TOEPLITZ OPERATORS ON THE FOCK SPACE

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ABSTRACT. We study Toeplitz operators on the Fock space with positive measures as symbols. Results obtained include characterizations of Fock-Carleson measures, bounded Toeplitz operators, compact Toeplitz operators, and Toeplitz operators in the Schatten p-classes.

1. INTRODUCTION

Throughout the paper we fix a positive parameter α and consider the Gaussian measure

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2}\, dA(z)$$

on the complex plane $\mathbb{C}$, where $dA(z) = dxdy$ is the ordinary area measure.

For any $p > 0$, the Fock space F^p_α consists of all entire functions f with the property that the function $f(z)e^{-\alpha|z|^2/2}$ is in $L^p(\mathbb{C}, dA)$. For $f \in F^p_\alpha$, we write

$$\|f\|_p = \left[\frac{p\alpha}{2\pi}\int_{\mathbb{C}} \left|f(z)e^{-\alpha|z|^2/2}\right|^p\, dA(z)\right]^{1/p}.$$

In particular, F^2_α is just the subspace (with inherited norm and inner product) of all entire functions in $L^2(\mathbb{C}, d\lambda_\alpha)$. The inner product in $L^2(\mathbb{C}, d\lambda_\alpha)$ (and hence in F^2_α as well) will be denoted by

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}\, d\lambda_\alpha(z).$$

It is well known that the orthogonal projection

$$P : L^2(\mathbb{C}, d\lambda_\alpha) \to F^2_\alpha$$

is an integral operator, namely,

$$Pf(z) = \int_{\mathbb{C}} K(z, w)f(w)\, d\lambda_\alpha(w),$$

where $K(z, w) = e^{\alpha z\overline{w}}$ is the reproducing kernel of the Hilbert space F^2_α. See [1, 2, 3, 7].

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Given $\varphi \in L^\infty(\mathbb{C})$, we can define a linear operator $T_\varphi : F^2_\alpha \to F^2_\alpha$ by

$$T_\varphi(f) = P(\varphi f), \qquad f \in F^2_\alpha.$$

We call T_φ the Toeplitz operator on F^2_α with symbol φ. It is clear that T_φ is bounded with $\|T_\varphi\| \le \|\varphi\|_\infty$. It is also clear that for any complex numbers a and b and for any bounded functions φ and ψ we have $T_{\overline{\varphi}} = T^*_\varphi$,

$$T_{a\varphi+b\psi} = aT_\varphi + bT_\psi,$$

and $T_\varphi \ge 0$ whenever $\varphi \ge 0$.

By the integral representation for the projection operator P, we can write

$$T_\varphi(f)(z) = \int_{\mathbb{C}} K(z,w)f(w)\varphi(w)\,d\lambda_\alpha(w).$$

This motivates us to define Toeplitz operators on F^2_α with much more general symbols. More specifically, if μ is a Borel measure on $\mathbb{C}$, we define the Toeplitz operator T_μ as follows:

$$T_\mu(f)(z) = \int_{\mathbb{C}} K(z,w)f(w)e^{-\alpha|w|^2}\,d\mu(w), \qquad z \in \mathbb{C}.$$

Note that T_μ is very loosely defined here, because it is not clear when the integrals above will converge, even if the measure μ is finite, as the kernel function $K(z,w)$ is unbounded for any fixed $z \ne 0$.

Suppose that μ is a Borel measure that satisfies the condition

$$\int_{\mathbb{C}} |K(z,w)|e^{-\alpha|w|^2}\,d|\mu|(w) < \infty \tag{1}$$

for all $z \in \mathbb{C}$. Then because of the exponential form of the kernel function, it is clear that condition (1) is equivalent to

$$\int_{\mathbb{C}} |K(z,w)|^2e^{-\alpha|w|^2}\,d|\mu|(w) < \infty \tag{2}$$

for all $z \in \mathbb{C}$.

If μ satisfies condition (1), then the Toeplitz operator T_μ is well-defined on a dense subset of F^2_α. In fact, if

$$f(w) = \sum_{k=1}^N c_k K(w, z_k)$$

is any finite linear combination of kernel functions in F^2_α, then it follows from (2) and the Cauchy-Schwarz inequality that $T_\mu(f)$ is well defined. It is easy to check that the set of all finite linear combinations of kernel functions is dense in F^2_α.

All measures used in the paper will be assumed to satisfy condition (1), so that all Toeplitz operators are well-defined. It also follows from condition (1) that we can define a function $\widetilde{\mu}$ on $\mathbb{C}$ as follows:

$$\widetilde{\mu}(z) = \int_{\mathbb{C}} |k_z(w)|^2 e^{-\alpha|w|^2}\, d\mu(w), \qquad z \in \mathbb{C}, \tag{3}$$

where

$$k_z(w) = K(w,z)/\sqrt{K(z,z)} = e^{\alpha w\bar{z} - \frac{\alpha}{2}|z|^2}$$

are called normalized reproducing kernels in F^2_α. Alternatively, we can write

$$\widetilde{\mu}(z) = \int_{\mathbb{C}} \frac{|K(z,w)|^2}{K(z,z)K(w,w)}\, d\mu(w) = \int_{\mathbb{C}} e^{-\alpha|z-w|^2}\, d\mu(w). \tag{4}$$

We call $\widetilde{\mu}$ the Berezin transform of μ. If the Toeplitz operator T_μ happens to be a bounded operator on F^2_α, then

$$\widetilde{\mu}(z) = \langle T_\mu k_z, k_z\rangle, \qquad z \in \mathbb{C}. \tag{5}$$

In the special case when

$$d\mu(z) = \frac{\alpha}{\pi}\varphi(z)\, dA(z),$$

we have $T_\mu = T_\varphi$ and we will write $\widetilde{\varphi}$ for $\widetilde{\mu}$. We also call $\widetilde{\varphi}$ the Berezin transform of φ in this case. Since

$$\widetilde{\varphi}(z) = \frac{\alpha}{\pi}\int_{\mathbb{C}} \varphi(w) e^{-\alpha|z-w|^2}\, dA(w),$$

we have that $\widetilde{\varphi}$ is simply φ convolved with the heat kernel

$$\Phi(w,t) = \frac{1}{4\pi t} e^{-\frac{|w|^2}{4t}}$$

at time $t = \frac{1}{4\alpha}$. Thus, the function $\widetilde{\varphi}$ is sometimes called the heat transform of φ. See [9] for example.

Given any two non-zero complex numbers z and w not on a same ray emanating from the origin, the set of points $nz + mw$, where n and m are arbitrary integers, is called the lattice generated by z and w. For any $r > 0$ we let $\{a_n\}$ denote any fixed arrangement into a sequence of the lattice generated by the points r and ri. The sequence $\{a_n\}$ is reserved for this lattice throughout the paper.

For any $z \in \mathbb{C}$ and $r > 0$ we use

$$B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$$

to denote the Euclidean disc centered at z with radius r. If μ is a locally finite Borel measure, the function $z \mapsto \mu(B(z,r))$ is the constant πr^2 times the average of μ over $B(z,r)$. Thus, we will call $\mu(B(z,r))$ an averaging function of μ.

We summarize the main results of the paper as Theorems A, B, and C below.

Theorem A. *Suppose $\mu \geq 0$ and $r > 0$. Then the following conditions are equivalent.*

(a) *The Toeplitz operator T_μ is bounded on F^2_α.*
(b) *The heat transform $\widetilde{\mu}$ is bounded on $\mathbb{C}$.*
(c) *The averaging function $\mu(B(z,r))$ is bounded on $\mathbb{C}$.*
(d) *The averaging sequence $\{\mu(B(a_n,r))\}$ is bounded.*

Theorem B. *Suppose $\mu \geq 0$ and $r > 0$. Then the following conditions are equivalent.*

(a) *T_μ is compact on F^2_α.*
(b) *$\widetilde{\mu}(z) \to 0$ as $z \to \infty$.*
(c) *$\mu(B(z,r)) \to 0$ as $z \to \infty$.*
(d) *$\mu(B(a_n,r)) \to 0$ as $n \to \infty$.*

Theorem C. *Suppose $\mu \geq 0$, $r > 0$, and $p > 0$. Then the following conditions are equivalent.*

(a) *The Toeplitz operator T_μ belongs to the Schatten class S_p.*
(b) *The heat transform $\widetilde{\mu}$ belongs to $L^p(\mathbb{C}, dA)$.*
(c) *The averaging function $\mu(B(z,r))$ belongs to $L^p(\mathbb{C}, dA)$.*
(d) *The averaging sequence $\{\mu(B(a_n,r))\}$ belongs to l^p.*

Theorems A and B would have been anticipated by anyone who is familiar with the theory of Toeplitz operators on Bergman spaces, as analogous results in the Bergman space setting are well-known. See [17] for example. However, the proof of Theorems A and B requires us to define T_μ and $\widetilde{\mu}$ in a way that is different from the standard approaches widely adopted for the Bergman space theory.

Theorem C differs from the Bergman space theory in one significant way. More specifically, the characterization of Schatten class Toeplitz operators on Bergman spaces in terms of the Berezin transform is not valid for the full range $0 < p < \infty$; there exists a so-called cut-off point. See [18]. But Theorem C here shows that there is no cut-off for the Fock space theory. This difference also shows up in the study of Hankel operators. See [6, 13, 14].

Theorems A, B, and C can easily be extended to the Fock space in $\mathbb{C}^n$. We chose to develop the theory in dimension one mainly for the sake of clarity. The extension to higher dimensions requires no new ideas. There is only some additional combinatorial complexity in handling the lattices in $\mathbb{C}^n$.

2. FOCK-CARLESON MEASURES

In this section we characterize Carleson-type measures for Fock spaces. The following pointwise estimate for functions in F^p_α will be needed on several occasions later.

Lemma 1. *For any $r > 0$ and $p > 0$, there exists a positive constant C such that*

$$|f(z)e^{-\alpha|z|^2/2}|^p \le C \int_{B(z,r)} |f(w)e^{-\alpha|w|^2/2}|^p \, dA(w)$$

for all $z \in \mathbb{C}$.

Proof. We apply a change of variables to the integral

$$I(z) = \int_{B(z,r)} \left|f(w)e^{-\alpha|w|^2/2}\right|^p \, dA(w)$$

to obtain

$$\begin{aligned} I(z) &= \int_{|u|<r} |f(z+u)|^p e^{-\frac{p\alpha}{2}|z+u|^2} \, dA(u) \\ &= e^{-\frac{p\alpha}{2}|z|^2} \int_{|u|<r} \left|f(z+u)e^{-\alpha\bar{z}u}\right|^p e^{-\frac{p\alpha}{2}|u|^2} \, dA(u). \end{aligned}$$

By polar coordinates and the subharmonicity of the function

$$h(u) = \left|f(z+u)e^{-\alpha\bar{z}u}\right|^p,$$

we have

$$I(z) \ge e^{-\frac{p\alpha}{2}|z|^2} |f(z)|^p \int_{|u|<r} e^{-\frac{p\alpha}{2}|u|^2} \, dA(u).$$

This gives the desired result. □

The following elementary estimate will also be used many times later on.

Lemma 2. *For any $r > 0$ there exists a positive constant $C = C_r$ such that*

$$\mu(B(z,r)) \le C\widetilde{\mu}(z)$$

for all $z \in \mathbb{C}$.

Proof. Given $z \in \mathbb{C}$ we have

$$\begin{aligned} \mu(B(z,r)) &= \int_{B(z,r)} d\mu(w) \le e^{\alpha r^2} \int_{B(z,r)} e^{-\alpha|z-w|^2} \, d\mu(w) \\ &= e^{\alpha r^2} \int_{B(z,r)} |k_z(w)|^2 e^{-\alpha|w|^2} \, d\mu(w) \\ &\le e^{\alpha r^2} \int_{\mathbb{C}} |k_z(w)|^2 e^{-\alpha|w|^2} \, d\mu(w) \\ &= e^{\alpha r^2} \widetilde{\mu}(z). \end{aligned}$$

Thus, the result holds with $C = e^{\alpha r^2}$. □

The main result of this section is the following.

Theorem 3. *Suppose $\mu \ge 0$, $p > 0$, and $r > 0$. Then the following conditions are equivalent.*

(a) *There exists a positive constant C such that*
$$\int_{\mathbb{C}} \left| f(w)e^{-\frac{\alpha}{2}|w|^2} \right|^p d\mu(w) \le C \int_{\mathbb{C}} \left| f(w)e^{-\frac{\alpha}{2}|w|^2} \right|^p dA(w)$$
for all entire functions f.

(b) *There exists a constant $C > 0$ such that $\mu(B(z,r)) \le C$ for all $z \in \mathbb{C}$.*

(c) *There exists a constant $C > 0$ such that $\mu(B(a_n,r)) \le C$ for all n.*

Proof. For any entire function f we set
$$I(f) = \int_{\mathbb{C}} \left| f(w)e^{-\frac{\alpha}{2}|w|^2} \right|^p d\mu(w).$$
Then
$$I(f) \le \sum_{n=1}^{\infty} \int_{B(a_n,r)} \left| f(w)e^{-\frac{\alpha}{2}|w|^2} \right|^p d\mu(w).$$
By Lemma 1 and the triangle inequality, there exists a constant $C_1 > 0$ such that
$$\left| f(w)e^{-\frac{\alpha}{2}|w|^2} \right|^p \le C_1 \int_{B(a_n,2r)} \left| f(u)e^{-\frac{\alpha}{2}|u|^2} \right|^p dA(u)$$
for all $w \in B(a_n, r)$. If condition (c) holds, then we can find a positive constant C_2 (independent of f) such that
$$I(f) \le C_2 \sum_{n=1}^{\infty} \int_{B(a_n,2r)} \left| f(u)e^{-\frac{\alpha}{2}|u|^2} \right|^p dA(u)$$
for all entire functions f. It is clear that there exists a positive integer m such that every point in the complex plane belongs to at most m of the disks $B(a_n, 2r)$. Therefore,
$$I(f) \le C_2 m \int_{\mathbb{C}} \left| f(u)e^{-\frac{\alpha}{2}|u|^2} \right|^p dA(u).$$
This shows that condition (c) implies (a).

To show that condition (a) implies (b), we consider the normalized reproducing kernels
$$k_z(w) = e^{\alpha w\bar{z} - \frac{\alpha}{2}|z|^2}, \qquad w \in \mathbb{C}.$$
Since
$$\int_{\mathbb{C}} \left| k_z(w)e^{-\frac{\alpha}{2}|w|^2} \right|^p dA(w) = \frac{2\pi}{p\alpha}$$

for every $z \in \mathbb{C}$, we see that condition (a) implies that there exists a positive constant C such that

$$\int_{\mathbb{C}} e^{-\frac{p\alpha}{2}|z-w|^2}\, d\mu(w) \le C$$

for all $z \in \mathbb{C}$. In particular,

$$\int_{B(z,r)} e^{-\frac{p\alpha}{2}|z-w|^2}\, d\mu(w) \le C$$

for all $z \in \mathbb{C}$. This clearly implies that

$$\mu(B(z,r)) \le C e^{\frac{p\alpha}{2}r^2}$$

for all $z \in \mathbb{C}$, so that condition (a) implies (b).

It is trivial that condition (b) implies (c). □

Note that Lemma 1 and the equivalence of (a) and (b) in Theorem 3 were proven in both [11] and [10], where the authors considered Fock-Carleson measures with respect to more general weighted Fock spaces. We have included proofs of these two results, because our arguments are more elementary and less complicated than the ones found in [11] and [10].

It is interesting to note that conditions (b) and (c) are independent of p and α. It follows that if condition (a) holds for some $p > 0$ and some α, then it holds for every p and every α (with the constant C dependent on p and α).

It is also interesting to note that condition (a) is independent of r. Therefore, if condition (b) or condition (c) holds for some $r > 0$, then it holds for every $r > 0$ (with the constant C dependent on r).

In view of the remarks above, we will call any positive Borel measure μ that satisfies condition (a), (b), or (c) a Fock-Carleson measure.

Similarly, we say that a positive Borel measure μ on $\mathbb{C}$ is a vanishing Fock-Carleson measure if

$$\lim_{n\to\infty} \int_{\mathbb{C}} \left| f_n(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p d\mu(z) = 0$$

whenever $\{f_n\}$ is a bounded sequence in F^p_α that converges to 0 uniformly on compact subsets of the complex plane. We proceed to show that being a vanishing Fock-Carleson measure is also independent of p and α.

Theorem 4. *Suppose $\mu \ge 0$, $p > 0$, and $r > 0$. Then the following conditions are equivalent.*

(i) *μ is a vanishing Fock-Carleson measure.*

(ii) *$\mu(B(z,r)) \to 0$ as $z \to \infty$.*

(iii) *$\mu(B(a_n,r)) \to 0$ as $n \to \infty$.*

Proof. By the proof of Lemma 2, there exists a positive constant C (independent of z) such that

$$\mu(B(z,r)) \le C \int_{\mathbb{C}} \left| k_z(w) e^{-\alpha|w|^2/2} \right|^p \, d\mu(w)$$

for all $z \in \mathbb{C}$. For any sequence $u_n \to \infty$, it is easy to see that the sequence of functions

$$f_n(w) = k_{u_n}(w) = e^{\alpha w \overline{u_n} - \frac{\alpha}{2}|u_n|^2}, \qquad w \in \mathbb{C},$$

satisfy $\|f_n\|_{p,\alpha}^p = 2\pi/p\alpha$ and $f_n(w) \to 0$ uniformly on compact sets. Therefore, if μ is a vanishing Fock-Carleson measure, then

$$\lim_{n\to\infty} \mu(B(u_n, r)) = 0.$$

Since $\{u_n\}$ is arbitrary, we conclude that

$$\lim_{z\to\infty} \mu(B(z, r)) = 0.$$

Thus, condition (i) implies (ii). That condition (ii) implies (iii) is obvious.

On the other hand, carefully examining the proof of Theorem 3, we see that there is a positive constant C (independent of f) such that the integral

$$I(f) = \int_{\mathbb{C}} \left| f(w) e^{-\alpha|w|^2/2} \right|^p \, d\mu(w)$$

satisfies

$$I(f) \le C \sum_{k=1}^{\infty} \mu(B(a_k, r)) \int_{B(a_k,2r)} \left| f(w) e^{-\alpha|w|^2/2} \right|^p \, dA(w). \tag{6}$$

If condition (iii) holds, then for any $\varepsilon > 0$, there exists a positive integer N such that $\mu(B(a_k, r)) < \varepsilon$ whenever $k > N$. Thus for any bounded sequence $\{f_n\}$ in F_α^p that converges to 0 uniformly on compact sets, we can estimate the sequence $I(f_n)$ according to (6) as follows:

$$\begin{aligned} I(f_n) \; &\le \; C' \sum_{k=1}^{N} \int_{B(a_k,2r)} \left| f_n(w) e^{-\alpha|w|^2/2} \right|^p \, dA(w) \\ &\quad + C\varepsilon \sum_{k=N+1}^{\infty} \int_{B(a_k,2r)} \left| f_n(w) e^{-\alpha|w|^2/2} \right|^p \, dA(w), \end{aligned}$$

where C and C' are positive constant independent of n. Since $f_n(w) \to 0$ uniformly on compact sets in $\mathbb{C}$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{N} \int_{B(a_k,2r)} \left| f_n(w) e^{-\alpha|w|^2/2} \right|^p \, dA(w) = 0.$$

Therefore,

$$\limsup_{n\to\infty} I(f_n) \le C\varepsilon \sum_{k=N+1}^{\infty} \int_{B(a_k,2r)} \left| f_n(w)e^{-\alpha|w|^2/2}\right|^p \, dA(w).$$

Since there is a positive integer m (depending on r only) such that every point in the complex plane belongs to at most m of the disks $D(a_k, 2r)$, we have that

$$\begin{aligned}\sum_{k=N+1}^{\infty} \int_{B(a_k,2r)} \left| f_n(w)e^{-\frac{\alpha}{2}|w|^2}\right|^p \, dA(w) &\le m \int_{\mathbb{C}} \left| f_n(w)e^{-\frac{\alpha}{2}|w|^2}\right|^p \, dA(w) \\ &\le C,\end{aligned}$$

where C is another positive constant independent of n. Therefore, we can find yet another positive constant C (independent of n and ε) such that

$$\limsup_{n\to\infty} \int_{\mathbb{C}} \left| f_n(w)e^{-\alpha|w|^2/2}\right|^p \, d\mu(w) \le C\varepsilon.$$

Since ε is arbitrary, we have

$$\lim_{n\to\infty} \int_{\mathbb{C}} \left| f_n(w)e^{-\alpha|w|^2/2}\right|^p \, d\mu(w) = 0.$$

This shows that condition (iii) implies condition (i), and thus completes the proof of the theorem. □

3. Boundedness and Compactness

The next result characterizes bounded Toeplitz operators with positive symbols and gives several other equivalent conditions for a positive Borel measure on $\mathbb{C}$ to be Fock-Carleson.

Theorem 5. *The following conditions are equivalent for* $\mu \ge 0$.

(a) *The Toeplitz operator* T_μ *is bounded on* F^2_α.
(b) *The Berezin transform* $\widetilde{\mu}$ *is bounded on* $\mathbb{C}$.
(c) *The measure* μ *is Fock-Carleson.*

Proof. If T_μ defines a bounded linear operator on F^2_α, then it is easy to check that

$$\langle T_\mu f, g\rangle = \int_{\mathbb{C}} f(w)\overline{g(w)}e^{-\alpha|w|^2}\, d\mu(w) \tag{7}$$

for all f and $g \in F^2_\alpha$. In particular,

$$\langle T_\mu f, f\rangle = \int_{\mathbb{C}} |f(w)|^2 e^{-\alpha|w|^2}\, d\mu(w) = \int_{\mathbb{C}} \left| f(w)e^{-\frac{\alpha}{2}|w|^2}\right|^2 \, d\mu(w) \tag{8}$$

for all $f \in F^2_\alpha$. If we set $f = k_z$ in (8), where k_z are the normalized reproducing kernels in F^2_α, then we obtain

$$\widetilde{\mu}(z) = \langle T_\mu k_z, k_z\rangle, \qquad z \in \mathbb{C}.$$

It follows from the Cauchy-Schwarz inequality that the boundedness of T_μ on F^2_α implies that

$$0 \le \widetilde{\mu}(z) \le \|T_\mu k_z\|\|k_z\| \le \|T_\mu\|$$

for all $z \in \mathbb{C}$, which proves that condition (a) implies condition (b).

That condition (b) implies (c) follows from Lemma 2 and Theorem 3.

Finally, if condition (c) holds, then by the definition of $\tilde{\mu}$, we have that $\tilde{\mu}$ is bounded. By an argument that is identical to the proof of Lemma 14 in [2], we have that $T_\mu : F^2_\alpha \to L^2(\mathbb{C}, d\lambda_\alpha)$ is bounded, with operator norm comparable to $\sup\{\tilde{\mu}(z) : z \in \mathbb{C}\}$. However, it is easy to check, by Fubini's theorem, that $PT_\mu = T_\mu$, which means that $T_\mu : F^2_\alpha \to F^2_\alpha$ is bounded. □

As a consequence of the characterization of vanishing Fock-Carleson measures in Theorem 4, we obtain several characterizations for compact Toeplitz operators on the Fock spaces.

Theorem 6. *The following conditions are equivalent for* $\mu \ge 0$*.*

(a) T_μ *is compact on* F^2_α*.*
(b) $\widetilde{\mu}(z) \to 0$ *as* $z \to \infty$*.*
(c) μ *is a vanishing Fock-Carleson measure.*

Proof. Recall that $\widetilde{\mu}(z) = \langle T_\mu k_z, k_z\rangle$, where k_z are the normalized reproducing kernels of F^2_α. It clear that we have $k_z \to 0$ weakly in F^2_α as $z \to \infty$, so the compactness of T_μ on F^2_α implies that $\widetilde{\mu}(z) \to 0$ as $z \to \infty$. This proves that condition (a) implies (b).

That condition (b) implies (c) follows from Lemma 2 and Theorem 4.

To finish the proof, let us assume that μ is a vanishing Fock-Carleson measure. In particular, μ is a Fock-Carleson measure, so T_μ is a bounded operator on F^2_α. To show that T_μ is actually compact, first recall that a positive operator T on a Hilbert space H is compact if and only if

$$\lim_{n\to\infty} \langle T f_n, f_n\rangle \to 0$$

whenever $\{f_n\}$ is a sequence in H that converges to 0 weakly, then observe that a sequence $\{f_n\}$ in F^2_α converges to 0 weakly if and only if $\{\|f_n\|\}$ is bounded and $f_n(w) \to 0$ uniformly on compact sets as $n \to \infty$. Thus, condition (c) along with the identity in (8) shows that

$$\lim_{n\to\infty} \langle T_\mu f_n, f_n\rangle = 0$$

whenever $\{f_n\}$ is a sequence in F^2_α that converges to 0 weakly. This together with the positivity of T_μ shows that T_μ is compact on F^2_α. □

Theorem 6 above also gives additional characterizations for vanishing Fock-Carleson measures in terms of Toeplitz operators and the Berezin transform.

Carefully examining the proof of Theorem 5, we see that the following quantities are equivalent.

(a) $\|\mu\|_1 = \|T_\mu\|$, the norm of T_μ on F^2_α.
(b) $\|\mu\|_2 = \sup\{\widetilde{\mu}(z) : z \in \mathbb{C}\}$.
(c) $\|\mu\|_3 = \sup\{\mu(B(z,r)) : z \in \mathbb{C}\}$, where r is any fixed positive radius.

The above quantities are also equivalent to

$$\|\mu\|_4 = \sup\left\{\int_{\mathbb{C}} \left|f(z)e^{-\alpha|z|^2/2}\right|^p \, d\mu(z) : \|f\|_p = 1\right\}.$$

From now on, whenever unspecified, the notation $\|\mu\|$ will denote (any) one of the above quantities.

Theorem 7. *The following conditions are equivalent for $\mu \geq 0$.*

(a) *μ is a vanishing Fock-Carleson measure.*
(b) *$\|\mu - \mu_R\| \to 0$ as $R \to \infty$, where μ_R is the truncation of μ on the disk $B(0,R)$.*
(c) *There exists a sequence of finite Borel measures μ_n, each with compact support, such that $\|\mu - \mu_n\| \to 0$ as $n \to \infty$.*

Proof. If μ is a vanishing Fock-Carleson measure, then $\mu(B(z,r)) \to 0$ as $z \to \infty$. It follows that $\|\mu - \mu_R\|_3 \to 0$ as $R \to \infty$. This shows that condition (a) implies (b).

Since each μ_R has compact support, it is obvious that condition (b) implies (c).

If μ_n is a finite Borel measure with compact support, then it is clear that T_{μ_n} is compact. Condition (c) implies that $\|T_{\mu_n} - T_\mu\| \to 0$ as $n \to \infty$, which in turn implies that T_μ is compact. Thus condition (c) implies (a). ☐

Finally in this section we specialize to the case of Toeplitz operators with nonnegative functions (instead of measures) as symbols. The following corollary is a consequence of Theorems 3 and 5.

Corollary 8. *Suppose $r > 0$ and $\varphi \geq 0$ on $\mathbb{C}$. Then the following conditions are equivalent.*

(a) *T_φ is bounded on F^2_α.*
(b) *There exists a positive constant C such that*

$$\int_{B(z,r)} \varphi(w)\, dA(w) \leq C$$

for all $z \in \mathbb{C}$.

(c) *There exists a positive constant C such that*

$$\int_{B(a_n,r)} \varphi(w)\, dA(w) \le C$$

for all $n \ge 1$.

Note that the conditions in (b) and (c) above are independent of α. Thus, if the Toeplitz operator on F^2_α induced by φ is bounded, then the Toeplitz operator on any other F^2_β induced by φ is also bounded.

The following corollary is a consequence of Theorems 4 and 6.

Corollary 9. *Suppose $r > 0$ and $\varphi \ge 0$ on $\mathbb{C}$. Then the following conditions are equivalent.*

(a) *T_φ is compact on F^2_α.*

(b)

$$\lim_{z\to\infty} \int_{B(z,r)} \varphi(w)\, dA(w) = 0.$$

(c)

$$\lim_{n\to\infty} \int_{B(a_n,r)} \varphi(w)\, dA(w) = 0.$$

Once again, the conditions in (b) and (c) are independent of α. Thus, T_φ is compact on F^2_α if and only if it is compact on any other F^2_β.

4. Toeplitz operators in S_p with $p \ge 1$

We are going to determine when a Toeplitz operator T_μ on F^2_α belongs to the Schatten class S_p. This section is devoted to the case $p \ge 1$, while the next section concerns the case $0 < p \le 1$. Background information about the Schatten classes S_p can be found in [17] for example.

For any bounded linear operator T on F^2_α, we can define the Berezin transform $\widetilde{T}$ by.

$$\widetilde{T}(z) = \langle Tk_z, k_z\rangle, \qquad z \in \mathbb{C}, \tag{9}$$

where k_z are the normalized reproducing kernels in F^2_α. If T is positive on F^2_α, then mimicking the proof of Theorem 6.4 in [17] we can show that

$$\mathrm{tr}\,(T) = \frac{\alpha}{\pi} \int_{\mathbb{C}} \widetilde{T}(z)\, dA(z). \tag{10}$$

In particular, T is in the trace class S_1 if and only if the integral above converges. As a consequence of (10), we obtain the following trace formula for Toeplitz operators on Fock spaces.

Proposition 10. *Suppose $\mu \ge 0$. Then T_μ is in the trace class S_1 if and only if μ is finite on $\mathbb{C}$. Moreover, $\mathrm{tr}\,(T_\mu) = \mu(\mathbb{C})$.*

Proof. Since all integrands below are nonnegative, we use Fubini's theorem to obtain

$$\begin{aligned} \mathrm{tr}\,(T_\mu) &= \frac{\alpha}{\pi}\int_{\mathbb{C}} \widetilde{\mu}(z)\,dA(z) \\ &= \int_{\mathbb{C}} e^{\alpha|z|^2}\,d\lambda_\alpha(z)\int_{\mathbb{C}} |e^{\alpha\bar{z}w}|^2 e^{-\alpha(|z|^2+|w|^2)}\,d\mu(w) \\ &= \int_{\mathbb{C}} e^{-\alpha|w|^2}\,d\mu(w)\int_{\mathbb{C}} |e^{\alpha\bar{z}w}|^2\,d\lambda_\alpha(z) \\ &= \int_{\mathbb{C}} d\mu(w) = \mu(\mathbb{C}). \end{aligned}$$

This also shows that $\mathrm{tr}\,(T_\mu) < \infty$ if and only if $\mu(\mathbb{C}) < \infty$. □

Lemma 11. *If $p \geq 1$ and $\varphi \in L^p(\mathbb{C}, dA)$, then $T_\varphi \in S_p$.*

Proof. This is proved in exactly the same way that Proposition 7.11 in [17] was proved. □

Lemma 12. *Suppose $r > 0$, μ is a positive Borel measure on $\mathbb{C}$, and $\widehat{\mu}_r(z) = \mu(B(z,r))/(\pi r^2)$. If $\widehat{\mu}_r$ is in $L^p(\mathbb{C}, dA)$, then T_μ and $T_{\widehat{\mu}_r}$ are bounded on F^2_α. Moreover, there exists a positive constant C (independent of μ) such that $T_\mu \leq CT_{\widehat{\mu}_r}$.*

Proof. Since $\widehat{\mu}_r$ is in $L^p(\mathbb{C}, dA)$, an easy application of Theorem 5 gives us that T_μ is bounded. Moreover, another application of Theorem 5 tells us that $T_{\widehat{\mu}_r}$ is bounded. Thus, given $f \in F^2_\alpha$, we use Fubini's theorem to obtain

$$\begin{aligned} \pi r^2\langle T_{\widehat{\mu}_r} f, f\rangle &= \pi r^2\int_{\mathbb{C}} |f(z)|^2\widehat{\mu}_r(z)\,d\lambda_\alpha(z) \\ &= \int_{\mathbb{C}} |f(z)|^2\mu(B(z,r))\,d\lambda_\alpha(z) \\ &= \int_{\mathbb{C}} |f(z)|^2\,d\lambda_\alpha(z)\int_{\mathbb{C}} \chi_{B(z,r)}(w)\,d\mu(w) \\ &= \int_{\mathbb{C}} d\mu(w)\int_{\mathbb{C}} |f(z)|^2\chi_{B(w,r)}(z)\,d\lambda_\alpha(z) \\ &= \frac{\alpha}{\pi}\int_{\mathbb{C}} d\mu(w)\int_{B(w,r)} |f(z)e^{-\alpha|z|^2/2}|^2\,dA(z). \end{aligned}$$

Combining the above identity with Lemma 1, we obtain a positive constant C such that

$$C\langle T_{\widehat{\mu}_r} f, f\rangle \geq \int_{\mathbb{C}} |f(w)|^2 e^{-\alpha|w|^2}\,d\mu(w) = \langle T_\mu f, f\rangle.$$

This proves the desired result. □

We are now ready to prove the main result of the section.

Theorem 13. *Suppose $\mu \geq 0$, $r > 0$, and $p \geq 1$. Then the following conditions are equivalent.*

(a) *The operator T_μ is in the Schatten class S_p.*
(b) *The function $\widetilde{\mu}(z)$ is in $L^p(\mathbb{C}, dA)$.*
(c) *The function $\mu(B(z,r))$ is in $L^p(\mathbb{C}, dA)$.*
(d) *The sequence $\{\mu(B(a_n, r))\}$ is in l^p.*

Proof. That (a) implies (b) follows from the first part of Theorem 6.6 in [17], and Lemma 2 shows that condition (b) implies (c).

If the averaging function $\widehat{\mu}_r(z) = \mu(B(z,r))/(\pi r^2)$ is in $L^p(\mathbb{C}, dA)$, then it follows from Lemma 11 that $T_{\widehat{\mu}_r}$ is in S_p. Combining this with Lemma 12, we conclude that T_μ is in S_p. This proves that (c) implies (a). Hence conditions (a), (b), and (c) are equivalent.

To prove that condition (d) is equivalent to the other conditions, we first assume that condition (b) holds, which implies that the function $\mu(B(z, 2r))$ is in $L^p(\mathbb{C}, dA)$. Choose a positive integer m such that each point in the complex plane belongs to at most m of the discs $B(a_n, r)$. Then

$$m \int_{\mathbb{C}} \mu(B(z, 2r))^p \, dA(z) \geq \sum_{n=1}^{\infty} \int_{B(a_n,r)} \mu(B(z, 2r))^p \, dA(z).$$

For each $z \in B(a_n, r)$ we deduce from the triangle inequality that

$$\mu(B(z, 2r)) \geq \mu(B(a_n, r)).$$

Therefore,

$$m \int_{\mathbb{C}} \mu(B(z, 2r))^p \, dA(z) \geq \pi r^2 \sum_{n=1}^{\infty} \mu(B(a_n, r))^p.$$

This shows that condition (b) implies (d).

To finish the proof, we assume that condition (d) holds, that is,

$$\sum_{n=1}^{\infty} \mu(B(a_n, r))^p < \infty.$$

It is easy to see that we also have

$$\sum_{n=1}^{\infty} \mu(B(z_n, r))^p < \infty$$

where $\{z_n\}$ is the lattice generated by $r/2$ and $ir/2$. In fact, for each point z_k that is not in the lattice $\{a_n\}$, the disc $B(z_k, r)$ is covered by six adjacent discs $B(a_k, r)$. Therefore,

$$\begin{aligned}\int_{\mathbb{C}} \mu(B(z, r/2))^p \, dA(z) &\le \sum_{n=1}^{\infty} \int_{B(z_n, r/2)} \mu(B(z, r/2))^p \, dA(z) \\ &\le \sum_{n=1}^{\infty} \int_{B(z_n, r/2)} \mu(B(z_n, r))^p \, dA(z) \\ &= \frac{\pi r^2}{4} \sum_{n=1}^{\infty} \mu(B(z_n, r))^p < \infty.\end{aligned}$$

This shows that condition (d) implies (c), as the equivalence of (c) to (b) implies that if condition (c) holds for one positive radius, then it will hold for any other positive radius. This completes the proof of the theorem. □

Specializing to the case when $d\mu(z) = (\alpha/\pi)\varphi(z) \, dA(z)$, we obtain the following corollary concerning Toeplitz operators induced by nonnegative functions.

Corollary 14. *Suppose $\varphi \ge 0$, $p \ge 1$, and $r > 0$. Then the following conditions are equivalent.*

(a) *The Toeplitz operator T_φ belongs to S_p.*
(b) *The Berezin transform $\widetilde{\varphi}$ belongs to $L^p(\mathbb{C}, dA)$.*
(c) *The averaging function*

$$\widehat{\varphi}_r(z) = \frac{1}{\pi r^2} \int_{B(z,r)} \varphi(w) \, dA(w)$$

belongs to $L^p(\mathbb{C}, dA)$.
(d) *The averaging sequence $\{\widehat{\varphi}_r(a_n)\}$ belongs to l^p.*

5. Toeplitz Operators in S_p with $0 < p \le 1$

We now turn our attention to the case $0 < p \le 1$, which requires new ideas and techniques.

Lemma 15. *Suppose $\mu \ge 0$, $r > 0$, and $0 < p \le 1$. Then the following conditions are equivalent.*

(a) *The function $\widetilde{\mu}(z)$ is in $L^p(\mathbb{C}, dA)$.*
(b) *The function $\mu(B(z, r))$ is in $L^p(\mathbb{C}, dA)$.*
(c) *The sequence $\{\mu(B(a_n, r))\}$ is in l^p.*

Proof. We begin with the inequality

$$\widetilde{\mu}(z) = \int_{\mathbb{C}} e^{-\alpha|z-w|^2} \, d\mu(w) \le \sum_{n=1}^{\infty} \int_{B(a_n,r)} e^{-\alpha|z-w|^2} \, d\mu(w).$$

For $w \in B(a_n, r)$ we have

$$|z-w|^2 \geq (|z-a_n| - |a_n - w|)^2 \geq |z-a_n|^2 - 2r|z-a_n|.$$

It follows that

$$\widetilde{\mu}(z) \leq \sum_{n=1}^{\infty} e^{-\alpha|z-a_n|^2 + 2\alpha r|z-a_n|} \mu(B(a_n, r)).$$

Since $0 < p \leq 1$, Hölder's inequality gives

$$\widetilde{\mu}(z)^p \leq \sum_{n=1}^{\infty} e^{-p\alpha|z-a_n|^2 + 2pr\alpha|z-a_n|} \mu(B(a_n, r))^p.$$

It follows from this and Fubini's theorem that

$$\int_{\mathbb{C}} \widetilde{\mu}(z)^p \, dA(z) \leq \sum_{n=1}^{\infty} \mu(B(a_n, r))^p \int_{\mathbb{C}} e^{-p\alpha|z-a_n|^2 + 2pr\alpha|z-a_n|} \, dA(z).$$

By an obvious change of variables, the integral above equals

$$\int_{\mathbb{C}} e^{-p\alpha|z|^2 + 2pr\alpha|z|} \, dA(z),$$

which is easily seen to be convergent. This shows that $\{\mu(B(a_n, r))\} \in l^p$ implies that $\widetilde{\mu} \in L^p(\mathbb{C}, dA)$.

On the other hand, there exists a positive integer m such that every point in the complex plane belongs to at most m of the discs $B(a_n, r)$. Thus,

$$m \int_{\mathbb{C}} \widetilde{\mu}(z)^p \, dA(z) \geq \sum_{n=1}^{\infty} \int_{B(a_n,r)} \widetilde{\mu}(z)^p \, dA(z).$$

For any $z \in B(a_n, r)$, we have

$$\begin{aligned} \widetilde{\mu}(z) &= \int_{\mathbb{C}} e^{-\alpha|z-w|^2} \, d\mu(w) \geq \int_{B(a_n,r)} e^{-\alpha|z-w|^2} \, d\mu(w) \\ &\geq e^{-4\alpha r^2} \mu(B(a_n, r)). \end{aligned}$$

It follows that

$$m \int_{\mathbb{C}} \widetilde{\mu}(z)^p \, dA(z) \geq \pi r^2 e^{-4p\alpha r^2} \sum_{n=1}^{\infty} \mu(B(a_n, r))^p.$$

Thus $\widetilde{\mu} \in L^p(\mathbb{C}, dA)$ implies that $\{\mu(B(a_n, r))\} \in l^p$, which proves the equivalence of conditions (a) and (c).

That condition (a) implies (b) follows from Lemma 2.

To prove that condition (b) implies (c), we assume that the function $\mu(B(z, r))$ is in $L^p(\mathbb{C}, dA)$. Consider the lattice generated by $r/2$ and $ri/2$ and arrange it into a sequence $\{z_n\}$. There exists a positive integer m such that every point in the complex plane belongs to at most m of the discs $B(z_n, r/2)$. Therefore,

$$m \int_{\mathbb{C}} \mu(B(z,r))^p \, dA(z) \geq \sum_{n=1}^{\infty} \int_{B(z_n,r/2)} \mu(B(z,r))^p \, dA(z).$$

For each $z \in B(z_n, r/2)$, the triangle inequality gives us that

$$\mu(B(z,r)) \geq \mu(B(z_n, r/2)).$$

Thus

$$m \int_{\mathbb{C}} \mu(B(z,r))^p \, dA(z) \geq \frac{\pi r^2}{4} \sum_{n=1}^{\infty} \mu(B(z_n, r/2))^p.$$

By the equivalence of conditions (a) and (c), the function $\widetilde{\mu}$ belongs to $L^p(\mathbb{C}, dA)$, and applying the equivalence of (a) and (c) once more, we conclude that $\{\mu(B(a_n, r))\} \in l^p$. This completes the proof of the lemma. □

Lemma 16. *Suppose $\mu \geq 0$, $0 < p \leq 1$, and the function $\widetilde{\mu}$ belongs to $L^p(\mathbb{C}, dA)$. Then the operator T_μ belongs to S_p.*

Proof. Since $\widetilde{\mu}$ belongs to $L^p(\mathbb{C}, dA)$, it is not difficult to see that T_μ is bounded, so that $\widetilde{T_\mu} = \tilde{\mu}$. Thus the lemma can be proven in exactly the same way that part (b) of Theorem 6.6 in [17] was proved. □

We will need the following Lemma, whose proof can be found in [17].

Lemma 17. *If $0 < p \leq 1$, then for any orthonormal basis $\{e_n\}$ of a separable Hilbert space H and any compact operator T on H, we have that*

$$\|T\|_{S_p}^p \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Te_n, e_k \rangle|^p.$$

We are now ready to characterize Toeplitz operators T_μ in S_p when $0 < p \leq 1$.

Theorem 18. *Suppose $\mu \geq 0$, $r > 0$, and $0 < p \leq 1$. Then the following conditions are equivalent.*

(a) *T_μ belongs to the Schatten class S_p.*
(b) *$\widetilde{\mu}$ belongs to $L^p(\mathbb{C}, dA)$.*
(c) *$\widehat{\mu}_r$ belongs to $L^p(\mathbb{C}, dA)$.*
(d) *$\{\widehat{\mu}_r(a_n)\}$ belongs to l^p.*

Proof. The equivalence of (b), (c), and (d) was proved in Lemma 15. That condition (b) implies condition (a) was proved in Lemma 16. Therefore, to finish the proof, we will show that condition (a) implies (d). In what follows, $C_1, C_2, \ldots$ will denote positive constants that only depend on p, α, and r. For convenience, we will use the norm $|z|_\infty = \max\{|x|, |y|\}$, where $z = x + iy$, and for the rest of the proof, we will let $B(z, r)$ denote the closed ball centered at z with radius r in this norm. If we can prove that condition (a) implies

$$\sum_{n=1}^{\infty} \mu(B(2a_n, r))^p < \infty,$$

then condition (d) will easily follow.

To this end, fix some large $R > 0$ and partition $\{2a_n\}$ into N subsequences such that the Euclidean distance between any two points in each subsequence is at least R. Let $\{\zeta_n\}$ be such a subsequence and let

$$\nu = \sum_{n=1}^{\infty} \mu\chi_n,$$

where χ_n is the characteristic function of $B(\zeta_n, r)$. Since $T_\mu \in S_p$ and $\mu \geq \nu$, we have $T_\nu \leq T_\mu$, and so $T_\nu \in S_p$ with $\|T_\nu\|_{S_p} \leq \|T_\mu\|_{S_p}$.

Let $\{e_n\}$ be an orthonormal basis for F^2_α and define a bounded linear operator A on F^2_α by $Ae_n = k_{\zeta_n}$, $n \geq 1$. Let $T = A^* T_\nu A$ so that $\|T\|_{S_p} \leq \|T_\mu\|_{S_p}$.

We split the operator T as $T = D + E$ where D is the diagonal operator defined on F^2_α by

$$Df = \sum_{n=1}^{\infty} \langle Te_n, e_n\rangle \langle f, e_n\rangle e_n,$$

and $E = T - D$. By the triangle inequality, we have

$$\|T\|^p_{S_p} \geq \|D\|^p_{S_p} - \|E\|^p_{S_p}. \tag{11}$$

Since D is a positive diagonal operator, we have

$$\begin{aligned} \|D\|^p_{S_p} &= \sum_{n=1}^{\infty} \langle Te_n, e_n\rangle^p = \sum_{n=1}^{\infty} \langle T_\nu k_{\zeta_n}, k_{\zeta_n}\rangle^p \\ &= \sum_{n=1}^{\infty} \left(\int_{\mathbb{C}} e^{-\alpha|z-\zeta_n|^2}\, d\nu(z) \right)^p \\ &\geq \sum_{n=1}^{\infty} \left(\int_{B(\zeta_n, r)} e^{-\alpha|z-\zeta_n|^2}\, d\nu(z) \right)^p \\ &\geq C_1 \sum_{n=1}^{\infty} \nu(B(\zeta_n, r))^p. \end{aligned} \tag{12}$$

On the other hand, by Lemma 17, we have

$$\|E\|_{S_p}^p \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle Ee_n, e_k\rangle|^p = \sum_{n\neq k} |\langle T_\nu k_{\zeta_n}, k_{\zeta_k}\rangle|^p$$

$$= \sum_{n\neq k} \left| \int_{\mathbb{C}} k_{\zeta_n}(z)\overline{k_{\zeta_k}(z)} e^{-\alpha|z|^2}\, d\nu(z)\right|^p. \tag{13}$$

A straightforward calculation shows that

$$k_{\zeta_n}(z)\overline{k_{\zeta_k}(z)} e^{-\alpha|z|^2} = e^{-\frac{\alpha|z-\zeta_n|^2}{2}} e^{-\frac{\alpha|z-\zeta_k|^2}{2}} e^{\alpha i \,\mathrm{Im}\,(z\overline{\zeta_n}+\bar{z}\zeta_k)},$$

so that (13) gives us

$$\|E\|_{S_p}^p \leq \sum_{n\neq k} \left(\int_{\mathbb{C}} e^{-\frac{\alpha|z-\zeta_n|^2}{2}} e^{-\frac{\alpha|z-\zeta_k|^2}{2}}\, d\nu(z)\right)^p. \tag{14}$$

If $n \neq k$, then $|\zeta_n - \zeta_k| \geq R$. Thus for $|z-\zeta_n| \leq \frac{R}{2}$ the triangle inequality gives us $|z-\zeta_k| \geq \frac{R}{2}$. Therefore, for each $z \in \mathbb{C}$, we have

$$e^{-\frac{\alpha|z-\zeta_n|^2}{2}} e^{-\frac{\alpha|z-\zeta_k|^2}{2}} \leq e^{-\frac{\alpha R^2}{16}} e^{-\frac{\alpha|z-\zeta_n|^2}{4}} e^{-\frac{\alpha|z-\zeta_k|^2}{4}}.$$

Plugging this into (14), we obtain

$$\|E\|_{S_p}^p \leq e^{-\frac{p\alpha R^2}{16}} \sum_{n\neq k} \left(\int_{\mathbb{C}} e^{-\frac{\alpha|z-\zeta_n|^2}{4}} e^{-\frac{\alpha|z-\zeta_k|^2}{4}}\, d\nu(z)\right)^p. \tag{15}$$

For each m in $\{0,1,2,\ldots\}$ and $n \in \mathbb{N}$, let

$$E_{m,n} = \{z : r(2m-1) \leq |z-\zeta_n|_\infty < r(2m+1)\}.$$

Since $0 < p \leq 1$, we have that

$$\sum_{n\neq k} \left(\int_{\mathbb{C}} e^{-\frac{\alpha|z-\zeta_n|^2}{4}} e^{-\frac{\alpha|z-\zeta_k|^2}{4}}\, d\nu(z)\right)^p$$

$$\leq \sum_{n\neq k}\sum_{m=0}^{\infty} \left(\int_{E_{m,n}} e^{-\frac{\alpha|z-\zeta_n|^2}{4}} e^{-\frac{\alpha|z-\zeta_k|^2}{4}}\, d\nu(z)\right)^p$$

$$\leq C_2 \sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n\neq k} \left(\int_{E_{m,n}} e^{-\frac{\alpha|z-\zeta_k|^2}{4}}\, d\nu(z)\right)^p \tag{16}$$

for some constant C_2.

For any fixed m and n we write $\mathbb{N} = \Omega_{m,n}^1 \cup \Omega_{m,n}^2$, where

$$\Omega_{m,n}^1 = \{k \in \mathbb{N} : |\zeta_n - \zeta_k|_\infty \leq 2rm\},$$

and

$$\Omega_{m,n}^2 = \{k \in \mathbb{N} : |\zeta_n - \zeta_k|_\infty > 2rm\}.$$

Therefore, we have that

$$\sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n\neq k} \left(\int_{E_{m,n}} e^{-\frac{\alpha|z-\zeta_k|^2}{4}}\, d\nu(z)\right)^p$$
$$\leq \sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n=1}^{\infty} \sum_{k\in\Omega^1_{m,n}} \left(\int_{E_{m,n}} e^{-\frac{\alpha|z-\zeta_k|^2}{4}}\, d\nu(z)\right)^p$$
$$+ \sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n=1}^{\infty} \sum_{k\in\Omega^2_{m,n}} \left(\int_{E_{m,n}} e^{-\frac{\alpha|z-\zeta_k|^2}{4}}\, d\nu(z)\right)^p$$
$$= S_1 + S_2. \tag{17}$$

Since card $\Omega^1_{m,n} \leq C_3(m+1)^2$ for some $C_3 > 0$, we have that

$$\begin{aligned} S_1 &\leq C_4 \sum_{m=0}^{\infty} (m+1)^2 e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n=1}^{\infty} \nu(E_{m,n})^p \\ &\leq C_4 \sum_{m=0}^{\infty} (m+1)^2 e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n=1}^{\infty} \sum_{\{k\,:\,|2a_k-\zeta_n|=2rm\}} \nu(B(2a_k,r))^p \\ &\leq C_5 \sum_{m=0}^{\infty} (m+1)^4 e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{k=1}^{\infty} \nu(B(2a_k,r))^p \\ &\leq C_6 \sum_{k=1}^{\infty} \nu(B(2a_k,r))^p \\ &= C_6 \sum_{k=1}^{\infty} \nu(B(\zeta_k,r))^p. \end{aligned} \tag{18}$$

Now we will estimate the sum S_2. Observe that if $k \in \Omega^2_{m,n}$, then $z \in E_{m,n}$ implies that

$$|z-\zeta_k|_\infty \geq |\zeta_k-\zeta_n|_\infty - |z-\zeta_n|_\infty > |\zeta_k-\zeta_n|_\infty - r(2m+1) > 0.$$

So we have that

$$\begin{aligned}
S_2 &\le \sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n=1}^{\infty} \nu(E_{m,n})^p \sum_{k\in\Omega^2_{m,n}} e^{-\frac{p\alpha(|\zeta_k-\zeta_n|_\infty-2rm)^2}{4}} \\
&\le C_7 \sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n=1}^{\infty} \nu(E_{m,n})^p \sum_{k=1}^{\infty} e^{-\frac{p\alpha(2rk)^2}{4}} (m+k+1)^2 \\
&\le C_8 \sum_{m=0}^{\infty} (m+1)^2 e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{n=1}^{\infty} \sum_{\{k\,:\,|2a_k-\zeta_n|=2rm\}} \nu(B(2a_k,r))^p \\
&\le C_9 \sum_{m=0}^{\infty} (m+1)^4 e^{-\frac{p\alpha r^2(2m-1)^2}{4}} \sum_{k=1}^{\infty} \nu(B(2a_k,r))^p \\
&\le C_{10} \sum_{k=1}^{\infty} \nu(B(2a_k,r))^p \\
&= C_{10} \sum_{k=1}^{\infty} \nu(B(\zeta_k,r))^p. \qquad (19)
\end{aligned}$$

Combining (18) and (19) with (15), (16), and (17), we obtain

$$\|E\|^p_{S_p} \le C_{11} e^{\frac{-p\alpha R^2}{16}} \sum_{k=1}^{\infty} \nu(B(\zeta_k,r))^p.$$

Going back to (11) and (12), we deduce that

$$\begin{aligned}
\|T_\mu\|^p_{S_p} &\ge \|D\|_{S_p} - \|E\|_{S_p} \\
&\ge (C_1 - C_{11} e^{\frac{-p\alpha R^2}{16}}) \sum_{k=1}^{\infty} \nu(B(\zeta_k,r))^p.
\end{aligned}$$

Since C_1 and C_{11} do not depend on R, setting $R > 0$ large enough gives us

$$\sum_{k=1}^{\infty} \nu(B(\zeta_k,r))^p \le C_{12} \|T_\mu\|^p_{S_p}.$$

Since this holds for each of the N subsequences of $\{2a_n\}$, we obtain

$$\sum_{n=1}^{\infty} \mu(B(2a_n,r))^p \le C_{12} N \|T_\mu\|^p_{S_p} \qquad (20)$$

for all positive Borel measures μ such that

$$\sum_{n=1}^{\infty} \mu(B(2a_n,r))^p < \infty.$$

However, an easy approximation argument shows that (20) holds for all positive Borel measures μ with $T_\mu \in S_p$. This completes the proof that condition (a) implies (d), and thus completes the proof of Theorem 18. □

Again, specializing to the case when $d\mu(z) = (\alpha/\pi)\varphi(z)\, dA(z)$, we obtain the following corollary concerning Toeplitz operators induced by non-negative functions.

Corollary 19. *Suppose $\varphi \geq 0$, $0 < p \leq 1$, and $r > 0$. Then the following conditions are equivalent.*

(a) *The Toeplitz operator T_φ belongs to S_p.*
(b) *The Berezin transform $\widetilde{\varphi}$ belongs to $L^p(\mathbb{C}, dA)$.*
(c) *The averaging function*

$$\widehat{\varphi}_r(z) = \frac{1}{\pi r^2} \int_{B(z,r)} \varphi(w)\, dA(w)$$

belongs to $L^p(\mathbb{C}, dA)$.
(d) *The sequence $\{\widehat{\varphi}_r(a_n)\}$ belongs to l^p.*

6. Some Examples

In this section we will present examples to illustrate that several results proved earlier in the paper are sharp.

First, we show that the conclusion in Lemma 11 is false for every $0 < p < 1$. To see this, consider the set $K \subset \mathbb{R}^2 = \mathbb{C}$ given by

$$K = \bigcup_{k=1}^{\infty} \left[2k, 2k + k^{-\frac{1}{2p}}\right] \times \left[2k, 2k + k^{-\frac{1}{2p}}\right],$$

and consider the characteristic function $\varphi = \chi_K$ of K. Clearly, φ is in $L^p(\mathbb{C}, dA)$ for $0 < p < 1$. However,

$$\sum_{n=1}^{\infty} \widehat{\varphi}_r(a_n)^p = +\infty$$

for every $r > 0$, which means that T_φ is not in S_p. Similar examples can easily be constructed in higher dimensions.

By an argument that is identical to the proof of Proposition 7.15 in [17], we have the following result.

Proposition 20. *Suppose $\varphi \geq 0$, $0 < p \leq 1$, and T_φ belongs to the Schatten class S_p. Then $\varphi \in L^p(\mathbb{C}, dA)$.*

Next we show that the conclusion in the proposition above is false for $p > 1$. If fact, if $p > 1$ and if we define

$$\varphi(z) = \chi_{[0,1]}(|z|)|z|^{-\frac{2}{p}}, \tag{21}$$

then $\varphi \notin L^p(\mathbb{C}, dA)$. On the other hand, as φ is radial, the operator T_φ is diagonal with respect to the standard orthonormal basis $\left\{\sqrt{\frac{\alpha^k}{n!}} z^k\right\}_{k\geq 0}$ of F^2_α. One can then easily check that $T_\varphi \in S_p$ for each $p > 1$.

Finally in this section we show that if we drop the assumption that f be non-negative, then Theorems 6, 13, and 18 can fail dramatically. To see this, we use the following example from [3],

$$f(z) = e^{\left(\frac{1}{5}+i\frac{2}{5}\right)|z|^2}.$$

It can be checked that the diagonal operator T_f (with respect to the standard monomial basis) is unitary, and that

$$\widetilde{f}(z) = \left(\frac{3}{5} + i\frac{4}{5}\right) e^{\left(-\frac{1}{5}+i\frac{2}{5}\right)|z|^2}.$$

Clearly $\widetilde{f}$ vanishes at infinity and $\widetilde{f}$ is in $L^p(\mathbb{C}, dA)$ for each $0 < p < \infty$. This tells us that the Berezin transform of f vanishing at infinity is in general not sufficient for T_f to be compact on F^2_α, and the Berezin transform being in $L^p(\mathbb{C}, dA)$ is in general not sufficient for T_f to be in the Schatten class S_p.

7. Further Remarks

The concept of Toeplitz operators can be introduced in many different contexts, including those of Hardy and Bergman spaces of various domains. In this section we point out several major differences between the various theories of Toeplitz operators.

One obvious difference between Toeplitz operators on the Fock space and those on Hardy and Bergman type spaces is the lack of bounded analytic and harmonic symbols in the Fock space setting. In fact, by the maximum modulus principle, if an analytic or harmonic function on $\mathbb{C}$ is bounded, it has to be a constant.

In the most classical case of Toeplitz operators on the Hardy space of the unit disk, a Toeplitz operator T_φ is bounded if and only if φ is bounded, and T_φ is compact if and only if $\varphi = 0$. See [17] for example. Therefore, the problem of boundedness and compactness in this case is not an issue at all. The situation is clearly different in the Bergman and Fock settings.

In the case of Toeplitz operators on the Bergman space of bounded symmetric domains, the boundedness and compactness of T_μ are characterized by Carleson and vanishing Carleson type measures in [15]. When $p \geq 1$, membership of T_μ in S_p is also characterized in [15] in terms of the Berezin transform $\widetilde{\mu}$ and the averaging functions $\widehat{\mu}_r$.

In the case of Bergman spaces on the unit disc, membership of T_μ in S_p, $0 < p < \infty$, was characterized in [8] in terms of the averaging sequence

$\{\widehat{\mu}_r(a_n)\}$, which is easily seen to be equivalent to the integral condition of the averaging function $\widehat{\mu}_r$. The Berezin transform was not considered in [8].

In the case of weighted Bergman spaces A^2_α on the unit ball in $\mathbb{C}^n$, membership of T_μ in S_p was characterized in [18] in terms of the averaging function $\widehat{\mu}_r$ when $0 < p < \infty$, and in terms of the Berezin transform $\widetilde{\mu}$ when $n/(n+1+\alpha) < p < \infty$. The lower bound $n/(n+1+\alpha)$ here is sharp and is often called a cut-off point. Our main result in this paper shows that there is no cut-off point for Schatten class Toeplitz operators on the Fock space. This is a major difference between the theory of Toeplitz operators on the Fock space and that on Bergman spaces. Moreover, this difference also shows up in the study of Hankel operators. See [6, 13, 14].

For Bergman and Hardy spaces, the kernel function has the property that whenever w is fixed, the function $z \mapsto K(z,w)$ is bounded. This is no longer true for the Fock space, and this unboundedness makes it impossible to extend many standard estimates in the Hardy and Bergman theory to the Fock theory.

Finally, we mention a slight difference in approach taken in this paper compared to the approach taken in the Bergman space theory. Starting with the paper [8], Toeplitz operators T_μ on the Bergman space A^2_α has been defined in the literature as

$$T_\mu f(z) = \int K(z,w) f(w)\, d\mu(w), \qquad f \in A^2_\alpha.$$

We did not use a direct carry-over of this definition to the Fock space setting. Instead, our definition of T_μ includes an extra weight factor $e^{-\alpha|w|^2}$. This minor modification in the definition makes several arguments significantly easier than their counter-parts in the Bergman space theory.

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