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SCHATTEN p CLASS HANKEL OPERATORS ON THE SEGAL–BARGMANN SPACE $H^2(\mathbb{C}^n, \mathrm{d}\mu)$ FOR $0 < p < 1$

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This paper is dedicated to the memory of Laura Jane Wisewell, whose kindness and generosity will always be remembered. May you finally rest in peace forever.

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ABSTRACT. We consider Hankel operators on the Segal–Bargmann space $H^2(\mathbb{C}^n, \mathrm{d}\mu)$. We obtain necessary and sufficient conditions for the simultaneous membership of H_f and $H_{\overline{f}}$ in the Schatten class S_p for $0 < p < 1$. In particular, we show that the necessary and sufficient conditions obtained by J. Xia and D. Zheng for the case $1 \leqslant p < \infty$ extends to the case $0 < p < 1$.

KEYWORDS: *Schatten class, Hankel operators, Segal–Bargmann space.*

MSC (2000): 47B32, 32A36.

1. INTRODUCTION

Let $\mathrm{d}\mu$ be the normalized Gaussian measure on $\mathbb{C}^n$ centered at 0, so that

$$\mathrm{d}\mu(z) = \pi^{-n} \mathrm{e}^{-|z|^2} \mathrm{d}V(z).$$

Recall that the Segal–Bargmann space $H^2(\mathbb{C}^n, \mathrm{d}\mu)$ is defined as $\{f \in L^2(\mathbb{C}^n, \mathrm{d}\mu) : f \text{ is analytic on } \mathbb{C}^n\}$. It is well known that

$$\{(k_1! \cdots k_n!)^{-1/2} z_1^{k_1} \cdots z_n^{k_n} : k_1 \geqslant 0, \dots, k_n \geqslant 0\}$$

forms an orthonormal basis for $H^2(\mathbb{C}^n, \mathrm{d}\mu)$ and that the orthogonal projection $P : L^2(\mathbb{C}^n, \mathrm{d}\mu) \to H^2(\mathbb{C}^n, \mathrm{d}\mu)$ is an integral operator on $L^2(\mathbb{C}^n, \mathrm{d}\mu)$ with kernel $\mathrm{e}^{\langle z, w\rangle}$. Here and in what follows, we write

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n.$$

For each $\nu \in \mathbb{C}^n$, let $\tau_\nu : \mathbb{C}^n \to \mathbb{C}^n$ be the translation

$$\tau_\nu(w) = w + \nu \quad w \in \mathbb{C}^n,$$

and define

$$\mathcal{T}(\mathbb{C}^n) = \{f \in L^2(\mathbb{C}^n, \mathrm{d}\mu) : f \circ \tau_\nu \in L^2(\mathbb{C}^n, \mathrm{d}\mu) \text{ for every } \nu \in \mathbb{C}^n\}.$$

It is easy to see that a measurable function f on $\mathbb{C}^n$ belongs to $\mathcal{T}(\mathbb{C}^n)$ if and only if the function $w \mapsto f(w)\mathrm{e}^{\langle w,\nu\rangle}$ belongs to $L^2(\mathbb{C}^n, \mathrm{d}\mu)$ for every $\nu \in \mathbb{C}^n$. This means that if $f \in \mathcal{T}(\mathbb{C}^n)$, then the set $\{h \in H^2(\mathbb{C}^n, \mathrm{d}\mu) : fh \in L^2(\mathbb{C}^n, \mathrm{d}\mu)\}$ is a dense, linear subspace of $H^2(\mathbb{C}^n, \mathrm{d}\mu)$.

Recall that the Hankel operator $H_f : L^2(\mathbb{C}^n, \mathrm{d}\mu) \to L^2(\mathbb{C}^n, \mathrm{d}\mu)$ with symbol f is defined by the formula

$$H_f = (I - P)M_f P.$$

Thus, if $f \in \mathcal{T}(\mathbb{C}^n)$, then H_f has at least a dense domain in $L^2(\mathbb{C}^n, \mathrm{d}\mu)$.

Given a $\varphi \in L^2(\mathbb{C}^n, \mathrm{d}\mu)$, let $\mathrm{SD}(\varphi)$ denote its standard deviation with respect to the probability measure $\mathrm{d}\mu$, which is defined by

$$\mathrm{SD}(\varphi) = \Big\{ \int \Big|\varphi - \int \varphi \mathrm{d}\mu\Big|^2 \mathrm{d}\mu \Big\}^{1/2} = \Big\{ \int |\varphi|^2 \mathrm{d}\mu - \Big| \int \varphi \mathrm{d}\mu \Big|^2 \Big\}^{1/2}.$$

When $f \in \mathcal{T}(\mathbb{C}^n)$, it was shown in [1] that H_f and $H_{\overline{f}}$ are simultaneously bounded if and only if $\xi \mapsto \mathrm{SD}(f \circ \tau_\xi)$ is a bounded function on $\mathbb{C}^n$, and H_f and $H_{\overline{f}}$ are simultaneously compact if and only if $\lim\limits_{|\xi|\to\infty} \mathrm{SD}(f \circ \tau_\xi) = 0$ (it should be noted that the later was proved in [2] for bounded measurable symbols, where in this setting, it was also proved that H_f is compact if and only if $H_{\overline{f}}$ is compact). Therefore, for $f \in \mathcal{T}(\mathbb{C}^n)$, it is reasonable to think that the simultaneous Schatten p class membership of H_f and $H_{\overline{f}}$ for $1 \leqslant p < \infty$ would be characterized by an L^p condition involving the standard deviation. In fact, it was shown in [6] that for $f \in \mathcal{T}(\mathbb{C}^n)$ and $1 \leqslant p < \infty$, H_f and $H_{\overline{f}}$ are simultaneously members of S_p if and only if

$$\int\limits_{\mathbb{C}^n} \{\mathrm{SD}(f \circ \tau_\xi)\}^p \mathrm{d}V(\xi) < \infty.$$

With this in mind, the following is the main result of this paper.

THEOREM 1.1. *Let $0 < p < 1$ and $f \in \mathcal{T}(\mathbb{C}^n)$. Let H_f and $H_{\overline{f}}$ be the corresponding Hankel operators from $L^2(\mathbb{C}^n, \mathrm{d}\mu)$ to $L^2(\mathbb{C}^n, \mathrm{d}\mu)$. Then we have the simultaneous membership of H_f and $H_{\overline{f}}$ in S_p if and only if*

$$\int\limits_{\mathbb{C}^n} \{SD(f \circ \tau_\xi)\}^p \mathrm{d}V(\xi) < \infty.$$

In particular, we will show that the quantity $\int\limits_{\mathbb{C}^n} \{\mathrm{SD}(f \circ \tau_\xi)\}^p \mathrm{d}V(\xi)$ is comparable to $\|H_f\|_{S_p} + \|H_{\overline{f}}\|_{S_p}$ with a constant that is independent of f.

We close this section with a sketch of the proof. The sufficiency direction is proved by an argument that is identical to the proof for the case $1 \leqslant p < 2$, and so we refer the reader to [6] for the details (more precisely, the proof follows from standard reproducing kernel Hilbert space techniques that are specialized to the Segal–Bargmann space.) For the other direction, let $\mathbb{Z}^{2n}$ be treated as a lattice in $\mathbb{C}^n$. We will first prove that

$$\int_{\mathbb{C}^n} \{\mathrm{SD}(f \circ \tau_\xi)\}^p \mathrm{d}V(\xi) \leqslant C \sum_{b \in \frac{1}{N}\mathbb{Z}^{2n}} \Big\{ \int_{Q_N} \int_{Q_N} |f(z+b) - f(w+b)|^2 \mathrm{d}V(w) \mathrm{d}V(z) \Big\}^{p/2}$$

where Q_N is the cube $[-\frac{1}{N}, \frac{2}{N})^{2n}$ in $\mathbb{R}^{2n}$ and C depends only on n, N, and p. The proof is very similar to the proof of Lemma 3.4 in [6], though we include it for the sake of the reader. We will next show that for large enough N and each $b \in \frac{1}{N}\mathbb{Z}^{2n}$,

$$\int_{Q_N} \int_{Q_N} |f(z+b) - f(w+b)|^2 \mathrm{d}V(w) \mathrm{d}V(z)$$
$$\leqslant C \int_{Q_N+b} \Big| \int_{Q_N+b} (f(z) - f(w)) \mathrm{e}^{\langle z,w \rangle} \mathrm{e}^{-|w|^2/2} \mathrm{e}^{-\mathrm{iIm}\langle b,w \rangle} \mathrm{d}V(w) \Big|^2 \mathrm{e}^{-|z|^2} \mathrm{d}V(z)$$

where C only depends on n, N, and p. To prove this, we will make crucial use of the fact that N is chosen to be sufficiently large, and it is for this reason alone that we work with the lattice $\frac{1}{N}\mathbb{Z}^{2n}$ rather than $\mathbb{Z}^{2n}$.

Next, it is easy to see that H_f and $H_{\overline{f}}$ are both in S_p for all $0 < p < \infty$ if and only if the first order commutator $[M_f, P]$ is in S_p. With this in mind, we will fix some N large enough and estimate $\|W\|_{S_p}$ directly, where $W = A[M_f, P]B$ and where A and B are some bounded operators that will depend on our fixed N. This will be done by the standard general method employed to handle estimating the Schatten p quasinorm of special classes of integral operators for $0 < p < 1$ (for example, see [4] and [5]). For each $M \in \mathbb{N}$, we will first appropriately decompose $\frac{1}{N}\mathbb{Z}^{2n}$ as the disjoint union of lattices $\{\Lambda_j^M\}_{j \in \{1,\dots,M\}^{2n}}$, and analogously decompose $W = \bigoplus_{j \in \{1,\dots,M\}^{2n}} W_j$. We will break up each $W_j = D_j + E_j$ where D_j is a diagonal operator and E_j is an off-diagonal operator, so that $\|W_j\|_{S_p}^p \geqslant \|D_j\|_{S_p}^p - \|E_j\|_{S_p}$. Finally, we will show that our choices of A and B, and the results from above, give us that $\sum_{j \in \{1,\dots,M\}^{2n}} \|D_j\|_{S_p}^p$ is bounded below by $C \int_{\mathbb{C}^n} \{\mathrm{SD}(f \circ \tau_\xi)\}^p \mathrm{d}V(\xi)$, and that $\sum_{j \in \{1,\dots,M\}^{2n}} \|E_j\|_{S_p}^p$ is bounded above by $C_M \int_{\mathbb{C}^n} \{\mathrm{SD}(f \circ \tau_\xi)\}^p \mathrm{d}V(\xi)$, where $C > 0$ is a constant that only depends on n, N and p, and C_M is a constant depending on n, N, p, and M with $\lim_{M \to \infty} C_M = 0$. Thus, with this fixed N, we can complete the proof by setting M large enough.

2. PRELIMINARIES

For each $N \in \mathbb{N}$, let $\frac{1}{N}\mathbb{Z}^{2n}$ denote the set $\{(\frac{k_1}{N}, \dots, \frac{k_{2n}}{N}) \in \mathbb{R}^{2n} : k_i \in \mathbb{Z}\}$. A subset $S = \{p_0, \dots, p_k\}$ of $\frac{1}{N}\mathbb{Z}^{2n}$ with $k \geqslant 1$ is said to be a *discrete segment* in $\frac{1}{N}\mathbb{Z}^{2n}$ if there exists $j \in \{1, \dots, 2n\}$ and $z \in \mathbb{Z}^{2n}$ such that

$$p_i = z + \frac{i}{N} e_j, \quad 0 \leqslant i \leqslant k$$

where e_j is the standard j^{th} basis vector of $\mathbb{R}^{2n}$. In this setting, we say that p_0 and p_k are the *endpoints* of S. Also, we define the *length* of S to be $|S| = k$. Let $\nu = (\frac{\nu_1}{N}, \dots, \frac{\nu_{2n}}{N})$ and $\nu' = (\frac{\nu'_1}{N}, \dots, \frac{\nu'_{2n}}{N})$ be elements of $\frac{1}{N}\mathbb{Z}^{2n}$ where $\nu \neq \nu'$. We can enumerate the integers $\{j : \nu_j \neq \nu'_j, 1 \leqslant j \leqslant 2n\}$ as $j_1, \dots, j_m$ in ascending order, so that $j_1 < \cdots < j_m$ when $m > 1$. Set $z_0(\nu, \nu') = \nu$, and inductively define $z_t(\nu, \nu') = z_{t-1}(\nu, \nu') + \frac{\nu'_{j_t} - \nu_{j_t}}{N} e_{j_t}$ for $t \in \{1, \dots, m\}$. Note that $z_m(\nu, \nu') = \nu'$. Let $S_t(\nu, \nu')$ be the discrete segment in $\frac{1}{N}\mathbb{Z}^{2n}$ which has $z_{t-1}(\nu, \nu')$ and $z_t(\nu, \nu')$ as its endpoints. The union of the discrete segments $S_1(\nu, \nu'), \dots, S_m(\nu, \nu')$ will be denoted by $\Gamma(\nu, \nu')$. We call $\Gamma(\nu, \nu')$ *the discrete path in* $\frac{1}{N}\mathbb{Z}^{2n}$ *from* ν *to* ν'. Furthermore, we define the *length* $|\Gamma(\nu, \nu')|$ of $\Gamma(\nu, \nu')$ to be $|S_1(\nu, \nu')| + \cdots + |S_m(\nu, \nu')|$. That is, the length of $\Gamma(\nu, \nu')$ is just the sum of the lengths of the discrete segments which make up $\Gamma(\nu, \nu')$. If $\nu' = 0$, we let $\Gamma(\nu)$ denote $\Gamma(\nu, \nu')$. In the case $\nu = \nu'$, we define the discrete path from ν to ν to be the singleton set $\Gamma(\nu, \nu) = \{\nu\}$.

Let S_N denote the cube $S_N = [0, \frac{1}{N})^{2n}$ and let Q_N be the cube $[-\frac{1}{N}, \frac{2}{N})^{2n}$. For any $f \in L^2_{\text{loc}}(\mathbb{C}^n, \mathrm{d}V)$, write

$$J_N(f) = \int_{Q_N} \int_{Q_N} |f(z) - f(w)|^2 \mathrm{d}V(z) \mathrm{d}V(w).$$

If E is a Borel set with $0 < V(E) < \infty$, we will denote the mean value of f on E by f_E. That is,

$$f_E = \frac{1}{V(E)} \int_E f \mathrm{d}V.$$

Universal constants will be denoted by $C^1, C^2, \dots$ and will represent different values in the proofs of different results. To keep better track of the dependence of the various constants encountered, we will use subscripts to denote what a particular constant depends on (though we implicitly assume that all universal constants may depend on n and p).

Finally, we conclude this section by reviewing some necessary facts about Schatten class ideals, all of which can be found in [3]. Recall that for any $0 < p < \infty$, the Schatten p class $S_p \subset B(H)$ consists of operators T satisfying the condition $\|T\|_{S_p} < \infty$, where $\|\cdot\|_{S_p}$ is defined by

$$\|T\|_{S_p} = \{\operatorname{tr}(|T|^p)\}^{1/p} = \{\operatorname{tr}((T^*T)^{p/2})\}^{1/p}.$$

When $p \geqslant 1, \|\cdot\|_{S_p}$ defines a norm. However, when $0 < p < 1, \|\cdot\|_{S_p}$ only defines a quasinorm, which means that we have the following:

LEMMA 2.1. *If $0 < p < 1$, then for any Schatten p class operators T and S, we have that*

$$\|T+S\|_{S_p}^p \leqslant \|T\|_{S_p}^p + \|S\|_{S_p}^p.$$

For all $0 < p < \infty$, it is well known that S_p is a two sided ideal of the ring of bounded operators $B(H)$. More precisely, if $A, B \in B(H)$ and $T \in S_p$, then $ATB \in S_p$ with

$$\|ATB\|_{S_p} \leqslant \|A\|_{Op}\|T\|_{S_p}\|B\|_{Op}.$$

If $0 < p \leqslant 2$, then for any $T \in S_p$ and any orthonormal basis $\{f_n\}$ of H (where H is a separable Hilbert space), we have that

$$\|T\|_{S_p}^p \leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle Tf_n, f_k\rangle|^p.$$

3. MAIN RESULT : NECESSITY FOR $0 < p < 1$.

We will now follow the outline discussed in the introduction. As stated before, the details for sufficiency can be found in [6]. The results and proofs of the next three lemmas are very similar to Lemmas 3.2–3.4 in [6], though we include proofs for the sake of the reader.

LEMMA 3.1. *For any $f \in L^2_{\mathrm{loc}}(\mathbb{C}^n, \mathrm{d}V)$ and $\nu \in \frac{1}{N}\mathbb{Z}^{2n}$, we have*

$$\int_{S_N} |f\circ\tau_\nu - f_{S_N}|^2 \mathrm{d}V \leqslant \Big(N^{2n} + 2\frac{N^{4n}}{3^{2n}}|\Gamma(\nu)|\Big) \sum_{a\in\Gamma(\nu)} J_N(f\circ\tau_a). \tag{3.1}$$

Proof. The case $\nu = 0$ is trivial. If $\nu \neq 0$, enumerate the points in $\Gamma(\nu)$ as $a_0, a_1, \dots, a_\ell$ with $\ell = |\gamma(\nu)|$ in such a way that $a_0 = 0, a_\ell = \nu$, and

$$\{S_N + a_{j-1}\} \cup \{S_N + a_j\} \subset Q_N + a_{j-1}, \quad 1 \leqslant j \leqslant \ell.$$

By the triangle inequality,

$$\begin{aligned} &|(f\circ\tau_{a_j})_{S_N} - (f\circ\tau_{a_{j-1}})_{S_N}| \\ &\qquad \leqslant |(f\circ\tau_{a_j})_{S_N} - (f\circ\tau_{a_{j-1}})_{Q_N}| + |(f\circ\tau_{a_{j-1}})_{Q_N} - (f\circ\tau_{a_{j-1}})_{S_N}| \end{aligned} \tag{3.2}$$

for any $1 \leqslant j \leqslant \ell$. Since $V(S_N + a_j) = \frac{1}{N^{2n}}$ and $S_N + a_j \subset Q_N + a_{j-1}$, we have

$$|(f\circ\tau_{a_j})_{S_N} - (f\circ\tau_{a_{j-1}})_{Q_N}|^2 = N^{4n}\Big|\int_{S_N+a_j} \{f - f_{Q_N+a_{j-1}}\}\mathrm{d}V\Big|^2$$

$$\leqslant N^{2n} \int\limits_{Q_N+a_{j-1}} |f - f_{Q_N+a_{j-1}}|^2 \mathrm{d}V$$

$$= N^{2n} \int\limits_{Q_N} |f \circ \tau_{a_{j-1}} - (f \circ \tau_{a_{j-1}})_{Q_N}|^2 \mathrm{d}V = \frac{1}{2} \frac{N^{4n}}{3^{2n}} J_N(f \circ \tau_{a_{j-1}}).$$

Similarly,

$$|(f \circ \tau_{a_{j-1}})_{Q_N} - (f \circ \tau_{a_{j-1}})_{S_N}|^2 \leqslant \frac{1}{2} \frac{N^{4n}}{3^{2n}} J_N(f \circ \tau_{a_{j-1}}).$$

Thus, by (3.2),

$$|(f \circ \tau_{a_j})_{S_N} - (f \circ \tau_{a_{j-1}})_{S_N}| \leqslant \frac{N^{4n}}{3^{2n}} J_N(f \circ \tau_{a_{j-1}}) \quad 1 \leqslant j \leqslant \ell. \tag{3.3}$$

Now,

$$\int\limits_{S_N} |f \circ \tau_\nu - f_{S_N}|^2 \mathrm{d}V \leqslant 2 \int\limits_{S_N} \{ |f \circ \tau_\nu - (f \circ \tau_\nu)_{S_N}|^2 + |(f \circ \tau_\nu)_{S_N} - f_{S_N}|^2 \} \mathrm{d}V. \tag{3.4}$$

However,

$$2 \int\limits_{S_N} |f \circ \tau_\nu - (f \circ \tau_\nu)_{S_N}|^2 \mathrm{d}V = \frac{1}{V(S_N)} \int\limits_{S_N} \int\limits_{S_N} |f(w+\nu) - f(z+\nu)|^2 \mathrm{d}V(w) \mathrm{d}V(z)$$
$$\leqslant \frac{1}{V(S_N)} J_N(f \circ \tau_\nu) = N^{2n} J_N(f \circ \tau_{a_\ell})$$

and by (3.3),

$$\begin{aligned} |(f \circ \tau_\nu)_{S_N} - f_{S_N}|^2 &= |(f \circ \tau_{a_\ell})_{S_N} - (f \circ \tau_{a_0})_{S_N}|^2 \\ &\leqslant \Big\{ \sum_{j=1}^{\ell} |(f \circ \tau_{a_j})_{S_N} - (f \circ \tau_{a_{j-1}})_{S_N}| \Big\}^2 \\ &\leqslant \ell \sum_{j=1}^{\ell} |(f \circ \tau_{a_j})_{S_N} - (f \circ \tau_{a_{j-1}})_{S_N}|^2 \leqslant \ell \frac{N^{4n}}{3^{2n}} \sum_{j=1}^{\ell} J_N(f \circ \tau_{a_{j-1}}). \end{aligned} \tag{3.5}$$

But $\ell = |\Gamma(\nu)|$, so that (3.1) follows from (3.4) and (3.5). ∎

LEMMA 3.2. *For $f \in \mathcal{T}(\mathbb{C}^n)$, there exists $C_N > 0$ such that*

$$\sup_{z \in S_N} \int\limits_{\mathbb{C}^n} \Big| f \circ \tau_z - \int\limits_{\mathbb{C}^n} f \circ \tau_z \mathrm{d}\mu \Big|^2 \mathrm{d}\mu \leqslant C_N \sum_{\nu \in \frac{1}{N}\mathbb{Z}^{2n}} \sum_{a \in \Gamma(\nu)} \mathrm{e}^{-|\nu|^2/3} J_N(f \circ \tau_a). \tag{3.6}$$

Proof. For any $z \in S_N$, we have

$$\begin{aligned}
\int_{\mathbb{C}^n} \Big| f\circ\tau_z - \int_{\mathbb{C}^n} f\circ\tau_z \mathrm{d}\mu\Big|^2 \mathrm{d}\mu &\leqslant \int_{\mathbb{C}^n} |f\circ\tau_z - f_{S_N}|^2 \mathrm{d}\mu \\
&= \sum_{\nu\in\frac{1}{N}\mathbb{Z}^{2n}} \frac{1}{\pi^n} \int_{S_N+\nu} |f(w) - f_{S_N}|^2 \mathrm{e}^{-|w-z|^2} \mathrm{d}V(w) \\
&= \sum_{\nu\in\frac{1}{N}\mathbb{Z}^{2n}} \frac{1}{\pi^n} \int_{S_N} |(f\circ\tau_\nu)(w) - f_{S_N}|^2 \mathrm{e}^{-|(w-z)+\nu|^2} \mathrm{d}V(w) \qquad (3.7)\\
&\leqslant \sum_{\nu\in\frac{1}{N}\mathbb{Z}^{2n}} \frac{d(\nu)}{\pi^n} \int_{S_N} |f\circ\tau_\nu - f_{S_N}|^2 \mathrm{d}V,
\end{aligned}$$

where $d(\nu) = \exp\Big\{ -\inf_{w,\xi\in S_N} |(w-\xi)+\nu|^2 \Big\}$. Since $|(w-\xi)+\nu|^2 \geqslant |\nu|^2 + |w-\xi|^2 - 2|w-\xi||\nu| \geqslant \frac{|\nu|^2}{2} - |w-\xi|^2$, there exists $C_N^1 > 0$ such that $d(\nu) \leqslant C_N^1 \mathrm{e}^{-|\nu|^2/2}$. Obviously, $N|\nu|$ dominates the length of every discrete segment in $\Gamma(\nu)$, so that $|\Gamma(\nu)| \leqslant 2nN|\nu|$. Therefore, we have that

$$\Big(N^{2n} + 2\frac{N^{4n}}{3^{2n}}|\Gamma(\nu)|\Big) d(\nu) \leqslant C_N^1 \Big(N^{2n} + 4n\frac{N^{4n+1}}{3^{2n}}|\nu|\Big)\mathrm{e}^{-|\nu|^2/2} \leqslant C_N^2 \mathrm{e}^{-|\nu|^2/3}$$

and so (3.6) follows from the above inequality and plugging (3.1) into (3.7). ∎

LEMMA 3.3. *For $0 < p \leqslant 2$ and $f \in \mathcal{T}(\mathbb{C}^n)$, there exists $C_N > 0$ such that*

$$\int_{\mathbb{C}^n} \{SD(f\circ\tau_\xi)\}^p \mathrm{d}V(\xi) \leqslant C_N \sum_{b\in\frac{1}{N}\mathbb{Z}^{2n}} \{J_N(f\circ\tau_b)\}^{p/2}.$$

Proof. Since $\bigcup_{u\in\frac{1}{N}\mathbb{Z}^{2n}} \{S_N + u\} = \mathbb{C}^n$ and $V(S_N+u) = \frac{1}{N^{2n}}$, it is enough to show that

$$\begin{aligned}
\sum_{u\in\frac{1}{N}\mathbb{Z}^{2n}} \sup_{z\in S_N+u} \{SD(f\circ\tau_z)\}^p &= \sum_{u\in\frac{1}{N}\mathbb{Z}^{2n}} \sup_{z\in S_N} \{SD(f\circ\tau_z\circ\tau_u)\}^p \\
&\leqslant C_N \sum_{b\in\frac{1}{N}\mathbb{Z}^{2n}} \{J_N(f\circ\tau_b)\}^{p/2}.
\end{aligned}$$

Since $0 < p \leqslant 2$, Hölder's inequality applied to (3.6) gives that

$$\sup_{z\in S_N} \{SD(f\circ\tau_u\circ\tau_z)\}^p \leqslant C_N^1 \sum_{\nu\in\frac{1}{N}\mathbb{Z}^{2n}} \sum_{a\in\Gamma(\nu)} \mathrm{e}^{-p|\nu|^2/6} \{J_N(f\circ\tau_u\circ\tau_a)\}^{p/2}.$$

Since $\tau_u \circ \tau_a = \tau_{u+a}$, we have

$$\sum_{u\in\frac{1}{N}\mathbb{Z}^{2n}} \sup_{z\in S_N} \{SD(f\circ\tau_u\circ\tau_z)\}^p \leqslant C_N^1 \sum_{u\in\frac{1}{N}\mathbb{Z}^{2n}} \sum_{\nu\in\frac{1}{N}\mathbb{Z}^{2n}} \sum_{a\in\Gamma(\nu)} \mathrm{e}^{-p|\nu|^2/6}\{J_N(f\circ\tau_{u+a})\}^{p/2}$$
$$= C_N^1 \sum_{\nu\in\frac{1}{N}\mathbb{Z}^{2n}} \mathrm{e}^{-p|\nu|^2/6} \sum_{a\in\Gamma(\nu)} \sum_{u\in\frac{1}{N}\mathbb{Z}^{2n}} \{J_N(f\circ\tau_{u+a})\}^{p/2}$$
$$= C_N^1 \sum_{\nu\in\frac{1}{N}\mathbb{Z}^{2n}} \mathrm{e}^{-p|\nu|^2/6}\mathrm{card}(\Gamma(\nu)) \sum_{b\in\frac{1}{N}\mathbb{Z}^{2n}} \{J_N(f\circ\tau_b)\}^{p/2}.$$

Since $\mathrm{card}(\Gamma(\nu)) = 1 + |\Gamma(\nu)| \leqslant 1 + 2nN|\nu|$, it is clear that Lemma 3.3 holds. ∎

LEMMA 3.4. *There exists $N \in \mathbb{N}$ and $C_N > 0$ such that for any $f \in L^2_{\mathrm{loc}}(\mathbb{C}^n)$ and $\nu \in \frac{1}{N}\mathbb{Z}^{2n}$, we have*

$$\int_{Q_N+\nu} \Big| \int_{Q_N+\nu} (f(z)-f(w))\mathrm{e}^{\langle z,w\rangle}\mathrm{e}^{-|w|^2/2}\mathrm{e}^{-\mathrm{i}\mathrm{Im}\langle \nu,w\rangle}\mathrm{d}V(w)\Big|^2 \mathrm{e}^{-|z|^2}\mathrm{d}V(z) \geqslant C_N J_N(f\circ\tau_\nu).$$

Proof. Since

$$\mathrm{e}^{-|z|^2/2}\mathrm{e}^{\langle z,w\rangle}\mathrm{e}^{-|w|^2/2} = \mathrm{e}^{-|z|^2/2}\mathrm{e}^{\langle z,w\rangle/2}\mathrm{e}^{\langle z,w\rangle/2}\mathrm{e}^{-|w|^2/2}$$
$$= \mathrm{e}^{-|z|^2/2}|\mathrm{e}^{\langle z,w\rangle/2}|^2\mathrm{e}^{-|w|^2/2}\mathrm{e}^{\mathrm{i}\mathrm{Im}\langle z,w\rangle} = \mathrm{e}^{-|z-w|^2/2}\mathrm{e}^{\mathrm{i}\mathrm{Im}\langle z,w\rangle},$$

we have that

$$\int_{Q_N+\nu} \Big| \int_{Q_N+\nu} (f(z)-f(w))\mathrm{e}^{\langle z,w\rangle}\mathrm{e}^{-|w|^2/2}\mathrm{e}^{-\mathrm{i}\mathrm{Im}\langle \nu,w\rangle}\mathrm{d}V(w)\Big|^2 \mathrm{e}^{-|z|^2}\mathrm{d}V(z)$$
$$= \int_{Q_N+\nu} \Big| \int_{Q_N+\nu} (f(z)-f(w))\mathrm{e}^{-|z-w|^2/2}\mathrm{e}^{\mathrm{i}\mathrm{Im}\langle z,w\rangle}\mathrm{e}^{-\mathrm{i}\mathrm{Im}\langle \nu,w\rangle}\mathrm{d}V(w)\Big|^2 \mathrm{d}V(z)$$
$$= \int_{Q_N} \Big| \int_{Q_N} (f\circ\tau_\nu(z)-f\circ\tau_\nu(w))\mathrm{e}^{-|z-w|^2/2}\mathrm{e}^{\mathrm{i}\mathrm{Im}\langle z,w\rangle}\mathrm{d}V(w)\Big|^2 \mathrm{d}V(z).$$

Pick some $\delta > 0$ to be determined, and pick N large enough so that

$$\mathrm{e}^{-|z-w|^2/2}\mathrm{e}^{\mathrm{i}\mathrm{Im}\langle z,w\rangle} = 1 + \gamma_{z,w} \tag{3.8}$$

where $|\gamma_{z,w}| < \delta$ for any $(z,w) \in Q_N \times Q_N$. This implies that if $z \in Q_N$, then

$$\Big(\Big|\int_{Q_N} (f\circ\tau_\nu(z)-f\circ\tau_\nu(w))\mathrm{d}V(w)\Big| - \Big|\int_{Q_N} (f\circ\tau_\nu(z)-f\circ\tau_\nu(w))\gamma_{z,w}\mathrm{d}V(w)\Big|\Big)^2$$
$$\geqslant \Big|\int_{Q_N} (f\circ\tau_\nu(z)-f\circ\tau_\nu(w))\mathrm{d}V(w)\Big|^2$$
$$-2\Big|\int_{Q_N}(f\circ\tau_\nu(z)-f\circ\tau_\nu(w))\mathrm{d}V(w)\Big|\Big|\int_{Q_N}(f\circ\tau_\nu(z)-f\circ\tau_\nu(w))\gamma_{z,w}\mathrm{d}V(w)\Big|$$

$$\geqslant \Big| \int_{Q_N} (f \circ \tau_\nu(z) - f \circ \tau_\nu(w)) \mathrm{d}V(w) \Big|^2 - 2\delta \Big(\int_{Q_N} |f \circ \tau_\nu(z) - f \circ \tau_\nu(w)| \mathrm{d}V(w) \Big)^2.$$

Therefore, (3.8) and the triangle inequality implies that

$$\begin{aligned} &\int_{Q_N} \Big| \int_{Q_N} (f \circ \tau_\nu(z) - f \circ \tau_\nu(w)) \mathrm{e}^{-|z-w|^2/2} \mathrm{e}^{\mathrm{iIm}\langle z,w\rangle} \mathrm{d}V(w) \Big|^2 \mathrm{d}V(z) \\ &\geqslant \int_{Q_N} \Big[\Big| \int_{Q_N} (f \circ \tau_\nu(z) - f \circ \tau_\nu(w)) \mathrm{d}V(w) \Big|^2 \\ &\quad - 2\delta \Big(\int_{Q_N} |f \circ \tau_\nu(z) - f \circ \tau_\nu(w)| \mathrm{d}V(w) \Big)^2 \Big] \mathrm{d}V(z). \end{aligned} \tag{3.9}$$

However,

$$\begin{aligned} &\frac{1}{(V(Q_N))^2} \int_{Q_N} \Big| \int_{Q_N} (f \circ \tau_\nu(z) - f \circ \tau_\nu(w)) \mathrm{d}V(w) \Big|^2 \mathrm{d}V(z) \\ &= \int_{Q_N} |f \circ \tau_\nu - (f \circ \tau_\nu)_{Q_N}|^2 \mathrm{d}V \\ &= \frac{1}{2V(Q_N)} \int_{Q_N} \int_{Q_N} |f \circ \tau_\nu(z) - f \circ \tau_\nu(w)|^2 \mathrm{d}V(w) \mathrm{d}V(z) \end{aligned} \tag{3.10}$$

so that

$$\begin{aligned} &\int_{Q_N} \Big| \int_{Q_N} (f \circ \tau_\nu(z) - f \circ \tau_\nu(w)) \mathrm{d}V(w) \Big|^2 \mathrm{d}V(z) \\ &= \frac{1}{2} V(Q_N) \int_{Q_N} \int_{Q_N} |f \circ \tau_\nu(z) - f \circ \tau_\nu(w)|^2 \mathrm{d}V(w) \mathrm{d}V(z) \end{aligned} \tag{3.11}$$

whereas the Cauchy–Schwarz inequality gives us

$$\begin{aligned} &\int_{Q_N} \Big(\int_{Q_N} |f \circ \tau_\nu(z) - f \circ \tau_\nu(w)| \mathrm{d}V(w) \Big)^2 \mathrm{d}V(z) \\ &\leqslant V(Q_N) \int_{Q_N} \int_{Q_N} |f \circ \tau_\nu(z) - f \circ \tau_\nu(w)|^2 \mathrm{d}V(z) \mathrm{d}V(w). \end{aligned} \tag{3.12}$$

Finally, plugging (3.11) and (3.10) into (3.9) gives us that

$$\int_{Q_N} \Big| \int_{Q_N} (f \circ \tau_\nu(z) - f \circ \tau_\nu(w)) \mathrm{e}^{-|z-w|^2/2} \mathrm{e}^{\mathrm{iIm}\langle z,w\rangle} \mathrm{d}V(w) \Big|^2 \mathrm{d}V(z)$$

$$\geqslant \Big(\frac{1}{2} - 2\delta\Big) V(Q_N) \int\limits_{Q_N} \int\limits_{Q_N} |f \circ \tau_\nu(z) - f \circ \tau_\nu(w)|^2 \mathrm{d}V(w) \mathrm{d}V(z)$$

$$= \frac{3^{2n}}{N^{2n}} \Big(\frac{1}{2} - 2\delta\Big) \int\limits_{Q_N} \int\limits_{Q_N} |f \circ \tau_\nu(z) - f \circ \tau_\nu(w)|^2 \mathrm{d}V(w) \mathrm{d}V(z)$$

since $V(Q_N) = \frac{3^{2n}}{N^{2n}}$. Therefore, picking $0 < \delta < \frac{1}{4}$ and N corresponding to δ completes the proof of Lemma 3.4. ∎

4. PROOF OF THE MAIN RESULT

We can now prove our main result.

THEOREM 4.1. *Let $0 < p < 1$ and $f \in \mathcal{T}(\mathbb{C}^n)$. If H_f and $H_{\overline{f}} \in S_p$, then there exists a constant $C > 0$ independent of f such that*

$$\int\limits_{\mathbb{C}^n} \{SD(f \circ \tau_\xi)\}^p \mathrm{d}V(\xi) < C(\|H_f\|_{S_p}^p + \|H_{\overline{f}}\|_{S_p}^p).$$

Proof. Fix $N \in \mathbb{N}$ such that Lemma 3.4 holds. For $M \in \mathbb{N}$ to be determined later and each $j = (j_1, \ldots, j_{2n}) \in \{1, \ldots, M\}^{2n}$, set $\Lambda_j^M = \{\nu \in \frac{1}{N}\mathbb{Z}^{2n} : \nu = (\frac{1}{N}\nu_1, \ldots, \frac{1}{N}\nu_{2n})$ with each $\nu_\ell \equiv j_\ell \bmod M\}$. Since $[M_f, P] = [M_f, P]P + [M_f, P](I - P) = H_f - (H_{\overline{f}})^*$, we have that $[M_f, P] \in S_p$ with $\|[M_f, P]\|_{S_p}^p \leqslant \|H_f\|_{S_p}^p + \|H_{\overline{f}}\|_{S_p}^p$.

Let $\{e_\nu\}_{\nu \in \Lambda_j^M}$ be an orthonormal basis for $L^2(\mathbb{C}^n, \mathrm{d}\mu)$. Let

$$h_\nu(w) = \mathrm{e}^{|w|^2/2} \mathrm{e}^{-\mathrm{i}\mathrm{Im}\langle \nu, w\rangle} \chi_{Q_N+\nu}(w) \quad \text{and} \quad \xi_\nu(z) = \frac{\chi_{Q_N+\nu}(z)([M_f, P]h\nu(z))}{\|\chi_{Q_N+\nu}[M_f, P]h\nu\|}.$$

Set $W_j = A_j^*[M_f, P]B_j$ where $A_j e_\nu = \xi_\nu$ and $B_j e_\nu = h_\nu$, so that

$$\sum_{j \in \{1,\ldots,M\}^{2n}} \|W_j\|_{S_p}^p \leqslant M^{2n} \frac{3^{np}}{N^{np}} \|[M_f, P]\|_{S_p}^p \leqslant M^{2n} \frac{3^{np}}{N^{np}} (\|H_f\|_{S_p}^p + \|H_{\overline{f}}\|_{S_p}^p). \tag{4.1}$$

Fix $R \in \mathbb{N}$ and let $Z = \{\nu = (\nu_1, \ldots, \nu_{2n}) \in \frac{1}{N}\mathbb{Z}^{2n}$ where each $|\nu_i| \leqslant R\}$ so that for any $\nu, \nu' \in Z$, we have $\Gamma(\nu, \nu') \subset Z$. Let $Z_j = \Lambda_j^M \cap Z$ and let P_{Z_j} denote the orthogonal projection onto $\mathrm{span}\{e_\nu : \nu \in Z_j\}$, so that clearly $P_{Z_j} W_j P_{Z_j} f = \sum\limits_{\nu, \widetilde{\nu} \in Z_j} \langle f, e_\nu\rangle \langle W_j e_\nu, e_{\widetilde{\nu}}\rangle e_{\widetilde{\nu}}$. Let D_j be defined by $D_j f = \sum\limits_{\nu \in Z_j} \langle f, e_\nu\rangle \langle W_j e_\nu, e_\nu\rangle e_\nu$ and set $E_j = P_{Z_j} W_j P_{Z_j} - D_j$ so that $\|W_j\|_{S_p}^p \geqslant \|P_{Z_j} W_j P_{Z_j}\|_{S_p}^p \geqslant \|D_j\|_{S_p}^p - \|E_j\|_{S_p}^p$.

Thus, since D_j is diagonal, we have that

$$
\begin{aligned}
\|D_j\|_{S_p}^p &= \sum_{\nu\in Z_j} |\langle A_j^*[M_f,P]B_j e_\nu, e_\nu\rangle|^p = \sum_{\nu\in Z_j} \|\chi_{Q_N+\nu}[M_f,P]h\nu\|^p \\
&= \sum_{\nu\in Z_j}\Big(\int_{Q_N+\nu}\Big|\int_{Q_N+\nu}(f(z)-f(w))\mathrm{e}^{\langle z,w\rangle}\mathrm{e}^{-|w|^2/2}\mathrm{e}^{-\mathrm{iIm}\langle \nu,w\rangle}\mathrm{d}V(w)\Big|^2\mathrm{e}^{-|z|^2}\mathrm{d}V(z)\Big)^{p/2} \qquad (4.2)\\
&\geqslant C_N^1 \sum_{\nu\in Z_j}\{J_N(f\circ\tau_\nu)\}^{p/2},
\end{aligned}
$$

where the last inequality follows from Lemma 3.4.

We now get a upper bound for $\|E_j\|_{S_p}^p$. Since $0<p<1$, we have that

$$
\begin{aligned}
\|E_j\|_{S_p}^p &\leqslant \sum_{\nu\in\Lambda_j^M}\sum_{\nu'\in\Lambda_j^M} |\langle E_j e_\nu, e_{\nu'}\rangle|^p = \sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}} |\langle E_j e_\nu, e_{\nu'}\rangle|^p \\
&= \sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}} \Big|\frac{\langle [M_f,P]h_\nu, \chi_{Q_N+\nu'}[M_f,P]h_{\nu'}\rangle}{\|\chi_{Q_N+\nu'}[M_f,P]h_{\nu'}\|}\Big|^p \\
&\leqslant \sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}} \|\chi_{Q_N+\nu'}[M_f,P]h_\nu\|^p \qquad (4.3)\\
&= \sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}} \Big(\int_{Q_N+\nu'}\Big|\int_{Q_N+\nu}(f(z)-f(w))\mathrm{e}^{\langle z,w\rangle}\mathrm{e}^{-|w|^2/2} \\
&\qquad \mathrm{e}^{-\mathrm{iIm}\langle \nu,w\rangle}\mathrm{d}V(w)\Big|^2\mathrm{e}^{-|z|^2}\mathrm{d}V(z)\Big)^{p/2}.
\end{aligned}
$$

But by the Cauchy–Schwarz inequality, we have that

$$
\begin{aligned}
&\sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}} \Big(\int_{Q_N+\nu'}\Big|\int_{Q_N+\nu}(f(z)-f(w))\mathrm{e}^{\langle z,w\rangle}\mathrm{e}^{-|w|^2/2}\mathrm{e}^{\mathrm{iIm}\langle \nu,w\rangle}\mathrm{d}V(w)\Big|^2\mathrm{e}^{-|z|^2}\mathrm{d}V(z)\Big)^{p/2} \\
&\qquad\leqslant \frac{3^{np}}{N^{np}}\sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}}\Big(\int_{Q_N+\nu'}\int_{Q_N+\nu}|f(z)-f(w)|^2\mathrm{e}^{-|z|^2} \\
&\qquad\qquad |\mathrm{e}^{\langle z,w\rangle}|^2\mathrm{e}^{-|w|^2}\mathrm{d}V(w)\mathrm{d}V(z)\Big)^{p/2} \\
&\qquad= \frac{3^{np}}{N^{np}}\sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}}\Big(\int_{Q_N+\nu'}\int_{Q_N+\nu}|f(z)-f(w)|^2\mathrm{e}^{-|z-w|^2}\mathrm{d}V(w)\mathrm{d}V(z)\Big)^{p/2}
\end{aligned}
$$

$$\leqslant e^{-(np/4)((M-3)/N)^2}\frac{3^{np}}{N^{np}} \sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}}\Big(\int_{Q_N+\nu'}\int_{Q_N+\nu}|f(z)-f(w)|^2 e^{-|z-w|^2/2}dV(w)dV(z)\Big)^{p/2} \tag{4.4}$$

$$\leqslant C_N^2 e^{-(np/4)((M-3)/N)^2}\sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}} e^{-p|\nu-\nu'|^2/5} \Big(\int_{Q_N}\int_{Q_N}|f\circ\tau_{\nu'}(z)-f\circ\tau_\nu(w)|^2 dV(w)dV(z)\Big)^{p/2}.$$

Now, if z and w are both in Q_N, then enumerating the points in $\Gamma(\nu',\nu)\subset Z$ as $\{a_0,\dots,a_\ell\}$ in such a way that $a_0=\nu'$ and $a_\ell=\nu$,

$$\{S_N+a_{j-1}\}\cup\{S_N+a_j\}\subset Q_N+a_{j-1},\quad 1\leqslant j\leqslant \ell,$$

we have that

$$\begin{aligned}|f\circ\tau_{\nu'}(z)-f\circ\tau_\nu(w)|\\ \leqslant &|f\circ\tau_{\nu'}(z)-(f\circ\tau_{\nu'})_{Q_N}|+|(f\circ\tau_\nu)_{Q_N}-f\circ\tau_\nu(w)|\\ &+\sum_{j=1}^{\ell}|(f\circ\tau_{a_{j-1}})_{Q_N}-(f\circ\tau_{a_j})_{Q_N}|.\end{aligned} \tag{4.5}$$

However,

$$\begin{aligned}&|f\circ\tau_{\nu'}(z)-(f\circ\tau_{\nu'})_{Q_N}|+\sum_{j=1}^{\ell}|(f\circ\tau_{a_{j-1}})_{Q_N}-(f\circ\tau_{a_j})_{Q_N}|+|(f\circ\tau_\nu)_{Q_N}-f\circ\tau_\nu(w)|\\ &\leqslant\{2nN|\nu-\nu'|+2\}^{1/2}(|f\circ\tau_{\nu'}(z)-(f\circ\tau_{\nu'})_{Q_N}|^2+|(f\circ\tau_\nu)_{Q_N}-f\circ\tau_{\nu(w)}|^2\\ &\qquad+\sum_{j=1}^{\ell}|(f\circ\tau_{a_{j-1}})_{Q_N}-(f\circ\tau_{a_j})_{Q_N}|^2)^{1/2}\end{aligned}$$

which means that

$$\begin{aligned}&\Big(\int_{Q_N}\int_{Q_N}|f\circ\tau_{\nu'}(z)-f\circ\tau_\nu(w)|^2dV(w)dV(z)\Big)^{p/2}\\ &\leqslant\frac{3^{np}}{N^{np}}\Big((2nN|\nu-\nu'|+2)\int_{Q_N}|f\circ\tau_{\nu'}-(f\circ\tau_{\nu'})_{Q_N}|^2dV\Big)^{p/2}\\ &\quad+\frac{3^{2np}}{N^{2np}}\Big((2nN|\nu-\nu'|+2)\sum_{j=1}^{\ell}|(f\circ\tau_{a_{j-1}})_{Q_N}-(f\circ\tau_{a_j})_{Q_N}|^2\Big)^{p/2}\\ &\quad+\frac{3^{np}}{N^{np}}\Big((2nN|\nu-\nu'|+2)\int_{Q_N}|f\circ\tau_\nu-(f\circ\tau_\nu)_{Q_N}|^2dV\Big)^{p/2}.\end{aligned} \tag{4.6}$$

Now plug the first term of (4.4) into (4.3) and noting that

$$\{2nN|\nu-\nu'|+2\}^{p/2}\mathrm{e}^{-p|\nu-\nu'|^2/5} \leqslant C_N^5\mathrm{e}^{-p|\nu-\nu'|^2/6},$$

we get

$$\begin{aligned}
&C_N^6\mathrm{e}^{-(np/4)((M-3)/N)^2}\sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}}\mathrm{e}^{-p|\nu-\nu'|^2/5}\\
&\qquad\Big((2nN|\nu-\nu'|+2)\int_{Q_N}|f\circ\tau_{\nu'}-(f\circ\tau_{\nu'})_{Q_N}|^2\mathrm{d}V\Big)^{p/2}\\
&\leqslant C_N^7\mathrm{e}^{-(np/4)((M-3)/N)^2}\sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}}\mathrm{e}^{-p|\nu-\nu'|^2/6}\\
&\qquad\Big(\int_{Q_N}|f\circ\tau_{\nu'}-(f\circ\tau_{\nu'})_{Q_N}|^2\mathrm{d}V\Big)^{p/2}\\
&= C_N^8\mathrm{e}^{-(np/4)((M-3)/N)^2}\sum_{\nu'\in Z_j}\Big(\int_{Q_N}|f\circ\tau_{\nu'}-(f\circ\tau_{\nu'})_{Q_N}|^2\mathrm{d}V\Big)^{p/2}\\
&= C_N^9\mathrm{e}^{-(np/4)((M-3)/N)^2}\sum_{\nu'\in Z_j}\{J_N(f\circ\tau_{\nu'})\}^{p/2}
\end{aligned}\tag{4.7}$$

and by symmetry, we get the exact same estimate by plugging the third term of (4.4) into (4.3).

Now we plug in the second term of (4.4) into (4.3). Since $0<p\leqslant 1$, we only need to estimate the quantity

$$\mathrm{e}^{-(np/4)((M-3)/N)^2}\sum_{\nu\in Z_j}\sum_{\substack{\nu'\in Z_j\\ \nu'\neq\nu}}\sum_{j=1}^{\ell_{(\nu',\nu)}}\mathrm{e}^{-p|\nu-\nu'|^2/6}|(f\circ\tau_{a_{j-1}})_{Q_N}-(f\circ\tau_{a_j})_{Q_N}|^p\tag{4.8}$$

where for each ν' and ν in the above sum, $\Gamma(\nu',\nu)=\{a_0,\dots,a_{\ell_{(\nu',\nu)}}\}$.

As in the computation from equation (3.3), we have

$$\begin{aligned}
|(f\circ\tau_{a_{j-1}})_{Q_N}-(f\circ\tau_{a_j})_{Q_N}|^p &= |f_{Q_N+a_{j-1}}-f_{Q_N+a_j}|^p\\
&\leqslant |f_{Q_N+a_{j-1}}-f_{S_N+a_j}|^p+|f_{S_N+a_j}-f_{Q_N+a_j}|^p\\
&\leqslant C_N^{10}[(J_N(f\circ\tau_{a_j-1}))^{p/2}+(J_N(f\circ\tau_{a_j}))^{p/2}].
\end{aligned}$$

Now, if $\nu=(y_1,\dots,y_{2n})$ and $\nu'=(z_1,\dots,z_{2n})$ with $\nu\neq\nu'$, then by definition,

$$\Gamma(\nu',\nu)=\bigcup_{\ell=1}^{2n}\{(y_1,\dots,y_{\ell-1},u,z_{\ell+1},\dots,z_{2n})\in Z:\min\{y_\ell,z_\ell\}\leqslant u\leqslant\max\{y_\ell,z_\ell\}\}.$$

Therefore, combining the three summations in (4.5) into one single sum, this sum is taken over the set $\{u, v, v' : u \in Z,\ (v, v') \in \Gamma_u\}$ where

$$\Gamma_u \subseteq \bigcup_{\ell'=1}^{2n} \bigcup_{\ell=1}^{2n} \Gamma_u^{\ell'_+} \cup \Gamma_u^{\ell_-},$$

with

$$\Gamma_u^{\ell-} = \{v, v' \in Z : u = (u_1, \dots, u_{2n}),\ v = (u_1, \dots, u_{\ell-1}, u', x_{\ell+1}, \dots, x_{2n}),\\ v' = (y_1, \dots, y_{\ell-1}, u'', u_{\ell+1}, \dots, u_{2n}) \text{ where } u' \geqslant u_\ell \geqslant u''\}$$

and

$$\Gamma_u^{\ell+} = \{v, v' \in Z : u = (u_1, \dots, u_{2n}), v = (u_1, \dots, u_{\ell-1}, u', x_{\ell+1}, \dots, x_{2n}),\\ v' = (y_1, \dots, y_{\ell-1}, u'', u_{\ell+1}, \dots, u_{2n}) \text{ where } u' \leqslant u_\ell \leqslant u''\}.$$

Thus, after switching the order of summation, (4.5) is smaller than

$$C_N^{11} e^{-(np/4)((M-3)/N)^2} \sum_{u \in Z} \{J_N(f \circ \tau_u)\}^{p/2} \sum_{(v,v') \in \Gamma_u} e^{-p|v-v'|^2/6}. \tag{4.9}$$

If we denote $v = (v_1, \dots, v_{2n})$ and $v' = (v'_1, \dots, v'_{2n})$, and let $\pi_i : \mathbb{C}^n \leftarrow \mathbb{R}$ be the canonical projection onto the i^{th} factor, then

$$\sum_{(v,v') \in \Gamma_u} e^{-p|v-v'|^2/6} = \Big(\sum_{(v_1, v'_1) \in \pi_1 \times \pi_1(\Gamma_u)} e^{-p|v_1 - v'_1|^2/6} \Big) \cdots \Big(\sum_{(v_{2n}, v'_{2n}) \in \pi_{2n} \times \pi_{2n}(\Gamma_u)} e^{-p|v_{2n} - v'_{2n}|^2/6} \Big).$$

For some $\ell \in \{1, \dots, 2n\}$, we now get an estimate on

$$\Big(\sum_{(v_1, v'_1) \in \pi_1 \times \pi_1(\Gamma_u^{\ell+})} e^{-p|v_1 - v'_1|^2/6} \Big) \cdots \Big(\sum_{(v_{2n}, v'_{2n}) \in \pi_{2n} \times \pi_{2n}(\Gamma_u^{\ell+})} e^{-p|v_{2n} - v'_{2n}|^2/6} \Big) \tag{4.10}$$

for fixed u. The terms that involve some $\Gamma_u^{\ell-}$'s are treated similarly. Assume ℓ is neither 1 nor $2n$, since the case where $\ell = 1$ and $\ell = 2n$ is very similar. If $j \in \{1, \dots, \ell - 1\}$, then $|v_j - v'_j| = |u_j - v'_j|$, and if $j \in \{\ell + 1, \dots, 2n\}$, then $|v_j - v'_j| = |u_j - v_j|$. Moreover, since $v_\ell \leqslant u_\ell \leqslant v'_\ell$, (4.7) is smaller than

$$\Big(\sum_{v'_1 \in \frac{1}{N}\mathbb{Z}} e^{-p|u_1 - v'_1|^2/6} \Big) \cdots \Big(\sum_{v_\ell \leqslant u_\ell \leqslant v'_\ell} e^{-p|v_\ell - v_{\ell'}|^2/6} \Big) \cdots \Big(\sum_{v_{2n} \in \frac{1}{N}\mathbb{Z}} e^{-p|v_{2n} - u_{2n}|^2/6} \Big).$$

Thus, as we are holding u fixed, each of the factors corresponding to $j \neq \ell$ in the product above converge to a positive number that depends only on N and p. For the factor that corresponds to $j = \ell$, we have

$$\sum_{v_\ell \leqslant u_\ell \leqslant v'_\ell} e^{-p|v_\ell - v_{\ell'}|^2/6} = \sum_{v_\ell = -\infty}^{u_\ell} \sum_{k=(u_\ell - v_\ell)}^{\infty} e^{-pk^2/6N^2} \leqslant \sum_{v_\ell \in \mathbb{Z}} \sum_{k=|u_\ell - v_\ell|}^{\infty} e^{-pk^2/6N^2}\\ \leqslant C_N^{12} \sum_{v_\ell \in \mathbb{Z}} e^{-p|u_\ell - v_\ell|^2/12N^2}.$$

Since each $\sum\limits_{\nu_\ell \in \mathbb{Z}} \mathrm{e}^{-p|u_\ell - \nu_\ell|^2/12N^2}$ converges to a number that only depends on p and N, we have that (4.6) is smaller than

$$C_N^{13} \mathrm{e}^{-(np/4)((M-3)/N)^2} \sum_{u \in Z} \{J_N(f \circ \tau_u)\}^{p/2}$$

which therefore gives us that

$$\|E_j\|_{S_p}^p \leqslant C_N^{14} \mathrm{e}^{-(np/4)((M-3)/N)^2} \sum_{u \in Z} \{J_N(f \circ \tau_u)\}^{p/2}. \tag{4.11}$$

Going back to (4.1) and combining (4.2) with (4.8), we have that

$$\begin{aligned} M^{2n} \frac{3^{np}}{N^{np}} (\|H_f\|_{S_p}^p + \|H_{\overline{f}}\|_{S_p}^p) \geqslant \|[M_f, P]\|_{S_p}^p \geqslant \sum_{j \in \{1,\dots,M\}^{2n}} \|W_j\|_{S_p}^p \geqslant \sum_{j \in \{1,\dots,M\}^{2n}} \|P_Z W_j P_Z\|_{S_p}^p \\ \geqslant \sum_{j \in \{1,\dots,M\}^{2n}} (\|D_j\|_{S_p}^p - \|E_j\|_{S_p}^p) \\ \geqslant (C_N^3 - C_N^{14} M^{2n} \mathrm{e}^{-(np/4)((M-3)/N)^2}) \sum_{u \in Z} \{J_N(f \circ \tau_u)\}^{p/2}. \end{aligned}$$

Pick $M > 0$ large enough so that $C_N^3 - C_N^{14} M^{2n} \mathrm{e}^{-(np4)((M-3)/N)^2} > 0$. Thus, since the above estimate holds for any $R \in \mathbb{N}$ (and recalling that $Z = \{\nu = (\nu_1, \dots, \nu_{2n}) \in \frac{1}{N}\mathbb{Z}^{2n}$ with each $|\nu_i| \leqslant R\}$), Lemma 3.3 gives us

$$(\|H_f\|_{S_p}^p + \|H_{\overline{f}}\|_{S_p}^p) \geqslant C_N^{15} \int_{\mathbb{C}^n} \{\mathrm{SD}(f \circ \tau_\xi)\}^p \mathrm{d}V(\xi).$$

For some constants $C_N^{15} > 0$, which proves the theorem. ∎

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