Invitation to the Proof of Fermat's Last "Theorem"

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March, 2011

Abstract

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Fermat's Last "Theorem" (ca. 1637) was finally proved in the mid-1990s by using the study of plane cubic curves of the form

$$y^2 \;=\; (x-A)(x-B)(x-C)$$

where A, B, and C are distinct integers.
This talk will provide an overview of the main ingredients.

1 The Statement

There is no solution in positive integers x, y, z of the equation

$$x^n + y^n \;=\; z^n$$

for $n \geq 3$.
Note: There are infinitely many essentially different solutions when $n = 1, 2$.

2 Old History

- The statement is equivalent to the statement that there are no non-zero integers x, y, z satisfying $x^n + y^n = z^n$ for $n \geq 3$.
- The statement is equivalent to the statement that there are no rational points off the coordinate axes on the plane curve $x^n + y^n = 1$ for $n \geq 3$.
- For odd exponents $n \geq 3$ the statement is equivalent to the statement that there are no non-zero integers such that $x^n + y^n + z^n = 0$.

3 Old History (continued)

- If the theorem is true when the exponent is a given n, then it is certainly true when the exponent is a multiple of that value of n.
- The case where n is 3 or 4 can be handled within the realm of "elementary" number theory. (See, for example, the classic text of Hardy & Wright.)
- Any integer $n \geq 3$ not divisible by 4 must be divisible by an odd prime.
- It remains to prove the theorem when the exponent n is a prime $p \geq 5$.

4 A Solution leads to a Cubic Curve

Let $p \geq 5$ be prime.
Suppose there are non-zero integers a, b, c such that

$$a^p + b^p = c^p \quad .$$

Then the plane cubic curve

$$y^2 = x(x - a^p)(x + b^p) \quad .$$

is an *elliptic curve* "defined over" $\mathbf{Q}$ — the Frey-Hellegouarch curve.

5 Cubic Curves

Over any field K, e.g., $\mathbf{Q}$, $\mathbf{C}$, or $\mathbb{F}_p = \mathbf{Z}/p\mathbf{Z}$, after a (projective) change of coordinates in K a non-singular cubic curve with at least one K-valued point may be brought into "generalized Weierstrass form"

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6 \;,$$

and over an algebraically closed field of characteristic $\neq 2, 3$ into an equation of the form

$$y^2 \;=\; (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \quad .$$

The latter is a non-singular cubic when

$$\Delta \;=\; \left(\prod_{i<j} (\lambda_i - \lambda_j)\right)^2 \neq 0 \quad .$$

Example: For the Frey-Hellegouarch curve

$$\Delta \;=\; (abc)^{2p} \quad .$$

6 Cubics over the Complex Numbers

Given a non-singular cubic curve C,

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6 \;,$$

with coefficients in $\mathbf{C}$ and $\Delta \neq 0$, the set of all solutions (x, y) in $\mathbf{C}^2$ together with the "distinguished point at infinity" forms a compact Riemann surface of genus one — a torus.

7 The Projective Plane

For a given field K

$$\mathbf{P}^2(K) \;=\; K^2 \cup \text{ (line at infinity)}$$

where

$$\begin{aligned}
\text{line at infinity} &= \{\text{classes of parallel lines}\} \text{ in } K^2 \\
&= \{\text{lines through } (0,0)\} \text{ in } K^2 \\
&= \{\text{slopes of lines}\} \;\cup\; (\infty) \\
&= K \;\cup\; (\infty)
\end{aligned}$$

Each line contains one and only one point (its parallel class) on the line at infinity. The "distinguished point at infinity" is the parallel class of vertical lines.

8 A Line Meets a Cubic in 3 Points

Given a non-singular cubic curve C,

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6 \;,$$

with coefficients in K, every line in K^2 passing through 2 points of C meets C in a third point, allowing for multiplicities.
Proof. Parameterize the line and get a cubic equation in the parameter with two known roots in K.

9 The Distinguished Point at Infinity

Given a non-singular cubic curve C,

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6 \;,$$

with coefficients in K, the distinguished point at infinity in $\mathbf{P}^2(K)$ lies on C.

Proof. Introduce homogeneous coordinates $(x, y, z) \neq (0, 0, 0)$ in $\mathbf{P}^2$ where:

- $(x_1, y_1, z_1) \equiv (x_2, y_2, z_2)$ if and only if $(x_2, y_2, z_2) \;=\; t(x_1, y_1, z_1)$ for some scalar $t \neq 0$.
- $(x, y, 1)$ is a homogeneous triple for the affine point (x, y).
- (x, y, z) is a homogenous triple for an affine point when $z \neq 0$.
- (x, y, z) represents a point on the line at infinity if $z \;=\; 0$.
- $(1, m, 0)$ represents "slope" m on the line at infinity.
- $(0, 1, 0)$ represents the "distinguished point at infinity".

In homogeneous coordinates the curve C has the equation

$$y^2z + a_1xyz + a_3yz^2 \;=\; x^3 + a_2x^2z + a_4xz^2 + a_6z^3 \quad .$$

In homogeneous coordinates the line at infinity has the equation $z \;=\; 0$.

The intersection of the line at infinity with C has the equation $x^3 \;=\; 0$. Thus, C meets the line at infinity "triply" in the distinguished point at infinity.

10 The Group Law

Given a non-singular cubic curve C,

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6 \;,$$

with coefficients in K, there is a unique "algebraic" group law on the points of C in $\mathbf{P}^2(K)$ characterized by the two conditions

1. The group origin 0 is the distinguished point at infinity.
2. For three points P, Q, R of C one has $P + Q + R \;=\; 0$ if and only if P, Q, R lie on a line.

Note: Although the commutative law is obviously automatic here, it is not easy to check the associative law.

11 The Group Negative

For a given point (c, d) on the cubic curve

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6 \;,$$

its negative in the group law is the point (c, d') where d, d' are the two roots of

$$y^2 + (a_1c + a_3)y \;=\; c^3 + a_2c^2 + a_4c + a_6 \;,$$

as a quadratic equation in y.

12 Elliptic Curves

- The non-singular cubic curves defined over K with at least one K-valued point are the "group objects" in the category of algebraic curves defined over K.
- For a curve in generalized Weierstrass form, the required K-valued point may always be taken to be the distinguished point at infinity.
- These are called elliptic curves.
- When $K = \mathbf{Q}$, much is known about them.
- *Modular forms* — objects associated with hyperbolic geometry — provide a dictionary for elliptic curves defined over $\mathbf{Q}$.
- The Frey-Hellegouarch curve cannot be in that dictionary.

13 The mod ℓ reduction of an elliptic curve

Let E be an elliptic curve of the form

$$y^2 \;=\; (x-A)(x-B)(x-C)$$

where A, B, C are distinct integers. When ℓ is a prime not dividing Δ (the square of the product of the root differences), E determines also a curve E_ℓ defined over the finite field $\mathbb{F}_\ell = \mathbf{Z}/\ell\mathbf{Z}$. E_ℓ is non-singular when l is not a factor of Δ.
For our purposes, i.e., in the case of the Frey-Hellegouarch curve, the *conductor* N of E may be defined to be

$$N \;=\; \prod_{\ell|\Delta} \ell \;,$$

the square-free part of Δ.
Let c_ℓ be defined by

$$c_\ell \;=\; 1 - |E(\mathbb{F}_\ell)| + \ell$$

when $l \not| N$. Here $|E(\mathbb{F}_\ell)|$ denotes the number of points of E_ℓ in the field $\mathbb{F}_\ell$.
c_ℓ is defined in a slightly more complicated way for each of the finitely many primes ℓ dividing N.

14 The L-series of E

One defines the "L-series" of E by forming the Euler product, indexed by primes ℓ as follows:

$$L(E,s) \;=\; \prod_{\ell|N} \frac{1}{1-c_\ell \ell^{-s}} \prod_{\ell \not| N} \frac{1}{1-c_\ell \ell^{-s} + \ell^{1-2s}}$$

Expanding the product, one obtains a Dirichlet series

$$L(E,s) \;=\; \sum_{k=1}^{\infty} \frac{c_k}{k^s} \;,$$

which converges for $\mathrm{Re}(s) > 3/2$
Series of this type have been seen in other contexts.

15 Isometries of the Upper-Half Plane

Let $\mathbf{H}$ be

$$\mathbf{H} \;=\; \{\tau \in \mathbf{C} \,|\, \mathrm{Im}(\tau) > 0\} \quad .$$

The group $G \;=\; \mathrm{SL}_2(\mathbf{R})$ operates via

$$M \cdot \tau \;=\; \frac{a\tau + b}{c\tau + d}, \quad M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a, b, c, d \in \mathbf{R}, \; ad - bc \;=\; 1$$

$G/\{\pm 1\}$ is the group of isometries (distance-preserving analytic maps) of $\mathbf{H}$ relative to the Poincaré metric

$$ds^2 \;=\; \frac{dx^2 + dy^2}{y^2} \;, \text{ for } \tau \;=\; x + iy \in H \quad .$$

(This is the connection with "hyperbolic geometry".)

16 Family of Elliptic Curves over C

Let G_w denote the Eisenstein series

$$G_w(\tau) \;=\; \text{const} \cdot \sum_{(m,n) \in \mathbf{Z}^2 - \{(0,0)\}} \frac{1}{(m\tau + n)^w} \;,$$

which converges normally for all $\tau \in \mathbf{H}, \; w \geq 4$.
$G_w(\tau)$ is not identically 0 for even $w \geq 4$, while it is self-cancelling for odd w.
For given τ with $g_4(\tau) \;=\; 60G_4(\tau)$, $g_6(\tau) \;=\; 140G_6(\tau)$ the equation

$$y^2 \;=\; 4x^3 - g_4(\tau)x - g_6(\tau)$$

gives rise to a cubic curve C_τ in classical Weierstrass form.
Every elliptic curve defined over $\mathbf{C}$ occurs this way, and one has

$$C_{\tau'} \cong C_\tau \;\Leftrightarrow\; \tau' \;=\; M \cdot \tau \text{ for} M \in \mathrm{SL}_2(\mathbf{Z}) \quad .$$

Thus, over $\mathbf{C}$

$$\{\text{isomorphism classes of elliptic curves}\} \;\cong \mathbf{H}/\mathrm{SL}_2(\mathbf{Z})$$

17 Modular Forms

The Eisenstein series are examples of modular forms: complex-valued holomorphic functions f in $\mathbf{H}$ satisfying

$$f(M \cdot \tau) \;=\; (c\tau + d)^w f(\tau)$$

for

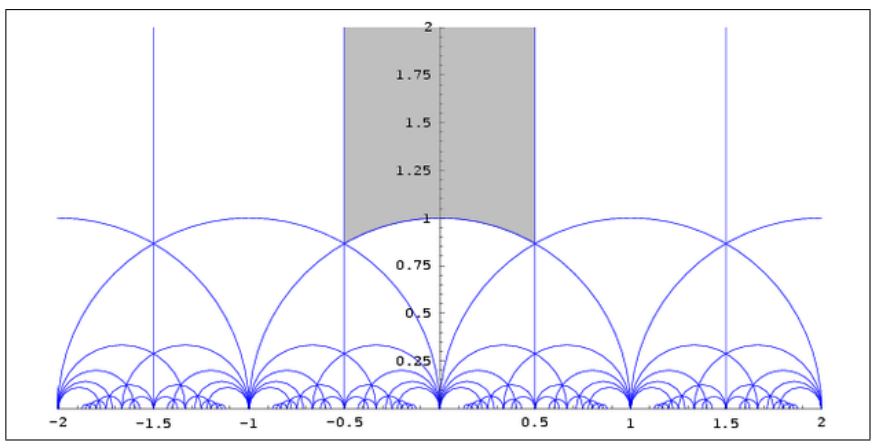
$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,, \quad \tau \in \mathbf{H} \quad .$$

where Γ is $\mathrm{SL}_2(\mathbf{Z})$ or a subgroup of finite index in $\mathrm{SL}_2(\mathbf{Z})$.

- The integer w is the *weight* of f.
- G_w is a modular form of weight k.
- A modular form is, more or less, a holomorphic section of a "line bundle" on the quotient space $\mathbf{H}/\Gamma$.
- Modular forms are also required to be "holomorphic at cusps", i.e., approach a finite limit at a "cusp" (see below).

18 Action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{H}$

The action of $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{H}$ is portrayed in this picture:



(Wikipedia image licensed under GFDL)

- The gray area is a fundamental domain. It has infinite extent.
- $\mathbf{H}/\Gamma$ is non-compact.

19 Cusps Compactify the Quotient

Let Γ be a subgroup of finite index in $\Gamma_0(1) = \mathrm{SL}_2(\mathbf{Z})$.

- $\mathbf{H}/\Gamma$ "covers" $\mathbf{H}/\Gamma_0(1)$
- Γ operates on $\mathbf{H}^* = \mathbf{H} \cup \mathbf{Q} \cup \{\infty\}$.
- For $\Gamma = \Gamma(1)$ the orbit of ∞ is $\mathbf{Q} \cup \{\infty\}$.
- For general Γ the number of orbits in $\mathbf{Q} \cup \{\infty\}$ is finite.
- $\mathbf{H}^*/\Gamma$ compactifies $\mathbf{H}/\Gamma$ by adjoining the finitely many "cusps".

20 Cusp Forms

Let Γ be a subgroup of finite index in $\Gamma_0(1) = \mathrm{SL}_2(\mathbf{Z})$. A modular form for Γ is a *cusp form* if its limiting value at each cusp of Γ is 0.
Example: For $\Gamma = \Gamma_0(1) = \mathrm{SL}_2(\mathbf{Z})$ the modular form

$$\lambda(\tau) = g_4(\tau)^3 - 27 g_6(\tau)^2$$

is a cusp form of weight 12 — the smallest weight of a cusp form for $\Gamma_0(1)$.

21 The Groups $\Gamma_0(N)$

Let $N \geq 1$ be a positive integer. The group $\Gamma_0(N)$ is given by

$$\left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\} \quad .$$

In particular

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N) \text{ for all } N \geq 1 \quad .$$

If f is a modular form, then

$$f(M_1 \cdot \tau) = f(\tau + 1) = f(\tau)$$

is periodic, so has a Fourier expansion

$$f(\tau) = \sum_{k \in \mathbf{Z}} c_k e^{2\pi i k \tau} \quad .$$

Because f is holomorphic at the cusp ∞ one has $c_k = 0$ for $k < 0$, and if f is a cusp form $c_0 = 0$ so that then

$$f(\tau) = \sum_{k=1}^{\infty} c_k e^{2\pi i k \tau} \quad .$$

N is called the *level*.

22 The Dirichlet Series

There are certain operators, called Hecke operators $\{T_w(m)\}_{m\geq 1}$, that act semi-simply on the space of cusp forms for $\Gamma_0(N)$ not coming from levels dividing N.
The structure of the algebra of these operators shows that if

$$f(\tau) \;=\; \sum_{k=1}^{\infty} c_k e^{2\pi i k\tau}$$

is a cusp form **of weight 2** that is a simultaneous eigenform of these operators then the corresponding Dirichlet series

$$\varphi_f(s) \;=\; \sum_{k=1}^{\infty} \frac{c_k}{k^s}$$

has an Euler product expansion just like the Euler product that is the L-function of an elliptic curve defined over $\mathbf{Q}$:

$$\varphi_f(s) \;=\; \prod_{\ell|N} \frac{1}{1-c_\ell \ell^{-s}} \prod_{\ell\not| N} \frac{1}{1-c_\ell \ell^{-s}+\ell^{1-2s}}$$

23 Cusp Forms of Weight 2 on $\Gamma_0(N)$

A cusp form f of weight 2 for $\Gamma_0(N)$ is essentially a regular differential on the quotient $X_0(N) = \mathbf{H}^*/\Gamma_0(N)$. When f, not coming from levels dividing N, is an eigenform of the Hecke operators, it determines in a straightforward way a 1-dimensional quotient variety of the Jacobian variety $J_0(N)$ of $X_0(N)$,

$$X_0(N) \quad\longrightarrow\quad J_0(N) \quad\longrightarrow E_f$$

which quotient is an elliptic curve E_f defined over $\mathbf{Q}$ with conductor N, and, therefore a regular map – the modular parametrization of E_f – from $X_0(N)$ to E_f with the property that the unique (up to a constant) regular differential on E_f pulls back to the differential on $X_0(N)$ determined by f.

24 The Dictionary for Elliptic Curves over Q

- Since the mid 20th century one has known that a cusp form f of weight 2 for $\Gamma_0(N)$, not also residing at a level dividing N, that is a simultaneous eigenform for the Hecke operators gives rise to an elliptic curve E_f defined over $\mathbf{Q}$ with conductor N.
- The L-function of E_f is the Dirichlet series $\varphi_f(s)$ associated with f.
- The "Modular Curve Conjecture", which originated in the mid 20th century, is that every elliptic curve defined over $\mathbf{Q}$ is isogenous to one of those obtained from such a cusp form. (Isogenous elliptic curves share L-functions.)
- In the mid 1980s it was shown using the theory of representations of the Galois group of the field of all algebraic numbers that the dictionary for elliptic curves defined over $\mathbf{Q}$ provided by the modular curve conjecture (and the extensive knowledge of modular forms) could not possibly contain the Frey-Hellegouarch curve.
- In the 1990s the "Modular Curve Conjecture" was proved.
- Fermat's Last Theorem is a corollary of that above.

25 Dictionary Trivia

- 11 is the smallest value of N for which there is a non-zero cusp form of weight 2 for the group $\Gamma_0(N)$. In this case the dimension of the space of cusp forms is 1. There are 3 non-isomorphic but isogenous elliptic curves with conductor 11:

$$\begin{aligned} y^2 + y &= x^3 - x^2 - 10x - 20 \\ y^2 + y &= x^3 - x^2 - 7820x - 263580 \\ y^2 + y &= x^3 - x^2 \end{aligned}$$

- The Cremona database — an encoding of the dictionary — has been built into *Sage* (`http://www.sagemath.org/`). Documentation for its use may be found at `http://www.sagemath.org/doc/reference/sage/databases/cremona.html`.
- The smallest conductor having more than 1 isogeny class is 26, which has 2.
- The smallest conductor having more than 2 isogeny classes is 57, which has 3.
- There are 38402 isogeny classes with conductors smaller than 10000.

26 For More Information

G. Cornell, J. Silverman, & G. Stevens,
Modular Forms and Fermat's Last Theorem,
Springer, 1997
— the record of an instructional conference held at Boston University in August, 1995

27 Acknowledgement

The XHTML + MathML version of these slides uses W3C's *Slidy* by Dave Raggett, a JavaScript/CSS package for sizing and flow control of an HTML or XHTML slide show.
(The slides were generated in a non-standard fashion from GELLMU source.)