore we may define \( w_{s,r} \) by

\[
  D_{r+1,s} = u_{r+1,s} G(u_{r+1,s+1}, w_{s,s+1}) \cdots G(u_{r+1,r}, w_{s,r}) w_{s,r+1} = 1.
\]

Since

\[
  \text{ord}(G(u_{r+1,k}, w_{s,k}) - 1) = \text{ord}(u_{r+1,k}) + \text{ord}(w_{s,k}) - \epsilon'
\]

for all \( k \) with \( r + 1 > k > s \), it follows that

\[
  \text{ord}(u_{r+1,s}) = \text{ord}(w_{s,r+1}) - 1
\]

by the isosceles triangle inequality.

In this way we may define the entries of \( W \) so that

\[
  D_{r,s} - 1 = 0
\]

for all \( r > s \), and so we obtain a dual pair of Hopf orders, \( H(U) \) and \( J(W') \), provided that both \( H(U) \) and \( J(W) \) are Hopf orders.

We collect the needed inequalities for both \( H(U) \) and \( J(W) \) to be Hopf orders and duals of each other.

**Theorem 5.** Let \( i_1, \ldots, i_n \) be valuation parameters satisfying \( 0 \leq i_r \leq e' \) for all \( r \) and \( i_r \leq pi_s \) and \( i_{s}' \leq pi_r' \) for all \( r > s \). Let \( U = (u_{r,s}) \) be a lower triangular \( n \times n \) matrix with entries in \( R \) and diagonal entries \( \zeta \). Define the upper triangular matrix \( W = (w_{s,r}) \) by \( w_{s,s} = \zeta \) and for \( r > s \),

\[
  u_{r,s} G(u_{r,s+1}, w_{s,s+1}) \cdots G(u_{r,r-1}, w_{s,r-1}) w_{s,r} = 1.
\]

Then \( H(U) \) and \( J(W) \) are a dual pair of Hopf orders with Larson parameters \( \mu_1 = i_1, \mu_2, \ldots, \mu_n, \delta_n' = i_{n}', \delta_{n-1}', \ldots, \delta_1' \) if the following
inequalities hold for all $1 \leq s < k < r \leq n$:

\begin{align*}
\text{ord}(u_{r,s} - 1) & > 0 \\
i_r & \leq p \mu_s \\
\text{ord}(u_{r,s} - 1) & \geq i_r + \frac{\mu_s}{p} \\
\text{ord}(u_{r,s} - 1) & \geq \frac{i_r}{p} + \mu_s \\
\mu_r & \leq i_r \\
\text{ord}(u_{r,s} - 1) & \geq \mu_r + \mu'_s \\
i'_s & \leq p \delta'_r \\
\text{ord}(u_{r,s} - 1) & \geq \frac{\delta_r}{p} + i'_s \\
\text{ord}(u_{r,s} - 1) & \geq \delta_r + \frac{i'_s}{p} \\
\delta'_s & \leq i'_s \\
\text{ord}(u_{r,s} - 1) & \geq \delta_r + \delta'_s
\end{align*}

\[(3) \quad \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) > e' + \text{ord}(u_{r,s} - 1).\]

When $i_1, \ldots, i_n$ satisfy the strict inequalities $0 < i_1, \ldots, i_n < e'$, then to obtain a dual pair of Hopf orders $H(U)$ and $J(W)$ in $KC^n_p$, $K\hat{C}^n_p$, resp., Theorem 5 requires that the $\binom{n}{2}$ variables $\{\text{ord}(u_{r,s} - 1) | r > s\}$ and the $2(n-1)$ variables $\mu_2, \ldots, \mu_n, \delta_{n-1}, \ldots, \delta_1$ must satisfy a system of

$$2(2n^2 - 2n) + \binom{n}{3} = \frac{n^3 + 21n^2 - 22n}{6}$$

linear inequalities.

For discussion on obtaining solutions to these inequalities, see Remark 11 and Section 4, below.

3. **Truncated exponentials**

Now we consider a variant construction of dual Hopf orders in $KC^n_p$, generalizing that of [GC98]. This construction uses the truncated exponential function,

$$\exp(x) = \sum_{i=0}^{p-1} \frac{x^i}{i!}.$$
which behaves well with respect to duality. (We never explicitly use the untruncated series and so will not require special notation.) Here is the first of two results that facilitate the use of the truncated exponential:

**Lemma 6.** For \( x, y \in \pi^l R \) with \( l \geq 1 \), we have
\[
\exp(x + y) \equiv \exp(x)\exp(y) \pmod{\pi^{ml} R}.
\]

**Proof.** As noted in [GC, Remark 1.1], the proof is a matter of showing the difference of the two sides can be written as a power series of order at least \( p \) with coefficients in \( R \) in which all terms have valuation \( \geq l \).

Define \( \lambda \in R \) by the equation \( \exp(\lambda) = \zeta \). (This can be done explicitly by solving \( \exp(x) = \zeta \) for \( x \) modulo higher and higher powers of \( \pi \).) From the definition of \( \exp(-) \), we get that \( \text{ord}(\lambda) = \text{ord}(\zeta - 1) = e' \).

Then for \( x \in K \), \( \text{ord}(\exp(\lambda x) - 1) = \text{ord}(x) + e' \). Thus \( \exp(\lambda x) \) is a unit of \( R \) iff \( \text{ord}(\lambda x) \geq 1 \), iff \( \text{ord}(x) \geq -e' + 1 \).

Let \( Y = (y_{r,s}) \) be a lower triangular matrix of elements of \( K \) with diagonal entries \( y_{r,r} = 1 \) and \( \text{ord}(y_{r,s}) > -e' \) for all \( r > s \). Then the matrix \( U = (u_{r,s}) \) with \( u_{r,s} = \exp(\lambda y_{r,s}) \) for \( r \geq s \), \( u_{r,s} = 0 \) for \( r < s \), is lower triangular with entries in \( R \) and diagonal entries \( \zeta \). The valuation conditions on \( U \) in order that \( H(U) \) be a Hopf order as in Theorem 3 translate immediately to valuation conditions on \( Y \), since \( \text{ord}(u_{r,s} - 1) = \text{ord}(y_{r,s}) + e' \).

Denote \( H(U) = H^e(Y) \).

Similarly, let \( Z = (z_{r,s}) \) be an upper triangular matrix of elements of \( K \) with diagonal entries \( z_{r,r} = 1 \) and \( \text{ord}(z_{s,r}) > -e' \) for all \( s < r \). Then the matrix \( W = (w_{s,r}) \) with \( w_{s,r} = \exp(\lambda z_{s,r}) \) for \( s \leq r \), \( z_{s,r} = 0 \) for \( s > r \), is upper triangular with entries in \( R \) and diagonal entries \( \zeta \). The valuation conditions on \( W \) in order that \( J(W) \) be a Hopf order as in Theorem 4 translate immediately to valuation conditions on \( Z \), since \( \text{ord}(w_{s,r} - 1) = \text{ord}(z_{s,r}) + e' \). Denote \( J(W) = J^e(Y) \).

The attractiveness of using matrices \( U \) and \( W \) with entries that are truncated exponentials of entries in \( Y, Z \), respectively, is that the dual of \( H^e(Y) \) is \( J^e(Z) \) where the transpose of \( Z \) is the inverse of \( Y \).

Along with Lemma 6 we need for duality the following extension of a result on Gauss sums ([GC98, Theorem 1.4]):

**Theorem 7.** Let \( x, y \) be elements of \( K \) with \( \min\{\text{ord}(\lambda x), \text{ord}(\lambda y)\} = g \) where \( 0 < g \leq e' \). Then
\[
G(\exp(\lambda x), \exp(\lambda y)) - \exp(\lambda xy) \in \pi^{(2p-1)g-(p-1)e'} R.
\]

**Proof.** Let \( P(X, Y) = G(\exp(\lambda X), \exp(\lambda Y)) \). Theorem 1.4 of [GC98] asserts that if \( X, Y \) are indeterminates, then the polynomial \( Q(X, Y) = G(\exp(\lambda X), \exp(\lambda Y)) \)
\[ P(X, Y) - \exp(\lambda XY) \] satisfies
\[ Q(X, Y) = \pi^{pe'} F(X, Y) \]
where \( F(X, Y) \) is a polynomial with coefficients in \( R \). Now
\[
Q(X, Y) = \frac{1}{p} \sum_{i,j=0}^{p-1} \sum_{m=0}^{p-1} \left( \frac{\lambda X}{m!} \right)^i \zeta^{-ij} \left( \sum_{n=0}^{p-1} \frac{\lambda Y}{n!} \right)^j - \sum_{k=0}^{p-1} \frac{(\lambda XY)^k}{k!}
\]
for some coefficients \( s_{m,n} \in R \). Since \( Q(X, Y) \) has coefficients in \( \pi^{pe'} R \), it follows that for all \( m + n = d \geq 0 \),
\[ \text{ord}(s_{m,n}) - \text{ord}(p) + de' \geq pe', \]
hence
\[ \text{ord}(s_{m,n}) \geq (2p - 1 - d)e'. \]
Since \( s_{m,n} \) is in \( R \), we also have
\[ \text{ord}(s_{m,n}) \geq 0 \]
for all \( m, n \).

Suppose \( \min\{\text{ord}(\lambda x), \text{ord}(\lambda y)\} = g \) with \( 0 < g \leq e' \). Then for each \( m, n \) with \( m + n = d \),
\[ \text{ord} \left( \frac{1}{p} s_{m,n}(\lambda x)^m(\lambda y)^n \right) \geq -(p - 1)e' + gd + ((2p - 1) - d)e' \]
for \( d \leq 2p - 1 \), and
\[ \text{ord} \left( \frac{1}{p} s_{m,n}(\lambda x)^m(\lambda y)^n \right) \geq -(p - 1)e' + gd \]
for \( d \geq 2p - 1 \). Thus the term with minimal valuation in \( Q(x, y) \) has valuation \( \geq -(p - 1)e' + (2p - 1)g \), completing the proof. \( \square \)

Now we repeat the construction in section 2.

Assuming that \( H^e(Y) \), \( J^e(Z) \) are Hopf orders in \( KG \) and \( K\hat{G} \), respectively, then by the choice of denominators (valuation parameters), \( J^e(Z) \) will be the dual of \( H^e(Y) \) iff \( \langle H^e(Y), J^e(Z) \rangle \subset R \). Since both are Hopf orders, it suffices that the duality map applied to generators lands in \( R \), that is, for all \( r, s \),
\[
\langle a_{r,1}a_{r,2} \cdots a_{r,r-1} \sigma_r - 1, b_{s,n}b_{s,n-1} \cdots b_{s,s+1} \gamma_s - 1 \rangle \subset R.
\]
As before, set
\[ D_{r,s} = \langle a_{r,1}a_{r,2} \cdots a_{r,r-1} \sigma_r, b_{s,n}b_{s,n-1} \cdots b_{s,s+1} \gamma_s \rangle. \]
We require that
\[ D_{r,s} - 1 \in \pi^{i_r + i_s'} R. \]
One sees easily that \( D_{r,s} = 1 \) if \( r < s \), and \( = \zeta \) if \( r = s \). For \( r > s \),
\[
D_{r,s} = \langle a_{r,s} \gamma_s \rangle \langle a_{r,s+1} b_{s,s+1} \rangle \cdots \langle a_{r,r-1} b_{r-1,r-1} \rangle \langle a_{r,r} \gamma_s \rangle
\]
\[ = P(y_{r,s}, z_{s,s}) P(y_{r,s+1}, z_{s,s+1}) \cdots P(y_{r,r-1}, z_{s,r-1}) P(y_{r,r}, z_{s,r}) \]
(c.f. [GC98, Lemma 2.1]), where \( P(X,Y) = G(\exp(\lambda X), \exp(\lambda Y)) \) as in Lemma 7. Thus we want
\[
P(y_{r,s}, z_{s,s}) P(y_{r,s+1}, z_{s,s+1}) \cdots P(y_{r,r-1}, z_{s,r-1}) P(y_{r,r}, z_{s,r}) \in \pi^{i_r + i_s'} R
\]
for all \( r > s \).
As in the previous section, in order to construct \( Z \) so that \( D_{r,s} \equiv 1 \) (mod \( \pi^{i_r + i_s'} R \)) for \( r > s \), we assume that for all \( r > k > s \), we have
\[
\text{ord}(y_{r,s}) < \text{ord}(y_{r,k}) + \text{ord}(y_{k,s}).
\]
If \( u_{r,s} = \exp(\lambda y_{r,s}) \), then this assumption is equivalent to the assumption
\[
\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) > e' + \text{ord}(u_{r,s} - 1)
\]
made in the construction of Theorem 5.

**Proposition 8.** Suppose \( Z^t = Y^{-1} \), and assume that for all \( r > k > s \),
\[
\text{ord}(y_{r,s}) < \text{ord}(y_{r,k}) + \text{ord}(y_{k,s}).
\]
Then for all \( r > s \), \( \text{ord}(z_{s,r}) = \text{ord}(y_{r,s}). \)

**Proof.** For \( r > s \) we have
\[
y_{r,s} + y_{r,s+1} z_{s,s+1} + \cdots + y_{r,r-1} z_{s,r-1} + z_{s,r} = 0.
\]
Thus \( \text{ord}(y_{r,r-1}) = \text{ord}(z_{s,r-1}) \) for all \( r \). Proceeding by induction, assume \( \text{ord}(z_{t,r}) = \text{ord}(y_{r,t}) \) for \( r - t < r - s \), then
\[
\text{ord}(y_{r,s}) < \text{ord}(y_{r,k}) + \text{ord}(y_{k,s}) = \text{ord}(y_{r,k} z_{s,k})
\]
for \( k = s + 1, \ldots, r - 1 \), so by the isosceles triangle inequality we have
\[
\text{ord}(y_{r,s}) = \text{ord}(z_{s,r}).
\]

Assume the entries of \( Y \) satisfy the hypotheses of Proposition 8. Then by Theorem 7, for all \( r > k > s \),
\[
P(y_{r,k}, z_{s,k}) \equiv \exp(\lambda y_{r,k} z_{s,k}) \pmod{\pi^{(2p-1)g_{r,k,s} - (p-1)e'}}
\]
where
\[
g_{r,k,s} = \min\{\text{ord}(\lambda y_{r,k}), \text{ord}(\lambda z_{s,k})\}
\]
\[
= \min\{\text{ord}(\lambda y_{r,k}), \text{ord}(\lambda y_{k,s})\}
\]
by Proposition 8. So assume

\[(2p - 1)\text{ord}(\lambda_y_{r,k}) - (p - 1)e' \geq i_r + i'_s\]

and

\[(2p - 1)\text{ord}(\lambda_y_{k,s}) - (p - 1)e' \geq i_r + i'_s\]

for all \(r > k > s\). Then modulo \(\pi^{i_r + i'_s} R\)

\[D_{r,s} = P(y_{r,s}, z_{s,s})P(y_{r,s+1}, z_{s,s+1}) \ldots \cdot P(y_{r,r-1}, z_{s,r-1})P(y_{r,r}, z_{s,r})\]

\[\equiv \exp(\lambda y_{r,s}) \exp(\lambda y_{r,s+1}z_{s,s+1}) \ldots \cdot \exp(\lambda y_{r,r-1}z_{s,r-1}) \exp(\lambda z_{s,r}).\]

Then

\[\exp(\lambda y_{r,s}) \exp(\lambda y_{r,s+1}z_{s,s+1}) \ldots \cdot \exp(\lambda y_{r,r-1}z_{s,r-1}) \exp(\lambda z_{s,r})\]

\[\equiv \exp(\lambda (y_{r,s} + y_{r,s+1}z_{s,s+1} + \ldots + y_{r,r-1}z_{s,r-1} + z_{s,r}))\]

\[= \exp(0) = 1 \pmod{\pi^{i_r + i'_s}}\]

and so for all \(r > s\),

\[D_{r,s} - 1 \in \pi^{i_r + i'_s} R,\]

which implies that

\[\langle H_e(Y), J_e(Z) \rangle \subseteq R.\]

We have shown nearly all of

**Theorem 9.** Suppose \(Y = (y_{r,s})\) is an \(n \times n\) lower triangular matrix with entries in \(\pi^{-e'+1} R\) and diagonal entries 1. Suppose for all \(r > k > s\),

\[\text{ord}(y_{r,k}) + \text{ord}(y_{k,s}) > \text{ord}(y_{r,s}).\]

Let \(Z^t = Y^{-1}\), let \(U = (u_{r,s})\) be lower triangular with \(u_{r,s} = \exp(\lambda y_{r,s})\)

for \(r \geq s\), and let \(W = (w_{r,s})\) be upper triangular with \(w_{r,s} = \exp(\lambda z_{s,r})\)

for \(r \geq s\). If \(\{u_{r,s}, \mu_r, \delta'_s\}\) satisfy the inequalities of Theorem 5 together with the inequalities

\[(2p - 1)\text{ord}(\lambda y_{r,k}) \geq (p - 1)e' + i_r + i'_s\]

\[(2p - 1)\text{ord}(\lambda y_{k,s}) \geq (p - 1)e' + i_r + i'_s\]

for all \(r > k > s\), then \(J(W)\) and \(H(U)\) are a dual pair of Hopf orders.

**Proof.** Most of this result follows from Theorem 5 and the discussion just above the statement of Theorem 9. The only remaining observation to make is that the inequality

\[\text{ord}(\lambda y_{r,k}z_{s,k}) \geq \frac{i_r + i'_s}{p}\]
required to apply Lemma 6 is equivalent to
\[ \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e' \geq \frac{i_r + i'_s}{p}, \]
and that follows from
\[ \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e' > \text{ord}(u_{r,s} - 1) \]
and the inequalities
\[ \text{ord}(u_{r,s} - 1) \geq \frac{i_r}{p} + \mu'_s \]
and
\[ \mu_s \leq i_s, \]
for then
\[ \frac{i_r}{p} + \mu'_s \geq \frac{i_r}{p} + i'_s \geq \frac{i_r}{p} + \frac{i'_s}{p}. \]

\[ \square \]

Remark 10. The extra inequalities
\[ (2p - 1)\text{ord}(\lambda y_{r,k}) \geq (p - 1)e' + i_r + i'_s \]
\[ (2p - 1)\text{ord}(\lambda y_{k,s}) \geq (p - 1)e' + i_r + i'_s \]
of Theorem 9 impose a mild extra restriction on the orders of the elements of \( Y \) beyond the inequalities of Theorem 5. From the inequalities of Theorem 5 we have
\[ \text{ord}(u_{r,k} - 1) \geq \frac{i_r}{p} + \mu'_k \]
\[ \text{ord}(u_{k,s} - 1) \geq \delta_k + \frac{i'_s}{p} \]
and so
\[ \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) \geq \frac{i_r}{p} + \mu'_k + \delta_k + \frac{i'_s}{p} \]
\[ \geq e' + \frac{i_r}{p} + \frac{i'_s}{p}. \]

The extra inequalities of Theorem 9 may be restated as
\[ \text{ord}(u_{r,k} - 1) \geq \frac{(p - 1)e'}{2p - 1} + \frac{i_r + i'_s}{2p - 1} \]
\[ \text{ord}(u_{k,s} - 1) \geq \frac{(p - 1)e'}{2p - 1} + \frac{i_r + i'_s}{2p - 1}. \]
which, when added, yield
\[ \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) \geq 2 \frac{(p-1)e'}{2p-1} + 2 \frac{i_r + i'_s}{2p-1} \]
an inequality that follows from the inequality (*) derived from those of Theorem 5, since
\[ e' + \frac{i_r}{p} + \frac{i'_s}{p} > 2 \frac{(p-1)e'}{2p-1} + 2 \frac{i_r + i'_s}{2p-1} \]

Remark 11. Since the inequalities in Theorem 9 are all linear, one can use the simplex algorithm to construct examples. We tried \( n = 5, p = 3, e' = 540, i_1 = 390, i_2 = 150, i_3 = 390, i_4 = 240, i_5 = 150 \), and looked at the minimum of the objective function
\[ S := \sum_{r=2}^{5} \sum_{s=1}^{r-1} \text{ord}(u_{r,s} - 1) \]
subject to the inequalities of Theorem 3. We found that the minimum value of \( S \) increased when we added the inequalities of Theorem 5 as constraints, and further increased when we added the inequalities of Theorem 9 as constraints. Thus the families of examples obtained by Theorem 3 is strictly larger than the family obtained from Theorem 5, which in turn is strictly larger than that obtained from Theorem 9. (The size of \( e' \) and the valuation parameters \( i_1, \ldots, i_5 \) was a result of rescaling so that the minimum would occur at integer values of the variables.) We note the viability of the simplex algorithm mainly as a way of pointing out that even though the number of inequalities looks forbidding (with \( n = 5 \) we have 14 variables and 40 constraints for Theorem 3, 18 and 90 for Theorem 5, and 18 and 110 for Theorem 9), the inequalities are computationally manageable. We’ll see in the next section that they are theoretically manageable as well.

4. Larsony

The first general class of Hopf orders in \( KC_p^m \) was that of Larson orders [La76]. In our construction, \( H(U) \) is a Larson order if \( U = (u_{r,s}) \) with \( u_{r,s} = 1 \) for all \( r > s \), hence
\[ H(U) = R[\frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{\sigma_2 - 1}{\pi^{i_2}}, \ldots, \frac{\sigma_n - 1}{\pi^{i_n}}]. \]
Then the dual is also Larson, as can be seen either directly or from Theorem 5.
We wish to show that there are examples obtainable from Theorems 5 and 9 that are not Larson. To do so, we first note the following criterion for a triangular Hopf order of rank $p^2$ to be Larson.

**Proposition 12.** For $j > 0$, the Hopf order $H = R[\sigma \tau^{-1}, \sigma^2 \tau^{-1}]$ is a Larson order iff $\text{ord}(v^{-1}) \geq i' + j$.

This criterion is more or less well-known, but for convenience we sketch a proof:

**Proof.** If $H$ is Larson, $H = R[\sigma \tau^{-1}, \sigma^2 \tau^{-1}]$ for some $\nu$, then $\nu = j$ by a discriminant argument. Also, if $R[\sigma \tau^{-1}, \sigma^2 \tau^{-1}] \subseteq H$, then we have equality by a discriminant argument. We have

$$\text{ord}(v^{-1}) \geq i' + j$$

iff

$$R[a_v \frac{\tau-1}{\pi j}] \subseteq R[\frac{\sigma-1}{\pi i}]$$

by [UC05, Prop. 2.1] (another discriminant argument). Now

$$\frac{a_v \tau - 1}{\pi j} = a_v (\frac{\tau - 1}{\pi j}) + a_v \frac{\tau - 1}{\pi j},$$

so if $\text{ord}(v - 1) \geq i' + j$, then $a_v \frac{\tau - 1}{\pi j}$ is in $R[\frac{\sigma - 1}{\pi i}]$, hence $a_v$ is a unit of $R[\frac{\sigma - 1}{\pi i}]$ and so $\frac{\tau - 1}{\pi j}$ is in $H$. Hence $H$ is Larson.

Conversely, if $H = R[\frac{\sigma - 1}{\pi i}, \frac{\sigma^2 - 1}{\pi j}]$, then

$$\tau^{-1}(a_v \frac{\tau - 1}{\pi j}) - \tau^{-1}(\frac{\tau - 1}{\pi j}) = a_v \frac{\tau - 1}{\pi j},$$

and so $a_v \frac{\tau - 1}{\pi j}$ is in $H \cap K[\sigma] = R[\frac{\sigma - 1}{\pi i}]$, and so by [Ch00, (31.8)] we have $\text{ord}(v - 1) \geq i' + j$. $\square$

**Petit Larsons.** As a first example, we show that there are Hopf orders constructible by Theorem 3 that are not Larson orders. Recall that $G = G_1 \times G_2 \times \ldots \times G_n$ with $G_i = \langle \sigma_i \rangle$.

**Proposition 13.** Let $n = 3$. If $i_3, i'_3 \geq 2$, or $\frac{i_3}{p} + 1 \leq i_2$ and $i_3 \geq 2, i'_3 \geq 1$, then there exist Hopf orders $H$ in $KC_n^3$ constructed as in Theorem 3 that are not Larson orders.

**Proof.** If $H$ is Larson, then

$$H_2 = H \cap K(C_1 \times C_2) = R[\frac{\sigma_1 - 1}{\pi i_1}, \frac{\sigma_2 - 1}{\pi i_2}]$$

is Larson, and

$$H_3 \cong R[\frac{\sigma_2 - 1}{\pi i_2}, \frac{\sigma_3 - 1}{\pi i_3}]$$

We have
is Larson. Hence to show $H$ is not Larson, it suffices to find $u_{2,1}$ with \( \text{ord}(u_{2,1} - 1) < i'_1 + i_2 \) or $u_{3,2}$ with \( \text{ord}(u_{3,2} - 1) < i'_2 + i_3 \), so that $H_2$ or $H_3$ is not Larson.

With $n = 3$, the only Larson parameter is $\mu_2$ such that $\frac{i_2}{p} \leq \mu_2 \leq i_2$.

Set
\[
\begin{align*}
\text{ord}(u_{2,1} - 1) &= i'_1 + i_2 - \varphi_{1,2}, \\
\text{ord}(u_{3,2} - 1) &= i'_2 + i_3 - \varphi_{2,3}.
\end{align*}
\]

We wish to show that either $\varphi_{1,2} > 0$ or $\varphi_{2,3} > 0$.

Since
\[
\begin{align*}
\text{ord}(u_{2,1} - 1) &\geq i'_1 + \frac{i_2}{p}, \\
\text{ord}(u_{3,2} - 1) &\geq i'_2 + \frac{i_3}{p},
\end{align*}
\]

we have
\[
\begin{align*}
\varphi_{1,2} &\leq \frac{p - 1}{p} i_2, \\
\varphi_{2,3} &\leq \frac{p - 1}{p} i_3.
\end{align*}
\]

The three inequalities of Theorem 3 involving \( \text{ord}(u_{r,r-1} - 1) \) and $\mu_2$ yield the inequalities
\[
\begin{align*}
i_2 - \mu_2 &\leq (p - 1)i'_2 - p\varphi_{2,3} \\
i_2 - \mu_2 &\leq (\frac{p - 1}{p})i_3 - \varphi_{2,3} \\
i_2 - \mu_2 &\geq \varphi_{1,2}
\end{align*}
\]

Thus $\varphi_{1,2}, \varphi_{2,3}$ are constrained by the additional inequalities
\[
\begin{align*}
(p - 1)i'_2 &\geq p\varphi_{2,3} + \varphi_{1,2} \\
(\frac{p - 1}{p})i_3 &\geq \varphi_{2,3} + \varphi_{1,2}.
\end{align*}
\]

If $\varphi_{1,2} = 0$ we may set $\mu_2 = i_2$, in which case to allow $\varphi_{2,3} \geq 1$ we need
\[
\begin{align*}
1 &\leq (\frac{p - 1}{p})i_3 \\
1 &\leq (\frac{p - 1}{p})i'_2,
\end{align*}
\]

hence we need $i_3, i'_2 \geq 2$. If $\varphi_{2,3} = 0$, then for $\varphi_{1,2} \geq 1$ it suffices that $\mu_2 \leq i_2 - 1$, hence $\frac{i_2}{p} + 1 \leq i_2$ and
\[
\begin{align*}
1 &\leq (\frac{p - 1}{p})i_3 \\
1 &\leq (p - 1)i'_2,
\end{align*}
\]

hence we need $\frac{i_2}{p} + 1 \leq i_2$ and $i'_2 \geq 1, i_3 \geq 2$. \( \Box \)
In general, a Hopf order $H$ in $KG$ is realizable if there exists a Galois extension $L$ of $K$ with Galois group $G$ so that the valuation ring $S$ of $L$ is $H$-Galois over $R$. Given a rank $p^2$ Hopf order $H = R[\frac{a_i-1}{\pi^i}, \frac{a_j-1}{\pi^j}]$, the unit parameter $v$ must satisfy

$$\operatorname{ord}(v - 1) \geq i' + j.$$

If $\operatorname{ord}(v - 1) \geq i' + j$, then $H$ is a Larson order. If $\operatorname{ord}(v - 1) = i' + \frac{j}{p}$, then $H$ is realizable. Thus realizable Hopf orders and Larson orders are at opposite ends of the range of Hopf orders, and a Larson order is realizable iff $j = 0$. We've shown that there are many Hopf orders constructed by Theorem 3 that are not Larson orders; on the other hand, we have

**Proposition 14.** If $n = 3$ and $0 \leq i_1, i_2, i_3 < e'$, then no Hopf order $H_3$ constructed by Theorem 3 is realizable.

*Proof.* Since $i_1, i_2, i_3 < e'$, $R[\frac{a_i-1}{\pi^i}]$ is a local ring (since modulo $\pi$, $\alpha_i = \frac{a_i-1}{\pi^i}$ is nilpotent), hence by [Ch00, (29.1)], $H_3$ is a local ring. If there exists a totally ramified Galois extension $L$ of $K$ with Galois group $G = G_1 \times G_2 \times G_3$ such that the valuation ring $S$ of $L$ is $H_3$-Galois, then setting $L_2 = L^{G_3}$, $L_1 = L^{G_3 \times G_2}$ and $S_2 = S^{G_1}$, $S_1 = S^{G_3 \times G_2}$, then $S_1 \otimes H_2$-Galois over $S_1$ and $S_2$ is $H_3$-Galois over $R$ by [Ch00, (28.1), (29.1) and (14.7)]. (Note: (14.7), (1) should read "$S$ is a tame $H$-extension of $R$.")

Now

$$\overline{H}_3 \cong R[\frac{\sigma_2 - 1}{\pi^{t_2}}, \frac{a_32\sigma_3 - 1}{\pi^{t_3}}]$$

and

$$H_2 \cong R[\frac{\sigma_1 - 1}{\pi^{t_1}}, \frac{a_21\sigma_2 - 1}{\pi^{t_2}}],$$

hence

$$S \otimes H_2 \cong S_1[\frac{\sigma_1 - 1}{\pi^{t_1}}, \frac{a_21\sigma_2 - 1}{\pi^{t_2}}],$$

where $\pi_1$ is a parameter for $S_1$. If $S$ is $S_1 \otimes H_2$-Galois, then by [By02], $\operatorname{ord}_{L_1}(u_{2,1} - 1) = pi_1 + i_2$, hence $\operatorname{ord}(u_{2,1} - 1) = i_1 + \frac{i_2}{p}$.

If $S_2$ is $\overline{H}_3$-Galois, then $\operatorname{ord}(u_{3,2} - 1) = i'_2 + \frac{i_2}{p}$. But then, in the proof of Proposition 13, we have

$$\varphi_{2,3} = (\frac{p-1}{p})i_3,$$

so $\varphi_{1,2} = 0$, hence $\operatorname{ord}(u_{2,1} - 1) = i'_1 + i_2$. Thus $S$ cannot be $S_1 \otimes H_2$-Galois, and so $S$ is not $H_3$-Galois. Hence $H_3$ is not realizable. $\square$
**Grand Larsony.** Now we show that if \( e' \) is sufficiently large, then for suitable choices of the valuation parameters \( i_1, \ldots, i_n \), we can find pairs \( H(U), J(W) \) of dual Hopf orders of rank \( p^n \) as constructed by Theorem 5 or Theorem 9 such that none of the rank \( p^2 \) subquotients naturally associated with \( H(U), J(W) \) are Larson. We begin by identifying the rank \( p^2 \) subquotients we will examine.

Recalling from Theorems 3 and 4 that \( H(U) = H_r, J(W) = J_s \) where \( H_r = R[t_1, \ldots, t_r], J_s = R[q_n, \ldots, q_s] \) with
\[
t_r = \frac{a_{r,1}a_{r,2} \cdots a_{r,r-1}\sigma_r - 1}{\pi^{i_r}},
\]
\[
q_r = \frac{b_{s,n}b_{s,n-1} \cdots b_{s,s+1}\gamma_s - 1}{\pi^{i_s}}.
\]
For \( 1 \leq k < r \) let \( \overline{H}_{k,r} \) be the image of \( H_r \) under the map from \( K(G_1 \times \ldots \times G_r) \) to \( K(G_k \times G_r) \) given by sending \( \sigma_\mu \) to 1 for \( \mu < r, \mu \neq k \). Then
\[
t_1, \ldots, t_{k-1} \mapsto 0
\]
\[
t_k \mapsto \frac{\overline{\sigma}_k - 1}{\pi^{i_k}}
\]
\[
t_r \mapsto \frac{a_{r,k}\overline{\sigma}_r - 1}{\pi^{i_r}}
\]
and for \( k < l < r \),
\[
t_l \mapsto \frac{a_{l,k} - 1}{\pi^{i_l}},
\]
and so
\[
\overline{H}_{k,r} \cong R[\frac{\overline{\sigma}_k - 1}{\pi^{i_k}}, \frac{a_{k+1,k} - 1}{\pi^{i_{k+1}}}, \ldots, \frac{a_{r-1,k} - 1}{\pi^{i_{r-1}}}, \frac{a_{r,k}\overline{\sigma}_r - 1}{\pi^{i_r}}].
\]
Similarly, for \( n \geq l > s \) we have \( \overline{J}_{l,s} \) obtained from \( J_s \) by sending \( \gamma_\mu \) to 1 for \( \nu \neq l, s \), and so
\[
\overline{J}_{l,s} \cong R[\frac{\overline{\gamma}_l - 1}{\pi^{i_l}}, \frac{b_{l,l-1} - 1}{\pi^{i_{l-1}}}, \ldots, \frac{b_{l,s}\gamma_s - 1}{\pi^{i_s}}].
\]
If
\[
v_{l,k} = \text{ord}(u_{l,k} - 1) < i_l + e',
\]
then by [UC05, Corollary 2.2],
\[
\overline{H}_{k,r} \cong R[\frac{\overline{\sigma}_k - 1}{\pi^{\alpha_{k,r}}}, \frac{a_{r,k}\overline{\sigma}_r - 1}{\pi^{i_r}}]
\]
with
\[
\alpha_{k,r} = \text{max}\{i_k, i_{k+1} + v_{k+1,k}', \ldots, i_{r-1} + v_{r-1,k}'\},
\]
that is,
\[
\alpha_{k,r}' = \text{min}\{i_k', v_{k+1,k} - i_{k+1}, \ldots, v_{r-1,k} - i_{r-1}\}.\]
Similarly,
\[ J_{l,s} \cong R[\frac{\gamma l - 1}{\pi^2}, \frac{b_l \gamma s - 1}{\pi^2}] \]
with
\[ \beta_{l,s}' = \max\{i'_l, i'_{l-1} + v'_{l-1,l}, \ldots, i'_{s+1} + v'_{l,s+1}\}, \]
that is,
\[ \beta_{l,s} = \min\{i_l, v_{l-1,l} - i'_{l-1}, \ldots, v_{l,s+1} - i'_{s+1}\}. \]

We'll say that \( H(U), J(W) \) are Larsonless if none of the rank \( p^2 \) subquotients \( \mathcal{H}_{k,r}, \mathcal{J}_{l,s} \) of \( H(U), J(W) \) are Larson orders. Then \( H(U) \) is Larsonless iff
\[ v_{r,k} < i_r + \alpha'_{k,r} \]
for all \( 1 \leq k < r \leq n \), and \( J(W) \) is Larsonless if
\[ v_{s,l} < i'_s + \beta_{l,s}' \]
for all \( n \geq s > l \geq 1 \). In particular, for \( k = r - 1, l = s + 1 \) we require
\[ v_{r,r-1} < i_{r-1} + i'_r \]
for \( H \), and
\[ v_{s+1,s} < i_s + i'_{s+1} \]
for \( J \) (which is the same condition as for \( H \)).

To construct Larsonless examples, t is convenient to introduce some new notation and reexamine the inequalities of Theorem 5.

Let \( v_{r,s} = \text{ord}(u_{r,s} - 1) \). For all \( r > s \) and all \( r > k > s \), Theorem 5 requires the following inequalities on the \( i_j, \mu_j \) and \( \delta_j' \):

(1) \[ i_r \leq p \mu_s, \quad \mu_r \leq i_r, \quad i'_s \leq p \delta'_r, \quad \delta'_s \leq i'_s; \]
also the following inequalities involving the \( v_{r,s} \):
\[ v_{r,s} \geq 0 \]
\[ \geq i_r + \frac{\mu_s'}{p} \]
\[ \geq \frac{i_r}{p} + \mu'_s \]
\[ \geq \mu_r + \mu'_s \]
\[ \geq \frac{\delta_r}{p} + i'_s \]
\[ \geq \frac{\delta_r}{p} + \frac{i'_s}{p} \]
\[ \geq \frac{\delta_r}{p} + \frac{i'_s}{p} \]
(2)
and

(3)  \( v_{r,k} + v_{k,s} > e' + v_{r,s} \).

(omit inequalities involving \( \delta'_1 \) and \( \mu_n \); note also, \( \delta'_n = i'_n, \mu_1 = i_1 \)).

Set

\[ \lambda_s = i_s - \mu_s, \quad \theta_s = i'_s - \delta'_s. \]

for \( r > s \). Then inequalities (1) require that \( \lambda_s \geq 0 \) and \( \theta_s \geq 0 \) for \( 2 \leq s \leq n - 1 \).

Also, set

\[ v_{r,s} = i_r + i'_s - \varphi_{r,s}. \]

We rewrite the inequalities (2) of Theorem 5 in terms of the \( \varphi_{r,s}, \lambda_s \) and \( \theta_s \) as follows:

\begin{align*}
\lambda_s & \leq p((\frac{p-1}{p})i'_s - \varphi_{r,s}) \text{ for all } r > s \\
\lambda_s & \leq (\frac{p-1}{p})i_r - \varphi_{r,s} \text{ for all } r > s \\
\theta_r & \leq p((\frac{p-1}{p})i'_r - \varphi_{r,s}) \text{ for all } r > s \\
\theta_r & \leq (\frac{p-1}{p})i'_s - \varphi_{r,s} \text{ for all } r > s
\end{align*}

(2a)

and

\begin{align*}
\lambda_r & \geq \lambda_s + \varphi_{r,s} \text{ for all } r > s \\
\theta_s & \geq \theta_r + \varphi_{r,s} \text{ for all } r > s.
\end{align*}

(2b)

Inequalities (3) become

\[ i_r + i'_k - \varphi_{r,k} + i_k + i'_s - \varphi_{k,s} > e' + i_r + i'_s - \varphi_{r,s}, \]

which is the same as

(3a)  \( \varphi_{r,k} + \varphi_{k,s} < \varphi_{r,s} \)

for \( r > k > s \). We wish to satisfy these inequalities with \( \lambda_s > 0, \theta_s > 0 \) and \( \varphi_{r,r-1} > 0 \).

**Example 15.** Let \( p \geq 3 \), let

\[ \frac{e'}{3} < i_1, \ldots, i_n < \frac{2e'}{3}; \]

then

\[ \frac{e'}{3} < i'_1, \ldots, i'_n < \frac{2e'}{3}. \]

Choose

\[ \varphi_{r,s} = 2(r - s) - 1 \]
for \( r > s \), a minimal solution to (3a) with all \( \varphi_{r,s} > 0 \). Let \( \lambda_2 = \lambda, \theta_{n-1} = \theta \), and let

\[
\lambda_r = \lambda_2 + \varphi_{r,2} = \lambda + (2r - 5);
\theta_s = \theta_{n-1} + \varphi_{n-1,s} = \theta + (2(n - s) - 3).
\]

Then \( \lambda_r, \theta_s \) satisfy inequalities (2b) for all \( \lambda, \theta \).

Suppose
\[
2n - 3 < \frac{e'}{18}.
\]
and
\[
\lambda_r, \theta_s \leq \frac{e'}{9}.
\]
Then inequalities (2a) follow because for all \( r, s \), \( \varphi_{r,s} \leq 2n - 3 < \frac{e'}{18} \), hence
\[
\frac{p-1}{p} i_r - \varphi_{r,s} \geq \frac{2e'}{3} - \frac{e'}{18} = \frac{e'}{6}
\]
and
\[
\frac{p-1}{p} i'_s - \varphi_{r,s} \geq \frac{e'}{6}.
\]
Since \( \mu_s = i_s - \lambda_s, \delta'_r = i'_r - \theta_r \), the conditions \( i_r \leq p\mu_s, i'_s \leq p\delta'_r \) become
\[
i_r \leq p(i_s - \lambda_s), i'_s \leq p(i'_r - \theta_r)
\]
for \( r > s \). In the worst case we have
\[
i_r = \frac{2e'}{3}, \quad i_s = \frac{e'}{3}, \quad \lambda_s = \theta_r = \frac{e'}{9},
\]
hence
\[
\frac{2e'}{3} \leq p\left(\frac{e'}{3} - \frac{e'}{9}\right) = \frac{2pe'}{9},
\]
which holds for \( p \geq 3 \).

So we may choose any \( \lambda, \theta \) so that
\[
\lambda + (2n - 5) \leq \frac{e'}{9},
\]
\[
\theta + (2n - 5) \leq \frac{e'}{9}.
\]
Since
\[
\frac{e'}{9} - (2n - 5) \geq \frac{e'}{9} - \frac{e'}{18} = \frac{e'}{18},
\]
we can choose any \( \lambda, \theta \) with
\[
0 < \lambda, \theta \leq \frac{e'}{18}.
\]
To satisfy Theorem 9, we need to satisfy the additional inequalities

\[
\begin{align*}
  i_r + i'_k - \varphi_{r,k} &\geq \frac{p - 1}{2p - 1} e' + \frac{i_r + i'_s}{2p - 1}, \\
  i_k + i'_s - \varphi_{k,s} &\geq \frac{p - 1}{2p - 1} e' + \frac{i_r + i'_s}{2p - 1}
\end{align*}
\]

(4)

for all \( r > k > s \). But if \( e' \leq i_r, i'_s \) for all \( r, s \) and \( \varphi_{k,s} = 2(k-s) - 1 < \frac{e'}{18} \), then the left sides of equations (4) are

\[
\geq \frac{2e'}{3} - \frac{e'}{18} = \frac{11e'}{18}
\]

while the right side of equations (4) is

\[
\leq \frac{1}{2} e' + \frac{12e'}{5} - \frac{17}{30} e' < \frac{11}{18} e'.
\]

Hence these examples satisfy the conditions of Theorem 9 as well.

Now \( \varphi_{r,r-1} > 0 \) for all \( r > 1 \), and so \( \overline{H}_{r,r-1} \) is not Larson. In general, \( \overline{H}_{r,s} \) is not Larson if

\[
v_{r,s} < i_r + \alpha'_{s,r},
\]

that is,

\[
v_{r,s} - i_r < \min\{i'_s, v_{s+1,s} - i_{s+1}, \ldots, v_{r-1,s} - i_{r-1}\}.
\]

Thus we need

\[
v_{r,s} < i_r + i'_s,
\]

equivalent to \( \varphi_{r,s} > 0 \), and for all \( r > k > s \),

\[
v_{r,s} - i_r < v_{k,s} - i_s.
\]

But

\[
v_{k,s} - i_k = i'_s - \varphi_{k,s},
\]

so this becomes

\[
i'_s - \varphi_{r,s} < i'_s - \varphi_{k,s}
\]

or

\[
\varphi_{k,s} < \varphi_{r,s}
\]

for \( r > k > s \), which follows from (3a). Similarly, for \( r > s \), \( \overline{J}_{s,r} \) is not Larson if

\[
v_{r,s} < i'_s + \beta_{r,s},
\]

which is equivalent to \( v_{r,s} < i_r + i'_s \), that is, \( \varphi_{r,s} > 0 \) for \( r > s \), and

\[
v_{r,s} - i'_s < v_{r,k} - i'_k
\]

for all \( r > k > s \). But again this last inequality is equivalent to

\[
i_r - \varphi_{r,s} < i_r - \varphi_{r,k},
\]
which is equivalent to $\varphi_{r,k} < \varphi_{r,s}$ for $r > k > s$, and this again follows from (3a). Thus these examples are all Larsonless.

Note that the lower bound on $i_1, \ldots, i_n$ was helpful: if we set $\lambda = 1$, then $\lambda_{n-1} = 2n - 4$ and the inequality

$$\lambda_{n-1} \leq \left(\frac{p-1}{p}\right)i_n - \varphi_{n,n-1}$$

becomes

$$2n - 5 \leq \left(\frac{p-1}{p}\right)i_n,$$

which cannot hold if $i_n$ is too small.

We note that for all of the examples in [GC98], the rank $p^2$ images $\overline{H}_{r,r+1}$ are all Larson, since in those examples,

$$i_r + i'_s \leq \epsilon' \leq \text{ord}(u_{r,s} - 1).$$

The results in this paper extend and refine results from [Sm97].

References


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