SPLITTING ASSEMBLY MAPS FOR ARITHMETIC GROUPS WITH LARGE ACTIONS AT INFINITY

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Abstract. We construct a new compactification of a non-compact rank one globally symmetric space X. The result is a non-metrizable space $\hat{X}$ which also compactifies the Borel–Serre enlargement $\bar{X}$ of X, contractible only in the appropriate Čech sense, and with the action of any arithmetic subgroup of the isometry group of X on $\bar{X}$ not being small at infinity. Nevertheless, we show that such a compactification can be used in the approach to Novikov conjectures developed recently by Gunnar Carlsson and Erik K. Pedersen. In particular, we study the non-trivial instance of the phenomenon of bounded saturation in the boundary $\hat{X} - \bar{X}$ and deduce that integral assembly maps split in the case of a torsion-free arithmetic subgroup or, in fact, any lattice in a semi-simple algebraic $\mathbb{Q}$-group of real rank one.

Using a similar construction we also split assembly maps for neat subgroups of Hilbert modular groups. Extending the results in another direction we do the same for torsion-free lattices in the semi-simple group SL_3 of split rank two.

Introduction

Let Γ be a discrete group. Consider the assembly map $\alpha\colon B\Gamma_+ \wedge S(R) \longrightarrow S(R\Gamma)$ where $S(R)$ is the K- or L-theory spectrum for a ring R. There are also A-theoretic and C^*-theoretic versions of this map. It is known that for $S = L$ and $R = \mathbb{Z}$, the splitting of this map implies the classical form of the Novikov conjecture on the homotopy invariance of the higher signature for manifolds with the fundamental group Γ. By analogy, each of the other versions is called the *Novikov conjecture in S-theory*, and there are separate reasons for proving each of them (see §3.2). In the presence of torsion, assembly maps do not split, so attention is naturally restricted to torsion-free groups.

G. Carlsson, E. K. Pedersen, and W. Vogell verified the conjecture in K-, L- and A-theories for groups satisfying certain conditions ([20, 22]). For the sake of simplicity we state only the K-theoretic version:

Theorem 1 (Carlsson–Pedersen)**.** *Suppose there exists $E\Gamma$ such that the Γ-action is cocompact and extends to a contractible, metrizable compactification $\hat{X}$ of $E\Gamma$ so that the action of Γ on $\hat{X}$ is small at infinity, then α is a split injection.*

If Γ acts on a space X with an equivariant compactification $\hat{X} = X \cup Y$, the action is called *small at infinity* if for every $y \in Y$, compact subset $Z \subseteq X$, and neighborhood $U \subseteq \hat{X}$ of y, there exists a neighborhood $V \subseteq \hat{X}$ of y such that if $gZ \cap V \neq \emptyset$ for some $g \in \Gamma$ then $gZ \subseteq U$. Examples of such situations are the ideal

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compactification of a complete non-positively curved manifold (with a cocompact action) or the analogous constuction of the Rips complex associated to a Gromov hyperbolic group. Notice that these are essentially geometric compactifications performed with one's eye toward extending (quasi-)isometries to the boundaries. In particular, this property of the action means that every subset of Y is boundedly saturated in the sense of the following definition. A set $A \subseteq Y$ is *boundedly saturated* if for every closed set C in $\hat{X}$ with $C \cap Y \subseteq A$ the closure of any of the d-neighborhoods of $C \backslash Y$ satisfies $\overline{(C\backslash Y)[d]} \cap Y \subseteq A$.

After a considerable refinement of the methods ([18, 21, 65]), this result has been improved to

Theorem 2 (Carlsson–Pedersen)**.** *Suppose there exists $E\Gamma$ with the one-point compactification $E\Gamma^+$ such that the Γ-action on $E\Gamma$ is cocompact and extends to a Čech-acyclic compactification $\hat{X} = E\Gamma \sqcup Y$ so that there is a Γ-invariant system $\{\alpha\}$ of coverings of Y by boundedly saturated open sets and a weak homotopy equivalence*

$$\underset{\mathcal{U}\in \mathcal{C}ov E\Gamma^+}{\underleftarrow{\operatorname{holim}}} (N\mathcal{U} \wedge KR) \simeq \Sigma \underset{\alpha}{\underleftarrow{\operatorname{holim}}} (N\alpha \wedge KR),$$

then α is a split injection.

This theorem is part of a very general approach initiated in [16, 17]. The statement of Theorem 2 and its modification that we actually use will be explained in more detail in §5.

The purpose of this article is to provide examples where these new phenomena appear and get used. A general torsion-free arithmetic group seems to admit similar constructions, but then their analysis becomes more involved. From such a perspective, this paper completes the first two steps in a general inductive argument. Here we prove

Theorem 3. *Let G be a semi-simple linear algebraic group defined over $\mathbb{Q}$ of real rank one. If Γ is a torsion-free arithmetic subgroup of G then α is a split injection.*

The main body of this paper deals with the K-theoretic assembly map. In §A.1 we discuss the extension of this result to other versions of the map. The arithmeticity hypothesis in Theorem 3 can also be dropped. When the construction of Borel and Serre in our argument is replaced by an "intrinsic" construction à la Grayson ([43]) and the reduction theory of Garland and Raghunathan ([39]) is used, the proof of Theorem 3 works verbatim for arbitrary torsion-free lattices in linear semi-simple Lie groups of rank one. This can be further generalized to non-linear rank one groups where the reduction still works (see [64, pp. 14–17]). All of these groups may be classified as hyperbolic relative to a finite family of nilpotent subgroups in the sense of [30]. It seems very plausible that using a cross of the constructions of Rips and Borel–Serre, our argument also applies in this combinatorial situation.

Many of our constructions and results can be done and hold in greater generality than needed for the proof of Theorem 3. For example, §6 compactifies $E\Gamma$ for a torsion-free finitely generated nilpotent group Γ. We could follow it by a proof of the Novikov conjecture for such groups which is not a new result by itself. The importance of §6 is the role as the base case it plays in the construction of $\hat{X}$ for an arbitrary arithmetic group. Here the action of Γ on $\hat{X}$ is already not small at infinity. This property is preserved in the ambient construction for G from

Theorem 3 where copies of $\hat{X}$ for certain one or two step nilpotent groups embed. §A.3 contains a discussion of this situation and its relation to other approaches to Novikov conjectures.

In §A.2 the argument is adjusted slightly to apply to lattices in semi-simple Lie groups of higher $\mathbb{R}$-rank:

Theorem 4. *If Γ is a neat arithmetic subgroup of a Hilbert modular group then α is a split injection.*

The use of a "topological" approach as in Theorem 2 seems to be essential in both of our applications. Recall that no $SL_2(\mathcal{O}_d)$ is bicombable ([40, Proposition 6.14]) and neither of the groups Γ in $G \neq SO(n,1)$ from Theorem 3 is combable ([29, Theorem 1.2]). These results make it doubtful that our groups have reasonable geometric compactifications with small actions at infinity for it is precisely the combings that are used to produce examples after Theorem 1.

Appendix B together with Examples 8.3.3 and 9.4.7 is an illustration of the constructions and arguments for the simplest case of a rank one group $G = SL_2$. It is also used as a model for the argument in Appendix C which proves Novikov conjectures for arithmetic subgroups of the standard parabolic subgroup P_1 of $G = SL_3$. The group P_1 is not reductive. In its turn, this material is used when we deal with arithmetic subgroups of SL_3 in Appendix D where we prove

Theorem 5. *If Γ is a torsion-free lattice in $SL_3(\mathbb{R})$ then α is a split injection.*

There is an index in the back which lists the unconventional notation.

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Contents

Part 1. **Preliminary Material**

This is a collection of definitions and results drawn together in order to establish uniform notation and make future referral convenient.

1. Homotopy Theoretic Preliminaries

1.1. **Simplicial Sets.** Let $\mathcal{C}$ be any category and $\mathcal{O}$ be the category of finite ordered sets $\Delta_n = (0, 1, \ldots, n)$, $n \geq 0$, and order preserving maps, that is, maps $\mu\colon \Delta_n \to \Delta_m$ such that $\mu(i) \leq \mu(j)$ for $i < j$. A *simplicial object in* $\mathcal{C}$ is a contravariant functor from $\mathcal{O}$ to $\mathcal{C}$. Taking natural transformations to be morphisms one forms the category of simplicial objects $\mathrm{S}{-}\mathcal{C}$. Since order-preserving maps are compositions of elementary morphisms ($0 \leq i \leq n$)

$$\delta_i\colon \Delta_{n-1} \longrightarrow \Delta_n \ \ni \ \delta_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases}$$

and

$$\sigma_i\colon \Delta_{n+1} \longrightarrow \Delta_n \ \ni \ \sigma_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i, \end{cases}$$

a simplicial object $F \in \mathrm{S}{-}\mathcal{C}$ can be specified by the object map F and all maps $\partial_i = F\delta_i$ and $s_i = F\sigma_i$ called *face* and *degeneracy operators*. The elements of $F(\Delta_n)$ are called *n-simplices*.

Example 1. Recall that a *simplicial complex* K with *vertices* K^0 is a set of finite subsets of K^0 called *simplices* such that every non-empty subset of an element of K is itself an element of K. Each such K gives rise to a simplicial set $\tilde{K}$ (i.e., an element of s$-$Sets) as follows: an n-simplex of $\tilde{K}$ is a sequence $(a_0, \ldots, a_n) \in (K^0)^n$ such that the set $\{a_0, \ldots, a_n\}$ is an m-simplex of K for some $m \leq n$, and

$$(F\delta_i)(a_0, \ldots, a_n) = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n),$$
$$(F\sigma_i)(a_0, \ldots, a_n) = (a_0, \ldots, a_i, a_i, a_{i+1}, \ldots, a_n).$$

The terminology and the directions of these maps may seem funny. The reason is that simplicial sets are generalizations not of simplicial complexes but of singular chain complexes as the next example explains.

Example 2. Let $S_n(X)$ be the set of singular n-simplices of a topological space X. The *total singular complex* $S(X) \in$ s$-$Sets is defined by setting $S(X)(\Delta_n) = S_n(X)$ and

$$(F\delta_i)(f)(t_0, \ldots, t_{n-1}) = f(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}),$$
$$(F\sigma_i)(f)(t_0, \ldots, t_{n+1}) = f(t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}),$$

where f is a singular n-simplex of X defined on

$$\{(t_0, \ldots, t_n) : 0 \leq t_i \leq 1, \textstyle\sum t_i = 1\} \subseteq \mathbb{R}^{n+1}.$$

A simplicial set F is *Kan* if for every collection $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in F(\Delta_n)$ which satisfies the compatibility condition $F(\delta_i)x_j = F(\delta_{j-1})x_i$, $i < j$, $i \neq k$, $j \neq k$, there exists $x \in F(\Delta_{n+1})$ such that $F(\delta_i)x = x_i$ for $i \neq k$.

There is a homotopy theory of simplicial sets ([63]) which is useful because it is equivalent to the homotopy theory of topological spaces (see [13, Part II]). The equivalence is given by precisely the total singular complex functor S and the *geometric realization* functor R ([63]) in the opposite direction. Thus F and $(S \circ R)(F)$ are weakly equivalent. The latter simplicial set is always Kan, however, while F may not be Kan.

For every small category $\mathcal{C}$ the *underlying simplicial set* $U(\mathcal{C})$ has any sequence of n composable morphisms in $\mathcal{C}$

$$u = (c_0 \xleftarrow{\alpha_1} \ldots \xleftarrow{\alpha_n} c_n)$$

for an n-simplex with the obvious faces and degeneracies (omitting the i-th object by composing the incoming and the outgoing morphisms or simply deleting when $i \in \{0, n\}$ for ∂_i; extending the sequence by composing with id_{c_i} for s_i). Functors $\mathcal{C} \to \mathcal{D}$ of small categories correspond precisely to simplicial maps $U(\mathcal{C}) \to U(\mathcal{D})$. Now homotopy theoretic notions can be transported to small categories. In particular, $\mathcal{C}$ is *contractible* if $R\big(U(\mathcal{C})\big)$ is contractible.

Given a functor $F \colon \mathcal{C} \to \mathcal{D}$, for every object $D \in \mathcal{D}$, let $F{\downarrow}D$ denote the *over category* consisting of all arrows $F(C) \to D$, $C \in \mathcal{C}$, in $\mathcal{D}$ with morphisms $F(c) \colon F(C) \to F(C')$ for some particular $c \colon C \to C' \in \mathcal{C}$ which make the resulting triangles of arrows in $\mathcal{D}$ commute. Inverting the arrows above gives the definition of the *under category* $D{\downarrow}F$.

1.2. **Homotopy Limits.** A functor from a small category $F \colon \mathcal{C} \to \mathcal{D}$ is also called a *$\mathcal{C}$-diagram in $\mathcal{D}$*. Recall that the limit and the colimit of F are objects of $\mathcal{D}$ characterized by certain universal properties. They may not exist for an arbitrary diagram in s−Sets. The homotopy limit and colimit are simplicial sets which exist for any diagram F and satisfy universal properties with homotopy theoretic flavor.

Homotopy limits are natural in both variables. Thus a natural transformation $\phi\colon F \to G$ of functors F, $G\colon \mathcal{C} \to$ s−Sets induces

$$\operatorname{holim}\phi\colon \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, F \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, G,$$

and a functor $\Phi\colon \mathcal{C} \longrightarrow \mathcal{D}$ induces a natural map

$$\operatorname{holim}\Phi\colon \underset{\overleftarrow{\mathcal{D}}}{\operatorname{holim}}\, F \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, F\circ\Phi.$$

Similar maps are naturally induced between homotopy colimits.

We now list basic properties of homotopy limits and colimits which will be referred to later.

Theorem 1.2.1 ([13, XI, §3])**.** *There are natural maps*

$$\underset{\overleftarrow{\mathcal{C}}}{\lim}\, F \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, F \quad \textit{or} \quad \underset{\overrightarrow{\mathcal{C}}}{\operatorname{hocolim}}\, F \longrightarrow \underset{\overrightarrow{\mathcal{C}}}{\operatorname{colim}}\, F$$

whenever the appropriate limit or colimit exists.

Theorem 1.2.2 (Homotopy Invariance, [13, XI, §5])**.** *Let $\phi\colon F \to G$ be a natural transformation of functors such that each $\phi(C)\colon F(C) \to G(C)$, $C \in \mathcal{C}$, is a weak equivalence. Then* $\operatorname{hocolim}\phi$ *is a weak equivalence. If $F(C)$ and $G(C)$ are Kan for all $C \in \mathcal{C}$ then* $\operatorname{holim}\phi$ *is also a weak equivalence.*

Theorem 1.2.3 (Cofinality Lemma, [13, XI, §9])**.** *Let $\Phi\colon \mathcal{C} \to \mathcal{C}'$ and $F\colon \mathcal{C}' \to$* s−Sets *be functors from small categories. If Φ is right cofinal (that is, $C'\downarrow\Phi$ is non-empty and contractible for every $C' \in \mathcal{C}'$) then* $\operatorname{hocolim}\Phi$ *is a weak equivalence. If Φ is left cofinal (that is, $\Phi\downarrow C'$ is non-empty and contractible for every $C' \in \mathcal{C}'$) and each $F(C')$, $C' \in \mathcal{C}'$, is Kan, then* $\operatorname{holim}\Phi$ *is a weak equivalence.*

Theorem 1.2.4 (Push Down Theorem, [24, §9])**.** *Let $\Phi\colon \mathcal{C} \to \mathcal{C}'$ and $F\colon \mathcal{C} \to$* s−Sets *be functors with each $F(C)$, $c \in \mathcal{C}$, Kan. Let $\Phi_*F\colon \mathcal{C}' \to$* s−Sets *be the functor given by*

$$\Phi_*(C') = \underset{\overleftarrow{C'\downarrow\Phi}}{\operatorname{holim}}\, F\circ j,$$

where j is the forgetful functor: $j\big(C' \to \Phi(C)\big) = C$. Then the obvious map

$$\Phi_!\colon \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, F \longrightarrow \underset{\overleftarrow{\mathcal{C}'}}{\operatorname{holim}}\, \Phi_*F$$

is a weak equivalence.

Theorem 1.2.5 ([24, §9])**.** *Let $\mathcal{C}$ be a contractible small category and $F\colon \mathcal{C} \to$* s−Sets *be a functor such that, for each morphism $c \in \mathcal{C}$, $F(c)$ is a weak equivalence. Then, for every object $C \in \mathcal{C}$, the obvious map*

$$\tau_F(C)\colon FC \longrightarrow \underset{\overrightarrow{\mathcal{C}}}{\operatorname{hocolim}}\, F$$

is a weak equivalence. If each $F(C)$, $C \in \mathcal{C}$, is Kan then

$$\tau^F(C)\colon \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, F \longrightarrow FC$$

is a weak equivalence.

Theorem 1.2.6 (Fibration Lemma, [17, I.1]). *Let $\phi_1 \colon F \to G$ and $\phi_2 \colon G \to H$ be natural transformations of F, G, $H \colon \mathcal{C} \to$ s$-$Sets so that for each $C \in \mathcal{C}$ the values $F(C)$, $G(C)$, $H(C)$ are all Kan, and $F(C) \to G(C) \to H(C)$ is a homotopy fibration. Then*

$$\underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\ F \xrightarrow{\mathrm{holim}\,\phi_1} \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\ G \xrightarrow{\mathrm{holim}\,\phi_2} \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\ H$$

is a homotopy fibration.

Theorem 1.2.7 ([13, XI, §5]). *Let $\phi_1 \colon F \to G$ and $\phi_2 \colon G \to H$ be natural transformations of $F, G, H \colon \mathcal{C} \to$ s$-$Sets so that, for each $C \in \mathcal{C}$, $F(C) \to G(C) \to H(C)$ is a cofibration sequence of spectra. Then*

$$\underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\ F \xrightarrow{\mathrm{holim}\,\phi_1} \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\ G \xrightarrow{\mathrm{holim}\,\phi_2} \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\ H$$

is a cofibration sequence.

We assume familiarity with the language of spectra ([1]). All of the results above generalize to simplicial spectra ([17]), the notion of homotopy (co)limit being extended via a level-wise construction in the obvious way. The foundational material on simplicial spectra can be found in [83, §5].

Theorem 1.2.8 (Bousfield–Kan Spectral Sequence [13, 17]). *Given a functor $F \colon \mathcal{C} \to$ Spectra, let $\pi_i \circ F \colon \mathcal{C} \to$ AbGroups be the composition with the stable π_i. Then there is a spectral sequence converging to*

$$\pi_*(\underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\ F) \quad \text{with} \quad E_2^{p,q} = \underset{\overleftarrow{\mathcal{C}}}{\lim}{}^p\ (\pi_q \circ F).$$

The following strengthening of the general Cofinality Lemma is extremely useful.

Theorem 1.2.9 (Modified Cofinality Lemma). *Let $\mathcal{P}$ be a left filtering partially ordered set viewed as a category, and let $\imath \colon \mathcal{P}^0 \hookrightarrow \mathcal{P}$ be the inclusion of a partially ordered subset, also left filtering. Let $F \colon \mathcal{P} \longrightarrow$ Spectra be a functor, and assume the following two hypotheses.*

1. *For every $x \in \mathcal{P}$ there exist $x' \in \mathcal{P}$ and $y \in \mathcal{P}^0$ so that $x' > x$, $x' > y$, and so that $F(x' > y)$ is a weak equivalence.*
2. *Let x, $x' \in \mathcal{P}$, y, $y' \in \mathcal{P}^0$, with $x > y$, $x' > y'$, and $F(x > y)$ and $F(x' > y')$ be weak equivalences. Then there exist $x'' \in \mathcal{P}$ and $y'' \in \mathcal{P}^0$ with $x'' > x$, x'; $y'' > y$, y', and $x'' > y''$, so that $F(x'' > y'')$ is a weak equivalence.*

Then the restriction map

$$\imath^* \colon \underset{\overleftarrow{\mathcal{P}}}{\mathrm{holim}}\ F \longrightarrow \underset{\overleftarrow{\mathcal{P}^0}}{\mathrm{holim}}\ F$$

is a weak equivalence.

This theorem is part of the work of Carlsson and Pedersen currently in progress. For a proof in the situation where we use the theorem see Lemma 4.4.2.

1.3. **Steenrod Functors.** Consider the category $\mathcal{M}$ of all compact metrizable spaces. A functor $F\colon \mathcal{M} \to \textsc{Spectra}$ is called a *Steenrod functor* if

(1) $F\big(\mathrm{Cone}(X)\big)$ is contractible for any $X \in \mathcal{M}$,
(2) $F(K) \to F(X) \to F(X/K)$ is a homotopy fibration sequence for any closed subset $K \subseteq X$, and
(3) given a compact metric space $\bigvee X_i$ which is a countable wedge of metric spaces X_i, the projections induce a weak homotopy equivalence $F(\bigvee X_i) \simeq \prod F(X_i)$.

Theorem 1.3.1 (Milnor). *A natural transformation of Steenrod homology theories is an isomorphism on $\mathcal{M}$ if and only if it is an isomorphism on points.*

For the more precise and general statement and the proof see Lemma 3.3 of [22].

1.4. **Algebraic K-theory.** This describes what we mean by K-theory here. In [69] Quillen constructed K-groups of a ring R, $K_n(R)$, $n \geq 0$. Before that the lower K-groups $K_n(R)$, $-\infty < n \leq 2$, were studied by Bass, Milnor, and others. The groups of Quillen can be obtained as stable homotopy groups of connective spectra. The most suitable delooping machine to use in this situation is Thomason's ([84]) functor Spt. Pedersen and Weibel ([66, 67]) used this functor and controlled algebra to produce a non-connective spectrum $K(R)$ whose homotopy groups are all $K_n(R)$, $n \in \mathbb{Z}$. They also show that this agrees with the non-connective spectrum of Gersten and Wagoner.

The functor of Thomason constructs a connective spectrum for every small symmetric monoidal category. A *symmetric monoidal category* is a category $\mathcal{C}$ with some extra structure: a functor $\oplus\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and natural isomorphisms

$$\alpha\colon (A \oplus B) \oplus C \xrightarrow{\ \simeq\ } A \oplus (B \oplus C)$$
$$\gamma\colon A \oplus B \xrightarrow{\ \simeq\ } B \oplus A$$

subject to certain coherence conditions. A *lax symmetric monoidal functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a functor which also maps the monoidal structures: it comes with a natural transformation

$$\bar{f}\colon FA \oplus FB \longrightarrow F(A \oplus B)$$

subject to certain conditions (see [84]). Let $\textsc{Sym}{-}\textsc{Mon}$ be the category of small symmetric monoidal categories and lax symmetric monoidal functors, then

$$\mathrm{Spt}\colon \textsc{Sym}{-}\textsc{Mon} \longrightarrow \textsc{Spectra}.$$

If $\mathcal{F}$ is the symmetric monoidal category of isomorphisms of free finitely generated R-modules then $K^{\mathrm{conn}}(R) = \mathrm{Spt}(\mathcal{F})$ is the connective K-theory spectrum of R. In general, if $\mathcal{A}$ is a small additive category then the category of isomorphisms $i\mathcal{A}$ of $\mathcal{A}$ is a symmetric monoidal category.

Let $\mathcal{C}_k(\mathcal{A})$ denote the category of $\mathcal{A}$-objects parametrized over the metric space $\mathbb{Z}^k$ and bounded morphisms (the prototype of the categories defined in §7) then [67] constructs functorial maps

$$\mathrm{Spt}\big(i\mathcal{C}_k(\mathcal{A})\big) \longrightarrow \Omega\, \mathrm{Spt}\big(i\mathcal{C}_{k+1}(\mathcal{A})\big).$$

Taking

$$K(\mathcal{A}) = \underset{\overrightarrow{n>0}}{\mathrm{hocolim}}\ \Omega^n\, \mathrm{Spt}\big(i\mathcal{C}_n(\mathcal{A})\big)$$

one gets a non-connective spectrum.

Again, if $\mathcal{A}$ is the category of free finitely generated R-modules then $K(R) = K(\mathcal{A})$ is the Gersten–Wagoner spectrum.

2. Symmetric Spaces. Algebraic Groups. Arithmetic Subgroups

2.1. **Symmetric Spaces.** A *symmetric space of non-compact type* is an Hadamard manifold X, that is, a complete simply connected Riemannian manifold of non-positive sectional curvature, such that for each point $x \in X$ the geodesic symmetry $s_x \colon X \to X$ given by $\exp_x(v) \mapsto \exp_x(-v)$ for all $v \in T_x X$ is an isometry of X, and X is not compact but contains no Euclidean space as a Riemannian factor.

The connected isometry group $G = I_0(X)$ is a semi-simple Lie group with no compact factors and with trivial center. It is transitive on X, and $X \cong G/K$ where K is the maximal compact subgroup of G stabilizing a point $x \in X$. If G is a semi-simple Lie group with finite center and no compact factors, and if K is a maximal compact subgroup, then the homogeneous space G/K is a symmetric space of non-compact type.

A *k-flat* in X is a complete totally geodesic k-dimensional submanifold with zero sectional curvature. The *rank* of X is the maximal dimension of a k-flat in X. Let $\mathfrak{g}$ denote the Lie algebra of G. The k-flats in X which contain x are the orbits $A \cdot x$, where $A = \exp(\mathfrak{a})$, and $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$ of dimension k. If $k = \operatorname{rank} X$ then every geodesic γ of X is contained in at least one k-flat of X. If γ is contained in exactly one k-flat then it is called *regular*, otherwise it is *singular*.

2.1.1. *Ideal Boundary.* Every non-positively curved manifold may be compactified by attaching the *ideal boundary* ∂X and introducing the *cone topology* on $\varepsilon X = X \cup \partial X$ (cf. [5]).

Points of ∂X are asymptotic classes of geodesic rays, so the action of $G = I_0(X)$ on X extends to ∂X. If $x \in \partial X$ then $\operatorname{stab}(x)$ acts transitively on X. So, if γ is regular (respectively singular) then any geodesic of X asymptotic to γ is also regular (respectively singular). One gets classes of regular and singular points in ∂X which are invariant under the action of G.

For $x \in X$ and z_1, $z_2 \in \varepsilon X$, $z_1 \neq x \neq z_2$, the *angle* $\measuredangle_x(z_1, z_2)$ is defined as $\measuredangle\big(\dot{c}_1(0), \dot{c}_2(0)\big)$, where c_i are the unique geodesics from x to z_i and $\dot{c}_i$ denote the derivative functions. The cone topology on εX is generated by open sets in X and the cones

$$C_x(z, \epsilon) \stackrel{\text{def}}{=} \{y \in \varepsilon X : y \neq x,\ \measuredangle_x(z, y) < \epsilon\},$$

where $x \in X$, $z \in \partial X$, $\epsilon > 0$. The space εX is homeomorphic to a closed ball in the Euclidean n-space, and the topology induced on ∂X is called the *sphere topology* .

2.1.2. *Roots and Walls.* Now let $\mathfrak{a} \subseteq \mathfrak{g}$ be a maximal abelian subalgebra. For $H \in \mathfrak{a}$ consider $\operatorname{ad}(H) \colon \mathfrak{g} \to \mathfrak{g}$ defined by $Y \mapsto [H, Y]$. Each $\operatorname{ad}(H)$ is symmetric on $\mathfrak{g}$ with respect to the canonical inner product $\langle\,\cdot\,,\cdot\,\rangle = \operatorname{trace}(\,\cdot\,\cdot^t)$, so $\{\operatorname{ad}(H) : H \in \mathfrak{a}\}$ is a commutative family of symmetric linear operators. Decompose $\mathfrak{g}$ into a direct sum of common eigenspaces $\{\mathfrak{g}_\lambda : \lambda \in \Lambda\}$ that are orthogonal relative to $\langle\,\cdot\,,\cdot\,\rangle$:

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \sum_{\lambda \in \Lambda} \mathfrak{g}_\lambda$$

called the *root space decomposition of* $\mathfrak{g}$ determined by $\mathfrak{a}$. Here

$$\lambda \in \operatorname{Hom}(\mathfrak{a}, \mathbb{R}),$$
$$\mathfrak{g}_\lambda = \{Y \in \mathfrak{g} : \forall H \in \mathfrak{a} \ni \operatorname{ad}(H)(Y) = [H, Y] = \lambda(H) \cdot Y\},$$
$$\Lambda = \{\lambda \in \operatorname{Hom}(\mathfrak{a}, \mathbb{R}) : \mathfrak{g}_\lambda \neq \{0\},\ \lambda \neq 0\}.$$

The members of Λ are called *roots*.

We recall the characterization of regular points of ∂X and rephrase: $H \in \mathfrak{a}$ is tangent to a singular geodesic in the flat $F = \exp(\mathfrak{a})$ if and only if there is $Y \notin \mathfrak{g}_0$ with $[H, Y] = 0$, i.e., $\lambda(H) = 0$ for a root $\lambda \in \Lambda$. So the singular elements of $\mathfrak{a}$ form the set

$$\mathfrak{a}_{\text{sing}} = \{H \in \mathfrak{a} : \exists \lambda \in \Lambda \text{ with } \lambda(H) = 0\}$$

which consists of finitely many hyperplanes

$$\{H \in \mathfrak{a} : \lambda(H) = 0\}, \quad \lambda \in \Lambda.$$

The complement is $\mathfrak{a}_{\text{reg}}$. Its components are called *Weyl chambers* of $\mathfrak{a}$. Fix one Weyl chamber $\mathfrak{a}^+$ and define

$$\Lambda^+ = \{\lambda \in \Lambda : \lambda(H) > 0 \text{ for } H \in \mathfrak{a}^+\}.$$

A subset $\{\lambda_1, \ldots, \lambda_m\} \subseteq \Lambda^+$ is a *fundamental system* if it is a basis for $\mathfrak{a}$, and every $\lambda \in \Lambda^+$ can be written as $\lambda = \sum_i s_i \lambda_i,\ s_i \in \mathbb{N}$. The Weyl chamber $\mathfrak{a}^+$ is bounded by hyperplanes $\{H \in \mathfrak{a} : \lambda_i(H) = 0\}$. The *walls* of $\mathfrak{a}^+$ are the sets

$$\{H \in \mathfrak{a} : \lambda_{i_j}(H) > 0,\ j = 1, \ldots, r,\ \lambda_{i_j}(H) = 0,\ j = r+1, \ldots, m\}.$$

There are $\binom{m}{r}$ r-dimensional walls.

2.1.3. *Tits Building.* The *Weyl chambers* and *walls at infinity* are defined to be the intersections of Weyl chambers and walls in maximal flats in X with ∂X which becomes the disjoint union of all the Weyl chambers and walls at infinity. In a sense, ∂X is "tesselated" by Weyl chambers and their walls (see Schroeder's survey in [5]). The *Tits building* T is the simplicial complex obtained by taking a simplex for each Weyl chamber or wall in X and the face relation induced by the incidence relation. The dimension of a simplex is the dimension of the corresponding chamber or wall at infinity.

A flat $F \subseteq X$ is *asymptotic* to a chamber $W \in T$ if $W \subseteq \partial F$. The union of such chambers and their walls forms a subcomplex $\partial F \subseteq T$ called an *apartment*.

Theorem 2.1.1 ([5])**.** (1) *Every regular point in ∂X belongs to a Weyl chamber at infinity. Every singular point belongs to a wall at infinity.* (2) *Any two chambers at infinity are contained in some apartment.* (3) *If x_1, x_2 belong to the same Weyl chamber at infinity then* $\operatorname{stab}(x_1) = \operatorname{stab}(x_2)$.

In the case of rank $X = 1$ the Tits building is discrete. The apartments are pairs of points at the opposite limit points of the same geodesic. Part (2) of Theorem 2.1.1 says that X is a *visibility* space. In fact, the geodesics connecting the chambers at infinity are unique here.

2.1.4. *Example: Positive Definite Bilinear Forms.* The most important example of a symmetric homogeneous space is $G = SL_n(\mathbb{R})$, $K = SO_n(\mathbb{R})$, $n \geq 2$: every irreducible symmetric space X of non-compact type can be embedded isometrically as a totally geodesic submanifold of $X_n = SL_n(\mathbb{R})/\mathbb{SO}_{\ltimes}(\mathbb{R})$, where $n = \dim I_0(X)$. Let $\mathcal{P}_n$ be the space of positive definite bilinear forms, i.e.,

$$\mathcal{P}_n = \{x \in GL_n(\mathbb{R}) : \curvearrowleft = \curvearrowleft^{\approx},\ \curvearrowleft > \nvdash,\ \det \curvearrowleft = \nvDash\}.$$

Now $SL_n(\mathbb{R})$ acts on $\mathcal{P}_n$ by conjugation: $g \cdot x = gxg^t$. The isotropy group of $I \in \mathcal{P}_n$ is $\mathrm{stab}(I) = SO_n(\mathbb{R})$, so $\mathcal{P}_n \cong X_n$. On the other hand, the exponential map gives an explicit diffeomorphism between the (Euclidean) space

$$\mathfrak{p} = \{x \in M_n(\mathbb{R}) : \curvearrowleft = \curvearrowleft^{\approx},\ \mathrm{trace}\, \curvearrowleft = \nvdash\}$$

of dimension $-1 + \frac{1}{2}n(n+1)$ and $\mathcal{P}_n$. The ideal boundary ∂X_n can be identified with unit vectors in

$$T_I(X) \cong \mathfrak{p}_1 = \{Y \in M_n(\mathbb{R}) : \mathbb{Y} = \mathbb{Y}^{\approx},\ \mathrm{trace}\, \mathbb{Y} = \nvdash,\ \mathrm{trace}(\mathbb{Y}^{\nvDash}) = \nvDash\}$$

via $x \mapsto Y(x)$, where Y is uniquely determined by $x = \gamma_Y(\infty)$ for $\gamma_Y(t) = \exp(tY)(I) = \exp(2tY)$. Note that x is regular if and only if the eigenvalues of $Y(x)$ are all distinct. So for $n \geq 2$ the regular points form an open dense subset of ∂X_n, and the singular points form a closed nowhere dense subset.

Given a vector $Y \in \mathfrak{p}_1$, let $\lambda_1(Y) > \cdots > \lambda_k(Y)$ be the distinct eigenvalues of Y. Let $E_i(Y)$ be the eigenspace of Y associated to $\lambda_i(Y)$ and

$$V_i(Y) = \bigoplus_{j=0}^{i} E_j(Y).$$

The symmetric matrix Y (and the corresponding $x \in \partial X_n$) is completely determined by the vector $\lambda(Y) = \big(\lambda_1(Y), \ldots, \lambda_k(Y)\big) =: \lambda(x)$ and the flag $F(Y) = \big(V_1(Y), \ldots, V_k(Y)\big) =: F(x)$ in $\mathbb{R}^{\ltimes}$.

Theorem 2.1.2 (Eberlein [25])**.** *The action of G on ∂X_n can be expressed by the formula*

$$g \cdot \big(\lambda(x), F(x)\big) = \big(\lambda(gx), F(gx)\big) = (\lambda(x), g \cdot F(x)),$$

where $g \cdot F(x)$ is the standard action of $g \in SL_n(\mathbb{R})$ on the flag in $\mathbb{R}^{\ltimes}$.

Corollary 2.1.3. $g \in \mathrm{stab}(x)$ *if and only if* $g \cdot F(x) = F(x)$.

The equivalence classes $W(F) = \{x : F(x) = F\} \subseteq \partial X_n$ are the Weyl chambers or walls at infinity depending on whether F is a complete flag or not. They form a tesselation of ∂X_n and correspond to simplices in the *Tits building* so that the boundaries of maximal flats in X are subcomplexes called *apartments*.

The Tits building for X_3 is a graph. The apartments are circles subdivided by six arcs. Such computations can be found in Schroeder's article in [5].

2.1.5. *Horocycles.* Recall that to each Weyl chamber at infinity corresponds an Iwasawa decomposition $G = K \cdot A \cdot N$, where N is a connected nilpotent subgroup of G (see [48]). The Lie algebra of N is given by $\mathfrak{n} = \sum_{\lambda > 0} \mathfrak{g}_\lambda \subseteq \mathfrak{g}$. The nilpotent subgroups in various Iwasawa decompositions of G are all conjugate in G, they are the maximal unipotent subgroups of G.

A *horocycle* in X is an orbit of a subgroup of the form gNg^{-1} for any element g of G. Another way to define a horocycle is as the intersection of all horospheres passing through the same $x_0 \in X$ and based at points $x \in \partial X$ contained in the same Weyl chamber. When $\operatorname{rank} X = 1$, all horocycles are horospheres. If M is the centralizer of A in K then G/MN is the space of all horocycles (see [47]).

2.2. **Affine and Linear Groups.** A general reference for the material in this section is [79]. An *affine algebraic group* is an affine variety G over a field k which is also a group with the operation given by a k-morphism $m\colon G \times G \to G$. The maps between them are equivariant k-morphisms with respect to the left translation action of G on itself. An algebraic group is *linear* if it is a subgroup of some $GL_n(k)$ defined by polynomial equations in the entries a_{ij} and $\det^{-1}$, i.e., there is a set of polynomials $\{p_\alpha\}$, $\alpha \in \mathcal{A}$, such that $M = (a_{ij}) \in G$ if and only if $p_\alpha(a_{11}, \ldots, a_{nn}, \det^{-1} M) = 0$, $\alpha \in \mathcal{A}$.

Examples .

(1) The special linear group $SL_n(k)$ is defined by the polynomial equation $p(a_{11}, \ldots, a_{nn}) = \det M - 1 = 0$.
(2) The upper triangular group $T_n(k)$ is defined by $p_{ij}(a_{11}, \ldots, a_{nn}) = a_{ij} = 0$ for all pairs $1 \le j < i \le n$.
(3) The strict upper triangular group $U_n(k) \subseteq T_n(k)$ is defined by the polynomial system $p_{ij}(a_{11}, \ldots, a_{nn}) = a_{ij} = 0$ for all $1 \le j < i \le n$ and $p_{ii}(a_{11}, \ldots, a_{nn}) = a_{ii} - 1 = 0$ for all $1 \le i \le n$.
(4) The special orthogonal group $SO_n(k)$ is defined by n^2 polynomial conditions encoded by $M \cdot M^t = I$.
(5) The diagonal group $D_n(k)$ is defined by $n^2 - n$ conditions $a_{ij} = 0$ for $i \ne j$.

Given a linear algebraic group $G \subseteq GL_n(k)$, for any extension K of k the polynomials p_α can be viewed as polynomials over K and define the group of K-points $G(K) \subseteq GL_n(K)$. Note that there is a rational embedding $GL_n(k) \subseteq k^{n^2+1}$ defined by $M = (a_{ij}) \mapsto (a_{11}, \ldots, a_{nn}, \det^{-1} M)$. So the linear groups are defined to become affine subgroups of $GL_n(k)$. This shows that all $G(K)$ are also affine algebraic groups. In the harder opposite direction one has

Theorem 2.2.1. *Every affine algebraic group is linear.*

2.2.1. *Radicals.* Recall that an abstract group G is *solvable* (respectively *nilpotent*) if it has a composition series

$$\{e\} \subseteq G_n \subseteq \ldots \subseteq G_1 = G$$

with $[G_i, G_i] \subseteq G_{i+1}$ (respectively $[G, G_i] \subseteq G_{i+1}$). The standard examples of Zariski-connected solvable (respectively nilpotent) subgroups are $T_n(k)$ (respectively $U_n(k)$) in $GL_n(k)$.

The maximal connected closed normal solvable subgroup $R(G)$ of G is called the *radical* of G. A linear algebraic group is called *semi-simple* if $R(G) = \{e\}$. For example, SL_n is semi-simple, because the only proper normal subgroups of SL_n are contained in the finite center, so the connected radical must be $\{e\}$.

Let V be a finite-dimensional k-vector space. An endomorphism $A \in \operatorname{End}(V)$ is *semi-simple* if there is a basis of V consisting of eigenvectors of A. A is *nilpotent* if $A^N = 0$ for some N, and A is *unipotent* if $A - I$ is nilpotent.

Theorem 2.2.2. *Let $A \in \operatorname{End}(V)$. Then there is a unique decomposition $A = A_s + A_n$ with A_s semi-simple, A_n nilpotent, and $A_s A_n = A_n A_s$. Let $g \in GL(V)$. Then there is a unique decomposition $g = g_s \cdot g_u$ with g_s semi-simple, g_u unipotent, and $g_s g_u = g_u g_s$.*

If G is a linear algebraic group and $g \in G$ then g_s and $g_u \in G$. They form subsets $G_s, G_u \subseteq G$. If G is connected and solvable then G_u is a connected nilpotent subgroup. The *unipotent radical* of a linear algebraic group G is $R_u(G) = R(G)_u$. G is called *reductive* if $R_u(G) = \{e\}$.

2.2.2. *Parabolic Subgroups.* Let G be a linear algebraic group and P be a closed subgroup.

Definition 2.2.3. P is called *parabolic* if G/P is a complete variety. The maximal connected solvable subgroups are called *Borel subgroups* of G. Example: $T_n \subseteq GL_n$. Such a subgroup must contain the maximal *normal* connected solvable subgroup, the radical $R(G)$.

The totality of all parabolic subgroups of G will be denoted by $\mathcal{P} = \mathcal{P}(G)$. If $k' \subseteq k$ is a subfield then $\mathcal{P}_{k'} = \mathcal{P}_{k'}(G)$ will denote all parabolic subgroups defined over k'. Similar notation $\mathcal{B} = \mathcal{B}(G)$ and $\mathcal{B}_{k'} = \mathcal{B}_{k'}(G)$ will be used for Borel subgroups.

Theorem 2.2.4. *Borel subgroups are minimal parabolic subgroups and are conjugate in G. A closed subgroup of G is parabolic if and only if it contains a Borel subgroup. If G is connected then every $g \in G$ belongs to a Borel subgroup; for $A \in Aut(G)$, $B \in \mathcal{B}$, $A|B = Id \iff A = Id$; $B = N_G(B)$ for $B \in \mathcal{B}$, and $P = N_G(P)$ for $P \in \mathcal{P}$ (here $N_G(H)$ is the normalizer of H in G).*

2.2.3. *Tori.* Linear algebraic groups isomorphic to closed subgroups of some D_n are called diagonalizable. A *torus* is a connected diagonalizable group.

Theorem 2.2.5. *All maximal tori $\mathcal{T}$ are conjugate in G. Every maximal torus T is contained in a Borel subgroup.*

All tori are reductive which shows that semi-simple groups form a proper subclass of the reductive groups. In fact, for any connected reductive group G, $RG = Z(G)^0$ is a torus.

2.3. **Arithmetic Groups.** Let G be a linear algebraic group defined over $\mathbb{Q}$ and write $G(\mathbb{Z}) = G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$.

Definition 2.3.1. A subgroup Γ of $G(\mathbb{Q})$ is *arithmetic* if Γ and $G(\mathbb{Z})$ are commensurable, that is, if the subgroup $\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$. A discrete group Γ is *arithmetic* if it is isomorphic to an arithmetic subgroup of some group G.

Examples of arithmetic groups include all finite groups, finitely generated free groups, and finitely generated torsion-free nilpotent groups. They appear naturally in number theory, automorphic functions, representation theory, algebraic geometry, and K-theory. An interesting topological characterization due to Sullivan and Wilkerson is that the group of homotopy classes of self homotopy equivalences of a simply connected finite complex is arithmetic, and every arithmetic group occurs up to commensurability in this way ([76]).

Consider the real points $G(\mathbb{R})$ of G. It is a real Lie group, and $\Gamma \subseteq G(\mathbb{R})$ is a discrete subgroup. When G is semi-simple, Γ acts freely and properly discontinuously on the symmetric space X associated to $G(\mathbb{R})$. The quotient manifold $M = X/\Gamma$ is not compact unless $\operatorname{rank} G = 0$ but has finite invariant volume, i.e., Γ is a non-uniform lattice in $G(\mathbb{R})$.

The most famous class of arithmetic groups are *congruence subgroups* defined as the kernels of surjective maps $G(\mathbb{Z}) \to G(\mathbb{Z}_\ell)$ induced by reduction mod ℓ for various *levels* ℓ. Every arithmetic group contains a torsion-free subgroup of finite index, but, according to Minkowski, the congruence subgroups of SL_n of all levels $\ell \neq 2$ are themselves torsion-free (see [15, p. 40]).

2.4. **Examples of Groups Covered by Theorem 3.** If G is a connected linear simple Lie group with $\mathbb{R}$-rank one, there is a complete classification available ([85]). Denoting the identity $n \times n$ matrix by I_n, set

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad K_{p,q} = \begin{pmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{pmatrix}.$$

The four possibilities are

- $SO_0(1,n) = \{g \in GL_{n+1}(\mathbb{R}) : \eth^{\approx}\mathbb{I}_{\nVdash,\ltimes}\eth = \mathbb{I}_{\nVdash,\ltimes},\ \det \eth = \nVdash\}$,
- $SU(1,n) = \{g \in GL_{n+1}(\mathbb{C}) : \eth^{\approx}\mathbb{I}_{\nVdash,\ltimes}\bar{\eth} = \mathbb{I}_{\nVdash,\ltimes}\}$,
- $Sp\,(1,n) = \{g \in GL_{2(n+1)}(\mathbb{C}) : \eth^{\approx}\mathbb{J}_{\ltimes+\nVdash}\eth = \mathbb{J}_{\ltimes+\nVdash},\ \eth^{\approx}\mathbb{K}_{\nVdash,\ltimes}\bar{\eth} = \mathbb{K}_{\nVdash,\ltimes}\}$, and
- $\mathbb{F}_{\nsupseteq}$, the automorphism group of the exceptional simple Jordan algebra or, equivalently, the group of isometries of the Cayley projective plane with the appropriate Riemannian metric (see [2]).

The symmetric spaces for the classical groups are

$$SO_0(1,n)/SO(n), \quad SU(1,n)/S\big(U(1) \times U(n)\big), \quad Sp\,(1,n)/\big(Sp\,(1) \times Sp\,(n)\big).$$

Examples of torsion-free arithmetic subgroups here are congruence subgroups of level $\ell \geq 3$ of $SL(n+1,\mathbb{Z}) \cap \mathbb{SO}_{\nvdash}(\nVdash,\ltimes)$. This identifies a particular system of torsion-free arithmetic groups to which our Theorem 3 applies.

Part 2. **Assembly Maps**

3. Assembly in Algebraic K-theory

3.1. **Definition.** Let Γ be a discrete group and R a ring. The assembly map in algebraic K-theory

$$\alpha_n \colon h_n(B\Gamma; KR) \longrightarrow K_n(R\Gamma)$$

was first constructed by Jean-Louis Loday. Let $\imath\colon \Gamma \to GL_n(R\Gamma)$ be the inclusion of Γ in $(R\Gamma)^\times = GL_1(R\Gamma)$. Then there is a map

$$\Gamma \times GL_n(R) \xrightarrow{\imath \times id} GL_1(R\Gamma) \times GL_n(R) \xrightarrow{\otimes} GL_n(R\Gamma)$$

defined by

$$g, (a_{ij}) \longmapsto (g \cdot a_{ij}).$$

One can apply the classifying space functor B, pass to the limit as $n \to \infty$, and apply Quillen's plus construction to induce the map

$$B\Gamma_+ \wedge BGL(R)^+ \xrightarrow{B\imath^+ \wedge id} BGL(R\Gamma)^+ \wedge BGL(R)^+ \xrightarrow{\gamma} BGL(R\Gamma)^+.$$

This product is compatible with the infinite loop space structure of $BGL(\_)^+$ ([59, §11.2.16]). Delooping of this map results in the assembly map of spectra

$$\alpha \colon B\Gamma_+ \wedge K(R) \longrightarrow K(R\Gamma),$$

where $B\Gamma_+$ is the classifying space together with a disjoint base point, and $K(R)$ is the Gersten–Wagoner non-connective K-theory spectrum (see §1.4). This is the *assembly map in algebraic K-theory*. Loday's assembly map is induced by taking the homotopy groups:

$$h_n(B\Gamma; KR) = \pi_n(B\Gamma_+ \wedge KR) \xrightarrow{\alpha_n} \pi_n\big(K(R\Gamma)\big) = K_n(R\Gamma).$$

3.2. **Motivation.** There is at least a couple of reasons why the study of this map is of importance in geometric topology. One is the involvement of $K(\mathbb{Z}\Gamma)$ in the description of the space of automorphisms of a manifold M with $\pi_1 M = \Gamma$. The other is the connection with Novikov and Borel conjectures.

It is known that the classical version of the Novikov conjecture, i.e., the conjecture that

$$f_*\big(L(M) \cap [M]\big) = f_* g_*\big(L(M') \cap [M']\big) \in H_*(B\Gamma; \mathbb{Q}) \qquad (\ddagger)$$

whenever $g\colon M' \to M$ is a homotopy equivalence, follows from the splitting of the rational assembly map α in L-theory. The assembly naturally maps the rational group homology containing the signature to the surgery L-group where the image is a priori homotopy invariant. If the assembly is actually an injection then the signature is homotopy invariant. This is the modern approach to proving the Novikov conjecture. In fact, stronger integral conjectures may be stated when integral group homology is used, and there are K-, A-theoretic, and C^*-algebraic analogues of these integral maps. For example, the statement parallel to ($\ddagger$) about classes in $KO[\frac{1}{2}]$ is equivalent to integral injectivity of α (see [87]). It makes sense, therefore, to call the conjecture that the assembly map in K-theory is injective for torsion-free Γ *the integral Novikov conjecture in K-theory*. A stronger and geometrically important conjecture that α is an isomorphism is then the K-theoretical analogue of the Borel conjecture. For example, the vanishing of $Wh(\Gamma)$ would follow as a corollary to this.

There is another very interesting geometric application. The splitting of the C^*-algebraic version of the assembly map which can be obtained by applying the same approach as taken here (Carlsson–Pedersen–Roe) gives what J. Rosenberg calls the *strong Novikov conjecture*. That is known to imply rigidity and vanishing results for higher elliptic genera ([58]).

4. Modified Čech Homology

This section explains the setup for the recent work of Gunnar Carlsson and Erik Pedersen referred to in the Introduction.

4.1. **Classical Čech Homology.** Let $\mathcal{U}$ be an open covering of a topological space X. The *nerve* $N\mathcal{U}$ of $\mathcal{U}$ is the simplicial complex with members of $\mathcal{U}$ as vertices and a simplex $\{U_1, \ldots, U_s\}$ for each subset with $U_1 \cap \cdots \cap U_s \neq \emptyset$. We may think of $N\mathcal{U}$ as a simplicial set $N_\bullet\mathcal{U}$ in the way of Example 1. If $\mathcal{V}$ is another open covering of X, and for each $U \in \mathcal{U}$ there is $V(U) \in \mathcal{V}$ so that $U \subseteq V(U)$ then one says that $\mathcal{U}$ *refines* $\mathcal{V}$ and writes $\mathcal{U} > \mathcal{V}$. If $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ and $\mathcal{V} = \{V_\beta\}_{\beta\in\mathcal{B}}$ then the *map* of coverings $\mathcal{U} \to \mathcal{V}$ is a set map $f\colon \mathcal{A} \to \mathcal{B}$ such that $U_\alpha \subseteq V_{f(\alpha)}$ for all $\alpha \in \mathcal{A}$. Clearly, this map of vertices extends to a map of nerves $Nf\colon N\mathcal{U} \to N\mathcal{V}$. All such maps for one particular refinement $\mathcal{U} > \mathcal{V}$ are contiguous (see [27]) and, therefore, induce the same map on homology or homotopy groups of the nerves. Consider the partially ordered system $\mathcal{C}\mathrm{ov}X$ of all finite open coverings of X. The resulting inverse system of abelian groups $\{H_n(N\_\,; S)\}_{\mathcal{C}\mathrm{ov}X}$ always produces the inverse limit $\check{H}_n(X; S)$ called the n-th *Čech homology group*. The contiguity property of the refinement maps implies that the same inverse system is obtained as $\{H_n(N\_\,; S)\}_{\mathcal{C}\mathrm{ov}^{\mathrm{s}}X}$, where $\mathcal{C}\mathrm{ov}^{\mathrm{s}}X$ is the category of coverings and maps even though the functor itself is no longer a pro-abelian group.

Given a map of spaces $f\colon X \to Y$, any $\mathcal{V} \in \mathcal{C}\mathrm{ov}Y$ pulls back to a covering $f^*\mathcal{V} \in \mathcal{C}\mathrm{ov}X$ in the obvious way. The injections $Nf^*\mathcal{V} \to N\mathcal{V}$ induce the universal map

$$\lim_{\substack{\longleftarrow \\ f^*\mathcal{C}\mathrm{ov}Y}} H_n(N\_\,; S) \longrightarrow \check{H}_n(Y; S).$$

But the inclusion of inverse systems $f^*\mathcal{C}\mathrm{ov}Y \subseteq \mathcal{C}\mathrm{ov}X$ induces another universal map

$$\check{H}_n(X; S) \longrightarrow \lim_{\substack{\longleftarrow \\ f^*\mathcal{C}\mathrm{ov}Y}} H_n(N\_\,; S),$$

and the composition of the two is denoted by f_*. This makes $\check{H}_*$ functors. Actually, $\check{H}_*$ are almost a homology theory: they do not satisfy the exactness axiom ([27]). This is the classical Čech homology theory.

4.2. **Modified Čech Homology.** Another way to construct a functor similar in spirit is to take the inverse limit of the diagram of nerves $\{N\_\}_{\mathcal{C}\mathrm{ov}^{\mathrm{s}}X}$, or spectra $\{N\_ \wedge S\}_{\mathcal{C}\mathrm{ov}^{\mathrm{s}}X}$, or simplicial spectra $\{N_{\bullet}\_ \wedge S\}_{\mathcal{C}\mathrm{ov}^{\mathrm{s}}X}$, and then take homology groups, or stable homotopy groups, of the result. However, for the limit above to always exist, it must be a homotopy inverse limit. Notice also that the functor $N\_$ can only be defined on $\mathcal{C}\mathrm{ov}^{\mathrm{s}}X$ and not on $\mathcal{C}\mathrm{ov}X$. The maps are induced just as above.

Notation . Whenever we write $\operatorname{holim}(N\_ \wedge KR)$ we understand a simplicial spectrum, where $N\_$ stands for the simplicial set generated by the classical nerve complex via the total singular complex functor. The maps are usually induced from PL maps.

Remark 4.2.1. The values of the functor $N\colon \mathcal{C}\mathrm{ov}^{\mathrm{s}}\hat{X} \to$ S−SETS which sends a covering $\mathcal{U}$ to the simplicial nerve $N\mathcal{U}$ are not necessarily fibrant. To improve homotopy invariance properties of homotopy limits we adopt a convention which is (or may be) used in [17] and [18, 65]. Recall that there is a functorial replacement $K^\infty Q\colon \mathcal{S} \to \omega\mathcal{S}$ of a spectrum by a weakly equivalent Kan Ω-spectrum. The convention is that if $F\colon \mathcal{C} \to \mathcal{S}$ is a diagram whose values are not Kan Ω-spectra then the notation $\operatorname{holim}(F)$ will mean $\operatorname{holim}(K^\infty QF)$. This convention simplifies hypotheses in standard results about homotopy limits (e.g., theorems in §1.2).

Recall that $\mathcal{C}$ is a *left filtered* category if for any two objects $C_1, C_2 \in \mathcal{C}$ there exists $C_3 \in \mathcal{C}$ with $\mathrm{Mor}(C_3, C_1) \neq \emptyset \neq \mathrm{Mor}(C_3, C_2)$. If, in addition, for any two morphisms $m_1, m_2 \in \mathrm{Mor}(C, C')$ in $\mathcal{C}$ there exists $C'' \in \mathcal{C}$ and $m \in \mathrm{Mor}(C'', C')$ with $m_1 \circ m = m_2 \circ m$, then $\mathcal{C}$ is called *left filtering*. Let us recall

Theorem 4.2.2 (Quillen [69])**.** *Every left filtering category is contractible.*

Note that the homotopy limit above is taken over the category $\mathcal{C}\mathrm{ov}^{\mathrm{s}}X$ with morphism sets $\mathrm{Mor}(\mathcal{U}, \mathcal{V})$ consisting of contiguous maps $\mathcal{U} \to \mathcal{V}$. It is easy to see that although $\mathcal{C}\mathrm{ov}^{\mathrm{s}}X$ is left filtered, it is not left filtering. Instead of this category Carlsson and Pedersen use, following Friedlander ([36]), the category of *rigid coverings*. This category is, in fact, a partially ordered set: morphism sets are either empty or singletons. One advantage of this choice is the ease with which the

Cofinality Lemma 1.2.3 and the Modified Cofinality Lemma 1.2.9 can be applied. The more important consequence is the exactness property for the resulting Čech homology (Definition 4.2.4).

Definition 4.2.3. A *finite rigid covering* of X is a set function β from X to open subsets of X which takes only finitely many values and satisfies (1) $x \in \beta x$ for all $x \in X$ and (2) $\overline{\beta^{-1}U} \subseteq U$ for all $U \in \operatorname{im}(\beta)$. Each finite rigid covering can be thought of as a covering in the usual sense. Set

$$N(\beta) \stackrel{\text{def}}{=} N\big(\{\beta(x) : x \in X\}\big).$$

This time the nerve is an infinite simplicial complex unless X is finite.

We will denote the category of finite rigid coverings by $\mathcal{C}ovX$. There is a unique map $\beta_1 \to \beta_2$ if $\beta_1(x) \subseteq \beta_2(x)$ for all $x \in X$. Now $\mathcal{C}ovX$ is left filtering, so the maps can be indicated simply: $\beta_1 > \beta_2$.

Define $F \colon \mathcal{C}ovX \to \mathcal{C}ov^{\mathrm{s}}X$ to be the forgetful functor $\beta \mapsto \{U_x = \beta(x)\}_{x\in X}$, where $\mathcal{C}ov^{\mathrm{s}}X$ is the category of open coverings of X which may be infinite as sets but employ only finitely many open subsets of X. In particular, $F\beta$, $\beta \in \mathcal{C}ovX$, are always infinite if X is infinite, but the covering sets come from the finite $\operatorname{im}\beta$. Now $N(\beta) = N\big(F(\beta)\big)$ is clearly a functorial construction.

Let us emphasize that the assignment $\beta \mapsto \operatorname{im}\beta \in \mathfrak{C}\mathrm{ov}^{\mathrm{s}}X$ is *not functorial.* However, the obvious projection $F(\beta) \to \operatorname{im}\beta$ induces a homotopy equivalence on nerves according to Quillen's Theorem A.

Definition 4.2.4. The *Čech homology* of X with coefficients in S is the simplicial spectrum valued functor

$$\check{h}(X;S) = \underset{\mathcal{C}ovX}{\underleftarrow{\operatorname{holim}}}\,(N\_ \wedge S).$$

We summarize the properties of $\check{h}$ (all proved in [21]).

Theorem 4.2.5 (Homotopy Invariance)**.** *$\check{h}(\_;S)$ is homotopy invariant.*

Theorem 4.2.6. *For any space X there is a natural map $h(X;S) \to \check{h}(X;S)$. If X is a finite CW-complex, this map is a weak equivalence.*

Theorem 4.2.7 (Excision)**.** *Suppose Y is covered by two open sets A and B, and there exist open subsets N_A and N_B in Y with $A - A\cap B \subseteq N_A \subseteq A$ and $B - A \cap B \subseteq N_B \subseteq B$, and $N_A \cap N_B = \emptyset$. Then the diagram*

$$\begin{tikzcd} \check{h}(A\cap B;S) \arrow[r] \arrow[d] & \check{h}(B;S) \arrow[d] \\ \check{h}(A;S) \arrow[r] & \check{h}(Y;S) \end{tikzcd}$$

commutes, and the universal map from the homotopy colimit of the diagram

$$\begin{tikzcd} \check{h}(A\cap B;S) \arrow[r] \arrow[d] & \check{h}(B;S) \\ \check{h}(A;S) & \end{tikzcd}$$

to $\check{h}(Y;S)$ is a weak equivalence.

Theorem 4.2.8. *$\check{h}(\_;S)$ is a Steenrod homology theory.*

4.3. **Other Modifications.** The construction of the modified Čech homology is almost what Edwards and Hastings did in [26, §8.2] to construct their Steenrod extension ${}^{s}h(X;S)$. They used a Vietoris functor $V\colon \textsc{Top} \to \textsc{pro}-\textsc{s}-\textsc{Sets}$, $X \mapsto \{VN(\mathcal{U}) : \mathcal{U}$ an open cover of $X\}$, where VN denotes the Vietoris nerve. The rigity of the Vietoris construction makes V land in a pro-category. On page 251 they say that "an interesting problem is the construction of a nerve that is small like the Čech nerve and rigid like the Vietoris nerve".

The modified Čech homology is one possible answer to this question. After all, the nerves of the underlying open coverings are small. Another somewhat thriftier way to rigidify the Čech construction is to mimic the construction of Chogoshvili ([23]). This was done in [73] after Edwards and Hastings: the Vietoris nerve is again replaced by the Čech nerve, but the category of coverings is arranged to be left filtering as follows. For a compact Hausdorff space X, let $\mathfrak{A}$ be the set of all finite decompositions of the set X. An element $\mathcal{E} = (E_1, \ldots, E_k) \in \mathfrak{A}$ consists of arbitrary subsets $E_i \subseteq X$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{k} = X$. $\mathfrak{A}$ is ordered by inclusions. Let $\mathcal{COV}X$ be the set of pairs $(\mathcal{E},\mathcal{U})$, where $\mathcal{E} = \{E_i\}$ is a decomposition from $\mathfrak{A}$, $\mathcal{U} = \{U_i\}$ is a finite open covering of X with $\overline{E_i} \subseteq U_i$ for all i so that this correspondence indices an isomorphism of the nerves $N\overline{\mathcal{E}} \cong N\mathcal{U}$. For two elements $\Sigma = (\mathcal{E},\mathcal{U})$, $\Delta = (\mathcal{D},\mathcal{V})$, say that $\Sigma > \Delta$ if $\mathcal{E} > \mathcal{D}$ and $U_j \subseteq V_i$ when $E_j \subseteq D_i$. With this ordering, $\mathcal{COV}X$ is left filtering. The projection to the second coordinate gives a system cofinal in $\mathcal{C}\mathrm{ov}^{\mathrm{s}}X$, and the nerves are defined by $N(\mathcal{E},\mathcal{U}) = N\mathcal{U}$.

4.4. **Cofinality in Čech Theory.** First, we define some operations in $\mathcal{C}ovX$.

(1) Given $\beta \in \mathcal{C}ovX$, define $\cap\beta \in \mathcal{C}ovX$ by

$$(\cap\beta)(x) = \bigcap_{x\in\beta(z)} \beta(z).$$

Notice that $\cap\beta > \beta$. Another way to view this construction is as a canonical rigidification of the classical finite open covering $\operatorname{im}\beta$.

(2) Given a finite subset $\{\beta_i\} \subseteq \mathcal{C}ovX$, define $\cap\beta_i \in \mathcal{C}ovX$ by

$$\big(\cap\beta_i\big)(x) = \bigcap_{i,x\in\beta_i(z)} \beta_i(z).$$

Notice that $\cap\beta_i > \beta_i$ for every index i, and $\cap\beta = \beta \cap \beta$.

(3) Given a finite subset $\{\beta_i\} \subseteq \mathcal{C}ovX$, define $\times\beta_i \in \mathcal{C}ovX$ by

$$\big(\times\beta_i\big)(x) = \bigcap_{i} \beta_i(x).$$

Again, $\times\beta_i > \beta_j$ for every index j.

Choose and fix a (left filtering) subcategory $\imath\colon \mathcal{C} \hookrightarrow \mathcal{C}ovX$ closed under the $\times$-operation.

It is necessary to enlarge morphism sets in $\mathcal{C}ovX$. Let $\textsc{Cov}X$ be the category of set maps $\beta\colon X \to \mathcal{O}(X)$, the open subsets of X, such that $\operatorname{im}\beta$ is a finite set satisfying the two conditions from Definition 4.2.3. Morphisms $\phi\colon \beta_1 \to \beta_2 \in \textsc{Cov}X$ are set endomorphisms $\phi\colon X \to X$ with the property that $\beta_1(x) \subseteq \beta_2\big(\phi(x)\big)$ for all $x \in X$; they will be called *soft refinements*. The existence of such a refinement is denoted by $\beta_1 \gtrsim \beta_2$. If ϕ is realized by the identity map, we call ϕ a *rigid refinement*, denoted by $\beta_1 \geq \beta_2$. The subcategory of $\mathrm{Cov}X$ with only rigid morphisms is precisely $\mathcal{C}ovX$.

For each morphism $\phi \in \mathrm{Cov}X$, let $\circledast\phi$ denote the domain and $\odot\phi$ the range of $\phi\colon \circledast\phi \to \odot\phi$. Consider the subcategory $\mathcal{M}'$ of the category of morphisms of $\mathrm{Cov}X$ such that $\phi \in \mathcal{M}'$ iff (1) $\circledast\phi \in \mathcal{C}$, and (2) $\mathrm{im}(\odot\phi) \circ \phi = \mathrm{im}\,\odot\phi$ and $\mu\colon \phi_1 \to \phi_2 \in \mathcal{M}'$ iff (1) $\circledast\mu\colon \circledast\phi_1 \to \circledast\phi_2 \in \mathcal{C}$ and (2) $\odot\mu\colon \odot\phi_1 \to \odot\phi_2 \in \mathcal{C}ovX$. Notice that $\mu\colon \phi_1 \to \phi_2 \in \mathcal{M}'$ forces $\phi_1 = \phi_2$ as set maps. This comes from the requirement that ($\pitchfork$) $\phi_2 \circ \circledast\mu = \odot\mu \circ \phi_1$. The same requirement implies that (3) $\odot\phi_1\big(\phi_1(x)\big) \subseteq \odot\phi_2\big(\phi_2(x)\big)$ which becomes simply the realization of $\odot\mu$. Let us form a new category $\mathcal{M}$ with same elements as in $\mathcal{M}'$ and morphisms $\mu\colon \phi_1 \to \phi_2$ being pairs $(\circledast\mu, \odot\mu)$ satisfying (1), (2), and (3). The essence is that the weaker property (3) replaces ($\pitchfork$) from $\mathcal{M}'$. Consider also the subcategory $\mathcal{P}$ of the category of morphisms of $\mathcal{C}ovX$ with $\phi \in \mathcal{P}$ iff $\circledast\phi \in \mathcal{C}$. It can be viewed as a subcategory of $\mathcal{M}$ with the inclusion denoted by $i\colon \mathcal{P} \to \mathcal{M}$.

Definition 4.4.1. Let $\Theta\colon \mathcal{M} \to \mathcal{P}$ be the functor determined by

$$\circledast\Theta(\mu) = \circledast\mu,$$
$$\odot\Theta(\mu)(x) = \odot\mu\big(\mu(x)\big).$$

Functoriality of the construction follows from property (3) of morphisms in $\mathcal{M}$.

There are two functorial projections,

$$\pi_1\colon \mathcal{M} \longrightarrow \mathcal{C},\ \mu \mapsto \circledast\mu, \text{ and}$$
$$\pi_2\colon \mathcal{M} \longrightarrow \mathcal{C}ovX,\ \mu \mapsto \odot\mu.$$

The same notation will be used for similar $\pi_1\colon \mathcal{P} \to \mathcal{C}$, $\pi_2\colon \mathcal{P} \to \mathcal{C}ovX$.

Lemma 4.4.2. *The induced map of homotopy limits*

$$\imath^*\colon \underset{\pi_2(\mathcal{P})}{\underleftarrow{\mathrm{holim}}}\,(N\_ \wedge KR) \longrightarrow \underset{\mathcal{C}}{\underleftarrow{\mathrm{holim}}}\,(N\_ \wedge KR)$$

is a weak homotopy equivalence.

Proof. Since $\imath\colon \mathcal{C} \hookrightarrow \pi_2(\mathcal{P})$ is cofinal in the classical sense, the Modified Cofinality Lemma 1.2.9 applies. Our special situation allows for several simplifications in Carlsson's proof, so we choose to present a proof in this case (which seems to be the only case we use in this work).

We would like to claim that $\pi_1\colon \mathcal{P} \to \mathcal{C}$ and $\pi_2\colon \mathcal{P} \to \mathcal{C}ovX$ are both left cofinal. Since $\mathcal{P}$ is a partially ordered set, we will interpret its elements as pairs: $(\sigma, \delta) \in \mathcal{P} \subseteq \mathcal{C} \times \mathcal{C}ovX$ iff $\sigma \in \mathcal{C}$, $\sigma \geq \delta$. So let $\sigma \in \mathcal{C}$ and suppose (σ_1, δ_1), $(\sigma_2, \delta_2) \in \mathcal{P}$ with $\sigma_1 \geq \sigma$, $\sigma_2 \geq \sigma$. To prove that $\pi_1 \downarrow \sigma$ is left filtered, we need to exhibit $(\sigma_3, \delta_3) \in \mathcal{P}$ with $\sigma_3 \geq \sigma$ and $(\sigma_3, \delta_3) \geq (\sigma_i, \delta_i)$, $i = 1, 2$. Our choice is $(\sigma_1 \times \sigma_2, \delta_1 \times \delta_2)$. (Notice that it is not always true that $\sigma_1 \cap \sigma_2 \geq \delta_1 \cap \delta_2$.) The existence of equalizers in $\pi_1 \downarrow \sigma$ is obvious, so $\pi_1 \downarrow \sigma$ is left filtering, hence contractible. The other functor $\pi_2\colon \mathcal{P} \to \pi_2(\mathcal{P})$ is left cofinal for very similar reasons.

Now the map $\imath^*$ can be embedded as the bottom row of the following commutative diagram:

$$\begin{tikzcd}
\underset{\mathcal{P}}{\underleftarrow{\operatorname{holim}}}\,(N\pi_2_ \wedge KR) \arrow[r] & \underset{\mathcal{P}}{\underleftarrow{\operatorname{holim}}}\,(N\pi_1_ \wedge KR) \\
\underset{\pi_2(\mathcal{P})}{\underleftarrow{\operatorname{holim}}}\,(N_ \wedge KR) \arrow[r] \arrow[u, "\pi_2^*"'] & \underset{\mathcal{C}}{\underleftarrow{\operatorname{holim}}}\,(N_ \wedge KR) \arrow[u, "\pi_1^*"']
\end{tikzcd}$$

The vertical arrows are weak homotopy equivalences by the Cofinality Lemma. The top arrow can be interpreted as follows. Consider the functor $\Phi\colon \mathcal{P} \to \mathcal{P}$ given by $(\sigma,\delta) \mapsto (\sigma,\sigma)$ and notice that $N\pi_1\_ \wedge KR = N\pi_2\Phi\_ \wedge KR$. So, if denote $N\pi_2\_ \wedge KR$ by $G\_$, the top arrow is clearly

$$\Phi^*\colon \underset{\mathcal{P}}{\underleftarrow{\operatorname{holim}}}\, G \longrightarrow \underset{\mathcal{P}}{\underleftarrow{\operatorname{holim}}}\,(G \circ \Phi)$$

from the commutative square $\pi_2 \circ \Phi = \imath \circ \pi_1$.

The proof is completed by checking that Φ is left cofinal. This is literally the same argument as for π_1 above. The over categories are non-empty since $(\sigma,\sigma) \geq (\sigma,\delta) \in \mathcal{P}$. $\square$

Analogous diagrams exist for the category $\mathcal{M}$. If $\mathcal{C}$ is cofinal in $\textsc{Cov}X$ then $\pi_2(\mathcal{M}) = \mathcal{C}ovX$, and we get the following commutative diagram:

$$\begin{tikzcd}
\underset{\mathcal{M}}{\underleftarrow{\operatorname{holim}}}\,(N\pi_2_ \wedge KR) \arrow[r, "\Psi^*"] & \underset{\mathcal{M}}{\underleftarrow{\operatorname{holim}}}\,(N\pi_2\Psi_ \wedge KR) \\
\underset{\mathcal{C}ovX}{\underleftarrow{\operatorname{holim}}}\,(N_ \wedge KR) \arrow[r, "\jmath"] \arrow[u, "\pi_2^*"'] & \underset{\mathcal{C}}{\underleftarrow{\operatorname{holim}}}\,(N_ \wedge KR) \arrow[u, "\pi_1^*"']
\end{tikzcd}$$

Proposition 4.4.3. *Suppose $\pi_2\colon \mathcal{M} \to \mathcal{C}ovX$ is an epimorphism. In particular, the inclusion $i = \ell \circ \jmath\colon \mathcal{C} \hookrightarrow \textsc{Cov}X$ is cofinal. Then the induced map of homotopy limits*

$$\jmath^*\colon \check{h}(X;KR) = \underset{\mathcal{C}ovX}{\underleftarrow{\operatorname{holim}}}\,(N\_ \wedge KR) \longrightarrow \underset{\mathcal{C}}{\underleftarrow{\operatorname{holim}}}\,(N\_ \wedge KR)$$

is a weak homotopy equivalence.

Proof. The hypothesis makes every over category associated to π_2 non-empty. It is also clear that each $\pi_2 \downarrow \delta$, $\delta \in \mathcal{C}ovX$, has equalizers. It will be contractible if $\mathcal{M}$ is shown to be left filtered.

Given $\phi_i\colon \sigma_i \to \delta_i \in \mathcal{M}$, $i = 1,2$, consider $\delta_1 \cap \delta_2 \in \mathcal{C}ovX$. When X is connected, $(\delta_1 \cap \delta_2)^{-1}(U)$ is uncountable for $U \in \operatorname{im}(\delta_1 \cap \delta_2)$, so there is $\phi_{\mathrm{aux}}\colon \sigma_{\mathrm{aux}} \to \delta_1 \cap \delta_2 \in \mathcal{M}$, where ϕ_{aux} is a set automorphism of X. Since $x \in (\sigma_1 \times \sigma_2)(x) \cap (\delta_1 \cap \delta_2)(\phi_{\mathrm{aux}}(x))$, ϕ_{aux} can be chosen with the property $\phi_{\mathrm{aux}}(x) \in (\sigma_1 \times \sigma_2)(x)$. This gives $x \in (\sigma_1 \times \sigma_2) \circ \phi_{\mathrm{aux}}^{-1}(x)$. Construct new coverings $\sigma_i' = \sigma_i \circ \phi_{\mathrm{aux}}^{-1} \in \textsc{Cov}X$, $i = 1,2$. Let $\sigma = \sigma_1 \times \sigma_2 \times \sigma_{\mathrm{aux}}$, $\delta = \sigma_1' \times \sigma_2' \times (\delta_1 \cap \delta_2)$, and define $\phi\colon \sigma \to \delta$ to be ϕ_{aux}. Now we check: (1) $\sigma(x) \subseteq (\sigma_1 \times \sigma_2)(x) = (\sigma_1' \times \sigma_2')(\phi_{\mathrm{aux}}(x))$, $\sigma(x) \subseteq \sigma_{\mathrm{aux}}(x) \subseteq (\delta_1 \cap \delta_2)(\phi_{\mathrm{aux}}(x))$. So $\phi\colon \sigma \gtrsim \delta$. (2) $\sigma \geq \sigma_1 \times \sigma_2 \geq \sigma_i$, $\delta \geq \delta_1 \cap \delta_2 \geq \delta_i$ for $i = 1,2$. (3) $\delta(\phi(x)) \subseteq (\sigma_1' \times \sigma_2')(\phi(x)) = (\sigma_1 \times \sigma_2)(x) \subseteq \sigma_i(x) \subseteq \delta_i(\phi_i(x))$ for $i = 1,2$. We get the desired morphisms $\phi \to \phi_1$, $\phi \to \phi_2$.

The projection $\pi_1\colon \mathcal{M} \to \mathcal{C}$ is also an epimorphism on objects. Similar reasoning shows that each over category of π_1 is also left filtering. In fact, it has even simpler structure: the pair $(\sigma \geq \mathcal{X}, \sigma \geq \sigma)$, where $\mathcal{X} \colon x \mapsto X$ for each $x \in X$, is the terminal object in $\pi_1 \downarrow \sigma$.

Now π_1 and π_2 are left cofinal functors. It follows also from the left filtering property of $\mathcal{M}$ and the very functoriality of the Θ-construction that $\Theta \colon \mathcal{M} \to \mathcal{P}$ is likewise left cofinal.

Consider the commutative diagram

$$\begin{array}{ccccc}
\operatorname*{holim}_{\overleftarrow{\mathcal{P}}} (N\pi_2 i_ \wedge KR) & \xrightarrow{\Phi^*} & \operatorname*{holim}_{\overleftarrow{\mathcal{P}}} (N\pi_2 i\Phi_ \wedge KR) & = & \operatorname*{holim}_{\overleftarrow{\mathcal{P}}} (N\pi_2 \Psi i_ \wedge KR) \\
\Big\uparrow {\scriptstyle i^*} & & & & \Big\uparrow {\scriptstyle i^*} \\
\operatorname*{holim}_{\overleftarrow{\mathcal{M}}} (N\pi_2_ \wedge KR) & \xrightarrow{\Psi^*} & & & \operatorname*{holim}_{\overleftarrow{\mathcal{M}}} (N\pi_2 \Psi_ \wedge KR)
\end{array}$$

Since Φ is left cofinal, we are done as soon as both inclusion induced maps are shown to be weak equivalences. Notice that in the natural transformation $T \colon N\pi_2\Theta \to N\pi_2$ induced by the vertex maps $\odot\Theta\phi(x) \mapsto \odot\phi(\phi(x))$ all of $T\phi$, $\phi \in \mathcal{M}$, are homotopy equivalences by Quillen's Theorem A. Since $\Theta \circ i = \mathrm{id}$, we get the commutative diagram

$$\begin{array}{ccc}
\operatorname*{holim}_{\overleftarrow{\mathcal{M}}} (N\pi_2\Theta_ \wedge KR) & \xrightarrow{T_*} & \operatorname*{holim}_{\overleftarrow{\mathcal{M}}} (N\pi_2_ \wedge KR) \\
\Big\downarrow {\scriptstyle i^*} & & \Big\downarrow {\scriptstyle i^*} \\
\operatorname*{holim}_{\overleftarrow{\mathcal{P}}} (N\pi_2\Theta i_ \wedge KR) & = & \operatorname*{holim}_{\overleftarrow{\mathcal{P}}} (N\pi_2 i_ \wedge KR)
\end{array}$$

The vertical map on the left is the left inverse of Θ^* which is a weak equivalence. This i^* and T_* being weak equivalences proves that i^* on the right is one, too. Similarly, the other map i^* in the previous diagram is a weak homotopy equivalence. $\square$

Remark 4.4.4. When the inclusion i of $\mathcal{C}$ in $\mathrm{Cov}X$ is left cofinal, there is a reason to expect that $\jmath^*$ is again a weak homotopy equivalence. Just as in the case of cofinal i, the evidence comes from the Bousfield–Kan spectral sequence (Theorem 1.2.8), since the induced homomorphisms

$$\jmath^* \colon \operatorname*{lim^p}_{\overleftarrow{\mathcal{C}ov X}} \ \pi_q(N\_ \wedge KR) \longrightarrow \operatorname*{lim^p}_{\overleftarrow{\mathcal{C}}} \ \pi_q(N\_ \wedge KR)$$

of the entries in the E_2-terms coincide with

$$(F' \circ \jmath)^* \colon \operatorname*{lim^p}_{\overleftarrow{\mathfrak{C}\mathrm{ov} X}} \ \pi_q(N\_ \wedge KR) \longrightarrow \operatorname*{lim^p}_{\overleftarrow{F'(\mathcal{C})}} \ \pi_q(N\_ \wedge KR).$$

Here $F' \colon \mathcal{C}ov X \to \mathfrak{C}\mathrm{ov}X$ is the obvious extension of $F \colon \mathcal{C}ov X \to \mathcal{C}ov^{\mathrm{s}} X$ (see 4.2). It follows from the homotopy theoretic interpretation of derived limits ([13, XI, 7.2]) and the weak equivalence

$$(F' \circ \jmath)^* \colon \operatorname*{holim}_{\overleftarrow{\mathfrak{C}\mathrm{ov} X}} K\big(\pi_q(N\_ \wedge KR), n\big) \longrightarrow \operatorname*{holim}_{\overleftarrow{F'(\mathcal{C})}} K\big(\pi_q(N\_ \wedge KR), n\big)$$

that all $\jmath^*_{p,q}$ are isomorphisms.

5. The Approach to Novikov Conjectures

This section restates Theorem 2 in a more precise form. In particular, the map is defined between the homotopy limits in the statement which is expected to be a weak homotopy equivalence.

5.1. **Continuous Control at Infinity.** First, we copy some definitions from [20]. Let E be a topological space, and let $R[E]^\infty$ denote the free R-module generated by $E \times \mathbb{N}$. The category $\mathcal{B}(E;R)$ is defined to consist of submodules A of $R[E]^\infty$ such that denoting $A \cap R[x]^\infty$, $x \in E$, by A_x we have $A = \oplus A_x$, each A_x is a finitely generated free R-module, and $\{x : A_x \neq 0\}$ is locally finite in E. Morphisms are all R-module homomorphisms. Note that a Γ-action on X always induces a Γ-action on $\mathcal{B}(E;R)$. Also $\mathcal{B}(E;R)$ is a small additive category.

If X is a topological space, Y a subspace, $E = X - Y$, $U \subseteq X$ is any subset, and $A \in \mathcal{B}(E;R)$, define $A|U$ by $(A|U)_x = A_x$ if $x \in U - Y$ and 0 if $x \in X - U - Y$. A morphism $\phi\colon A \to B$ in $\mathcal{B}(E;R)$ is called *continuously controlled* at $y \in Y$ if for every neighborhood U of y in X there is a neighborhood V so that $\phi(A|V) \subseteq B|U$ and $\phi(A|X - U) \subseteq B|X - V$.

Now let T be an open subset of X and $p\colon T \to K$ be a map with continuous $p|Y \cap T$. A morphism $\phi\colon A \to B \in \mathcal{B}(E;R)$ is *p-controlled* at $y \in Y \cap T$ if for every neighborhood U of $p(y)$ in K there is a neighborhood V of y in X so that $\phi(A|V) \subseteq B|p^{-1}(U)$ and $\phi\big(A|X - p^{-1}(U)\big) \subseteq B|X - V$.

The category $\mathcal{B}(X,Y;R)$ has the same objects as $\mathcal{B}(E;R)$ and morphisms which are continuously controlled at all $y \in Y$. The category $\mathcal{B}(X,Y,p;R)$ has the same objects as $\mathcal{B}(E;R)$ and morphisms which are continuously controlled at all $y \in Y - T$ and p-controlled at all $y \in T \cap Y$. These are small symmetric monoidal categories, so there are corresponding non-connective K-theory spectra defined as in §1.4. We will use the notation $K(\_)$ for $K(\mathcal{B}(\_))$.

5.2. **Proof of Theorem 1.** Here is the general scheme of the approach used in [20] to prove Theorem 1 from the Introduction. Let $C\hat{X}$ be the cone on $\hat{X}$ with $\hat{X} = \hat{X} \times \{1\}$, $Y = \hat{X} - E\Gamma$, and $p\colon \hat{X} \times (0,1) \to \hat{X}$ be the projection. The map $\pi\colon C\hat{X} \to \Sigma\hat{X}$ collapsing $\hat{X}$ induces a Γ-equivariant functor

$$\mathcal{B}(C\hat{X}, CY \cup \hat{X}, p; R) \xrightarrow{\pi_*} \mathcal{B}(\Sigma\hat{X}, \Sigma Y, p; R)$$

which in its turn induces a map of spectra

$$\mathcal{S} = \Omega K(C\hat{X}, CY \cup \hat{X}, p; R) \xrightarrow{\pi_*} \mathcal{T} = \Omega K(\Sigma\hat{X}, \Sigma Y, p; R).$$

Next they show that there is a commutative diagram

$$\begin{array}{ccc} B\Gamma_+ \wedge KR & \xrightarrow{\alpha} & K(R\Gamma) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \mathcal{S}^\Gamma & \xrightarrow{\pi_*^\Gamma} & \mathcal{T}^\Gamma \end{array}$$

Recall that the fixed point spectrum of a Γ-spectrum $\mathcal{A}$ can be defined as

$$\mathcal{A}^\Gamma = \mathrm{Map}_\Gamma(S^0, \mathcal{A}_+).$$

The homotopy fixed point spectrum can be defined analogously:

$$\mathcal{A}^{h\Gamma} = \mathrm{Map}_\Gamma(E\Gamma_+, \mathcal{A}_+).$$

The collapse $\rho\colon E\Gamma_+ \to S^0$ induces $\rho^*\colon \mathcal{A}^\Gamma \to \mathcal{A}^{h\Gamma}$. Such maps make the next diagram commute:

$$\begin{array}{ccc} \mathcal{S}^\Gamma & \xrightarrow{\pi_*^\Gamma} & \mathcal{T}^\Gamma \\ {\scriptstyle\rho^*}\big\downarrow & & \big\downarrow \\ \mathcal{S}^{h\Gamma} & \xrightarrow{\pi_*^{h\Gamma}} & \mathcal{T}^{h\Gamma} \end{array}$$

It is shown in [20] that $\rho^*\colon \mathcal{S}^\Gamma \simeq \mathcal{S}^{h\Gamma}$ and $\pi_*^{h\Gamma}\colon \mathcal{S}^{h\Gamma} \simeq \mathcal{T}^{h\Gamma}$. Putting the two diagrams together we see that this is enough to make α a split injection. Note that very little is known about the map $\mathcal{T}^\Gamma \to \mathcal{T}^{h\Gamma}$, but only being a part of the commutative diagram is required of it.

5.3. **Proof of Theorem 2.** Now let us consider the circumstances of Theorem 2. Consider another map κ with domain $C\hat{X}$ which contracts the subspace CY and produces the reduced cone $\tilde{C}E\Gamma^+$. It induces a Γ-equivariant functor

$$\mathcal{B}(C\hat{X}, CY \cup \hat{X}, p; R) \xrightarrow{\kappa_*} \mathcal{B}(\tilde{C}E\Gamma^+, E\Gamma^+; R).$$

Notice that each morphism from $\mathcal{B}(C\hat{X}, CY \cup \hat{X}, p; R) \subseteq \mathcal{B}(X \times \mathbb{R})$ is controlled at $E\Gamma^+$. This functor induces a map of spectra

$$\mathcal{S} = \Omega K(C\hat{X}, CY \cup \hat{X}, p; R) \xrightarrow{\kappa_*} \mathcal{R} = \Omega K(\tilde{C}E\Gamma^+, E\Gamma^+; R).$$

Proposition 5.3.1. *κ_* is a weak homotopy equivalence.*

$E\Gamma^+$ is metrizable, so, according to Theorem 1.36 of [20], $\mathcal{R}$ is a Steenrod functor, and $\mathcal{R}^\Gamma \simeq \mathcal{S}^\Gamma \longrightarrow \mathcal{T}^\Gamma$ is again the assembly map. Also $\mathcal{R}^\Gamma \simeq \mathcal{R}^{h\Gamma}$ as before. Another Steenrod functor is the Čech homology

$$\check{h}(E\Gamma^+; KR) = \underset{\substack{\longleftarrow \\ \mathcal{U} \in \mathcal{C}ov E\Gamma^+}}{\operatorname{holim}} (N\mathcal{U} \wedge KR),$$

where $\mathcal{C}ov E\Gamma^+$ is the category of finite rigid open coverings of $E\Gamma^+$. The nerve functor $N\colon \mathcal{C}ov E\Gamma^+ \to$ s$-$Sets above lands in the category of simplicial sets. So $N\mathcal{U} \wedge KR$ above is a simplicial spectrum (see §4.2).

The *support at infinity* of an object $A \in \mathcal{B}(X, Y; R)$ is the set of limit points of $\{x : A_x \neq 0\}$ in Y. The full subcategory of $\mathcal{B}(X, Y; R)$ of objects with support at infinity contained in $C \subseteq Y$ is denoted by $\mathcal{B}(X, Y; R)_C$. If U_1, U_2 are open sets in $E\Gamma^+$ then we get maps induced by inclusions:

$$K(\tilde{C}E\Gamma^+, E\Gamma^+; R)_{U_1 \cap U_2} \longrightarrow K(\tilde{C}E\Gamma^+, E\Gamma^+; R)_{U_i}.$$

In general, there is a functor $\mathcal{I}nt\,\mathcal{U} \to$ Spectra for any $\mathcal{U} \in \mathcal{C}ov E\Gamma^+$, where $\mathcal{I}nt\,\mathcal{U}$ is the partially ordered set of all multiple intersections of members of $\mathcal{U}$ (indexed by finite subsets of Y).

Proposition 5.3.2. *For a fixed $\mathcal{U} \in \mathcal{C}ov E\Gamma^+$ the universal excision map*

$$\underset{\substack{\longrightarrow \\ \mathcal{I}nt\,\mathcal{U}}}{\operatorname{hocolim}}\, K(\tilde{C}E\Gamma^+, E\Gamma^+; R)_{\cap U_i} \longrightarrow K(\tilde{C}E\Gamma^+, E\Gamma^+; R)$$

is a weak equivalence.

The spectrum $\Sigma\mathcal{R}$ on the right is a Γ-spectrum. To rediscover this aspect of the structure on the left, we can write

$$\underset{\mathcal{U}\in\mathcal{C}ovE\Gamma^+}{\underleftarrow{\operatorname{holim}}} \left(\underset{\mathcal{I}nt\mathcal{U}}{\underrightarrow{\operatorname{hocolim}}}\, K(\tilde{C}E\Gamma^+, E\Gamma^+; R)_{\cap U_i} \right) \simeq \Sigma\mathcal{R},$$

where the Γ-action on the left-hand side is induced from the obvious action on $\mathcal{C}ovE\Gamma^+$. Notice that we have used the fact that $\mathcal{C}ovE\Gamma^+$ is contractible in applying Theorem 1.2.5 to $\operatorname{holim}\Sigma\mathcal{R}$.

In the proper setup (essentially sending each non-empty $\cap U_i$ to a point) one gets maps

$$\underset{\mathcal{I}nt\mathcal{U}}{\underrightarrow{\operatorname{hocolim}}}\, K(\tilde{C}E\Gamma^+, E\Gamma^+; R)_{\cap U_i} \longrightarrow |\mathcal{I}nt\mathcal{U}| \wedge KR.$$

Finally, we get the induced equivariant map of homotopy limits

$$\pi\colon \mathcal{R} \longrightarrow \check{h}(E\Gamma^+; KR).$$

This map can be viewed as a component of a natural transformation of Steenrod functors which is an equivalence on points, hence $\mathcal{R} \simeq \check{h}(E\Gamma^+; KR)$, according to Theorem 1.3.1. This is enough to conclude that $\mathcal{R}^{h\Gamma} \simeq \check{h}(E\Gamma^+; KR)^{h\Gamma}$.

Returning to $\mathcal{T}$, there is an excision result analogous to Claim 5.3.2. There is no automatic p-control as there was before ([20, 22]) because the action at infinity is no longer small. In order to produce a natural transformation analogous to π, the covering sets $p(U) \subseteq Y$ must be boundedly saturated. See [21] for the construction of a map

$$\pi_\ell\colon \mathcal{T} \longrightarrow \Sigma \underset{\alpha\in\{\alpha\}}{\underleftarrow{\operatorname{holim}}}\,(N\alpha \wedge KR)$$

for each Γ-closed contractible system of coverings of Y by boundedly saturated open sets.

Again, this map is Γ-equivariant, so the composition induces a map

$$\mathcal{T}^{h\Gamma} \longrightarrow \left(\Sigma \underset{\alpha\in\{\alpha\}}{\underleftarrow{\operatorname{holim}}}\, N\alpha \wedge KR \right)^{h\Gamma}.$$

Since $\hat{X}$ is Čech-acyclic, there is a composite weak equivalence

$$\begin{tikzcd}
\underset{Cov(\hat{X}\cup CY)}{\underleftarrow{\operatorname{holim}}}\,(N_ \wedge KR) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \underset{Cov\Sigma Y}{\underleftarrow{\operatorname{holim}}}\,(N_ \wedge KR) \\
\underset{CovE\Gamma^+}{\underleftarrow{\operatorname{holim}}}\,(N_ \wedge KR) \arrow[r, "\exists"] & \Sigma \underset{CovY}{\underleftarrow{\operatorname{holim}}}\,(N_ \wedge KR) \arrow[u, "\simeq"']
\end{tikzcd}$$

There is a map

$$\theta\colon \underset{CovY}{\underleftarrow{\operatorname{holim}}}\,(N\_ \wedge KR) \longrightarrow \underset{\{\alpha\}}{\underleftarrow{\operatorname{holim}}}\,(N\_ \wedge KR)$$

induced by the inclusion of categories $\{\alpha\} \hookrightarrow \mathcal{C}ovY$; it makes the ambient diagram commutative. If θ is a weak equivalence then $\theta^{h\Gamma}$ is a weak equivalence. This would make α a split injection, since a weak equivalence would again be factored as the composition of α with another map.

Summary . Given a discrete group Γ, the method described here calls for a construction of a compact classifying space $B\Gamma$ and an equivariant compactification $\hat{X}$ of the universal cover $E\Gamma$, i.e., an open dense embedding $E\Gamma \hookrightarrow \hat{X}$ in a compact Hausdorff space. The space $\hat{X}$ itself may not be metrizable but it is required to be acyclic in the sense that its Čech homology is that of a point. Then a convenient metric must be introduced on $E\Gamma$. The action may not be small at infinity, but the choice of metric determines the family of boundedly saturated subsets of $Y = \hat{X} - E\Gamma$. One has to make a choice of a Γ-invariant collection of coverings of Y by such sets which preserves the Čech homotopy type of Y. Furthermore, the weak homotopy equivalence of Čech homology spectra has to be realized by the map θ defined above.

5.4. **Modification.** The flexibility of this approach is in the freedom of choice of the metric in $\bar{X}$ and the system of special coverings $\{\alpha\}$. It happens to be not enough for making θ a weak equivalence in a situation like ours when the choice of the metric is convenient and natural but makes the family of open boundedly saturated sets in $Y = \hat{X} - \bar{X}$ too coarse to preserve the Čech homotopy type.

Definition 5.4.1. Let C_1 and C_2 be two closed subsets of Y. The pair (C_1, C_2) is called *excisive* if there is an open subset $V \subseteq \hat{X}$ such that $C_2 - C_1 \subseteq V$ and $\overline{V} \cap C_1 \subseteq C_2$. For two arbitrary subsets U_1 and U_2, the pair (U_1, U_2) is *excisive* if every compact subset C of $U_1 \cup U_2$ is contained in $C_1 \cup C_2$ where (C_1, C_2) is an excisive pair of closed subsets with $C_i \subseteq U_i$. A collection of subsets $U_i \subseteq Y$ is called *excisive* if every pair in the Boolean algebra of sets generated by U_i is excisive.

It is easy to show that $\mathcal{C}\mathrm{ov}X$ for compact Hausdorff X consists of excisive coverings. It turns out that this property is sufficient for the excision result like Proposition 5.3.2 (see the proof in [21]). Our choice for $\{\alpha\}$ will be certain excisive coverings by boundedly saturated sets so that the category itself is contractible. This makes possible the construction of a map similar to π_ℓ above. Since $\Sigma\check{h}(Y; KR)$ is weakly equivalent to the domain of $\pi_* \colon \mathcal{S} \to \mathcal{T}$ such that $(\pi_*)^\Gamma$ is the assembly map, there must be a map

$$\theta \colon \check{h}(Y; KR) = \underset{\overleftarrow{\mathcal{C}ovY}}{\mathrm{holim}}\,(N\_ \wedge KR) \longrightarrow \underset{\overleftarrow{\{\alpha\}}}{\mathrm{holim}}\,(N\_ \wedge KR)$$

which completes the commutative diagram.

To create a natural target for a map from $\mathcal{T}$ we can “saturate” the open sets $U \subseteq Y$ by associating to U its envelope in a Boolean algebra of boundedly saturated subsets of Y thus mapping $\mathcal{C}ovY$ functorially onto the resulting category $\{\alpha\}$. Let us denote this functor by $\mathfrak{sat} \colon \beta \mapsto \alpha(\beta)$. Since $\mathfrak{sat}$ is left cofinal, and the construction $\beta \rightsquigarrow \alpha(\beta)$ above induces a natural transformation of the functors $N\beta \wedge KR \to N\alpha(\beta) \wedge KR$ from $\mathcal{C}ovY$ to s$-$Spectra, we can induce and compose the following maps:

$$\theta \colon \underset{\overleftarrow{\beta \in \mathcal{C}ovY}}{\mathrm{holim}}\,(N\beta \wedge KR) \xrightarrow{\ \mathfrak{sat}_*\ } \underset{\overleftarrow{\beta \in \mathcal{C}ovY}}{\mathrm{holim}}\, N\alpha(\beta) \wedge KR \xleftarrow{\ \simeq\ } \underset{\overleftarrow{\alpha \in \{\alpha\}}}{\mathrm{holim}}\,(N\alpha \wedge KR).$$

This is the correct map if we make sure that the analogue of the excision result from [21] works with $\{\alpha\}$. It is precisely the property of $A \in \alpha$ being excisive that we need here. This cannot be always guaranteed. However, one can often make a more intelligent choice of the Boolean subalgebra of boundedly saturated sets in the construction of $\{\alpha\}$. Taking envelopes in this algebra defines all the analogues of the maps above with all the same properties.

Now $\{\alpha\}$ may not be included in $\mathcal{C}ovY$ any longer. This is why one is forced to consider the more general situation. We will pass to a convenient intermediate category $\mathcal{C}$ of $\mathcal{C}ovY$ where the open covering sets have particularly nice nature so that it is easy to predict the saturation and see that it does not change the homotopy type of the nerve. The passage is achieved using the following diagram.

$$\begin{tikzcd} \underset{\beta\in\mathcal{C}ovY}{\underleftarrow{\mathrm{holim}}}\,(N\beta\wedge KR) \arrow[r, "\mathfrak{sat}_*"] \arrow[d, "\imath^*"'] & \underset{\beta\in\mathcal{C}ovY}{\underleftarrow{\mathrm{holim}}}\; N\alpha(\beta)\wedge KR \arrow[d] & \underset{\alpha\in\{\alpha\}}{\underleftarrow{\mathrm{holim}}}\,(N\alpha\wedge KR) \arrow[l, "\simeq"'] \arrow[d] \\ \underset{\mathcal{C}}{\underleftarrow{\mathrm{holim}}}\,(N\_\wedge KR) \arrow[r, "(\mathfrak{sat}|\mathcal{C})_*"] & \underset{\mathcal{C}}{\underleftarrow{\mathrm{holim}}}\; N\alpha(\_)\wedge KR & \underset{\{\alpha'\}}{\underleftarrow{\mathrm{holim}}}\,(N\_\wedge KR). \arrow[l, "\simeq"'] \end{tikzcd}$$

The vertical maps are induced by inclusions. Now the map $\mathcal{T}\to\mathrm{holim}(N\alpha\wedge KR)$ can be composed with the vertical map on the right, so in order to split the assembly map we need $\imath^*$ and $(\mathfrak{sat}|\mathcal{C})_*$ to be weak equivalences.

Example . If there is a Γ-closed contractible category $\mathcal{D}$ of finite rigid open coverings by boundedly saturated sets then it can be taken to play the role of $\mathcal{C}$. In this case $\mathfrak{sat}$ is an identity, so only $\imath^*$ needs to be an equivalence, and we recover the theorem of Carlsson and Pedersen.

Our own choice for $\mathcal{C}$ will be explained in §13.1.

Part 3. Nilpotent Groups

We have to start our inductive constructions with a study of simply connected nilpotent groups. It could culminate in a proof of the Novikov conjecture for the class $\mathcal{N}\imath\ell$ of torsion-free finitely generated nilpotent groups. It is possible, however, to deal with these groups using different approaches via reduction ([17, 35, 70]). We actually compactify a suitable $E\Gamma$, and it is this construction that we are really after. We also use this format to organize some information needed in Part IV.

6. Malcev Spaces and their Compactification

6.1. **Malcev Coordinates.** Let G be a real Lie group, and $\mathfrak{g}$ be its Lie algebra. There are two different ways to introduce a local coordinate system in a neighborhood of the identity $e\in G$. If $\{X_1,\ldots,X_d\}$ is a basis in $\mathfrak{g}$, introduce a coordinate system $\{u_1,\ldots,u_d\}$ in $\mathfrak{g}$ by mapping

$$X=\sum_{i=1}^{d}u_iX_i\mapsto(u_1,\ldots,u_d)\in\mathbb{R}^d.$$

For the usual norm $|X|=(\sum_{i=1}^{d}|u_i|^2)^{1/2}$ in $\mathfrak{g}$, there exists a number $\epsilon>0$ such that the exponential maps an open norm-metric ball at 0 in $\mathfrak{g}$ injectively and regularly into G. The image $U_e=\{\exp X:|X|<\epsilon\}$ is a neighborhood of e. If we take

$$x_k\left(\exp\sum_{i=1}^{d}u_iX_i\right)=u_k$$

then $\{x_1,\ldots,x_d\}$ is a local *canonical coordinate system of the first kind.*

If $\mathfrak{g}$ contains vector subspaces $\mathfrak{m}_i$, $1 \le i \le k$, such that $\mathfrak{g} = \bigoplus \mathfrak{m}_i$ then there are neighborhoods W_i of 0 in $\mathfrak{m}_i$ respectively such that

$$\varphi\colon (Y_1, \ldots, Y_k) \mapsto \exp Y_1 \cdots \exp Y_k$$

gives a diffeomorphism of $W_1 \times \cdots \times W_k$ onto a neighborhood V of $e \in G$. Indeed, take a basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{g}$ subordinate to the decomposition $\mathfrak{g} = \bigoplus \mathfrak{m}_i$, i.e.,

$$X_s \in \mathfrak{m}_i \quad \text{for} \quad \sum_{j=1}^{i-1} \dim \mathfrak{m}_j < s \le \sum_{j=1}^{i} \dim \mathfrak{m}_j,$$

and take the corresponding canonical local coordinate system of the first kind in each $\exp(\mathfrak{m}_i)$. For

$$Z_j = \sum_{i=1+\sum_1^{j-1} \dim \mathfrak{m}_s}^{\sum_1^j \dim \mathfrak{m}_s} u_i X_i$$

one can write

$$\prod_{j=1}^{k} \exp Z_j = \exp \left(\sum_{i=1}^{d} u_i X_i \right).$$

In particular, if each $\mathfrak{m}_i$ is spanned by X_i from a basis $\{X_1, \ldots, X_d\}$ for $\mathfrak{g}$ then there is a neighborhood $V \ni e$ such that for each $g \in V$ there is a unique expression

$$g = \exp t_1 X_1 \cdots \exp t_d X_d, \quad t_i \in \mathbb{R}, \quad 1 \le i \le d.$$

The d-tuple $\{t_1, \ldots, t_d\}$ gives a local coordinate system in V which is called the *canonical coordinate system of the second kind*.

It seems that this terminology first appeared in Malcev's work on rigidity in nilpotent groups ([60]). Recall that if G is a connected simply connected nilpotent group, the exponential map is a global diffeomorphism. Therefore the coordinate systems of both kinds are also global. The canonical system of the second kind satisfying another condition which can always be arranged in nilpotent groups now bears the name of *Malcev coordinates* ([45]). Since we use both of the systems, we find it convenient to use Malcev's original terminology.

Using the Campbell–Hausdorff formula it is easy to see that the group operation in G is given by polynomial maps with rational coefficients with respect to either coordinate system (see [60]). This can be used to give G the structure of a unipotent linear algebraic group defined over $\mathbb{Q}$.

Corollary 6.1.1. *G has a faithful rational unipotent representation $\phi\colon G \to U_n$.*

The two coordinate systems are certainly related: if $\{u_1, \ldots, u_d\}$ is a system of the first kind in G then taking the one-parameter subgroups $x_i(t) = tu_i$ we get the corresponding system of the second kind. Conversely, if $x_1(t), \ldots, x_d(t)$ is the system of the second kind then $x_1(1), \ldots, x_d(1)$ give the corresponding coordinate system of the first kind.

6.2. **Compactification.** A *lattice* Γ in a connected Lie group G is a discrete subgroup such that G/Γ has finite volume. Let us begin with

Theorem 6.2.1. *A group is isomorphic to a lattice in a simply connected nilpotent Lie group if and only if it is finitely generated, nilpotent, and torsion-free.*

This characterization is essentially due to A. I. Malcev who stated it for *uniform lattices* (i.e., those with compact G/Γ) in [60]. As in any solvable Lie group, these lattice subgroups are always uniform.

Let Γ be a torsion-free finitely generated nilpotent group which we embed in a connected simply connected group N produced by Theorem 6.2.1. This N will be the model for $E\Gamma$. By Lemma 4 of [60] the subgroup Γ has generators $\{\gamma_1, \ldots, \gamma_r\}$, where $r = \dim N$, with the three properties:

(1) each $\gamma \in \Gamma$ can be written as $\gamma = \gamma_1^{n_1} \cdots \gamma_r^{n_r}$,
(2) each subset $\Gamma_i = \{\gamma_i^{n_i} \cdots \gamma_r^{n_r}\}$ is a normal subgroup of Γ, and
(3) the quotients Γ_i/Γ_{i+1} are infinite cyclic for all $1 \leq i < r$.

This *canonical basis* is constructed inductively with respect to the length of the central series for N. It can be used to introduce coordinates of the second kind in the ambient group N.

Let $C_i = c_i(t)$ be the one-parameter subgroup of N with $c_i(1) = \gamma_i$, $1 \leq i \leq r$. It turns out that N satisfies analogues of the three properties of Γ:

(1) $N = C_1 \cdots C_r$, and the representation of $g \in N$ as $g = g_1 \cdots g_r$, $g_i \in C_i$, is unique,
(2) if $N_{r+1} = \{e\}$, $N_i = C_i \cdots C_r$, $1 \leq i \leq r$, then N_i are Lie subgroups of N, $\dim N_i = r - i + 1$, and $N_i \lhd N$ for $1 \leq i < r$,
(3) $C_i \cong \mathbb{R}$ for all $1 \leq i \leq r$.

If $\mathfrak{n}$ is the Lie algebra of N then $e_1 = \log \gamma_1, \ldots, e_r = \log \gamma_r$ becomes a basis in $\mathfrak{n}$ such that each set

$$\mathfrak{n}_\mathfrak{i} = \{\alpha_\mathfrak{i} \mathfrak{e}_\mathfrak{i} + \alpha_{\mathfrak{i}+1} \mathfrak{e}_{\mathfrak{i}+1} + \cdots + \alpha_\mathfrak{r} \mathfrak{e}_\mathfrak{r}\} \subseteq \mathfrak{n}$$

is an ideal. So $\{\gamma_i\}$ produce canonical coordinates of the first kind according to Malcev. The correspondences

$$\log \colon g \mapsto \log g,$$

$$\sigma \colon \sum_{k=1}^{r} \alpha_k e_k \mapsto \sum_{k=1}^{r} \alpha_k (0, \ldots, \overset{k}{\widehat{1}}, \ldots, 0)$$

define diffeomorphisms between N, $\mathfrak{n}$ and $\mathbb{R}^r$ and induce flat metrics in N and $\mathfrak{n}$ from the standard Euclidean metric in $\mathbb{R}^r$.

The Lie algebra $\mathfrak{u}_n$ corresponding to the nilpotent group $U_n = U_n(\mathbb{R})$ consists of upper triangular matrices with zeros on the diagonal; it has a basis consisting of elementary matrices. By applying the above construction we see that U_n and $\mathfrak{u}_n$ are diffeomorphic to $\mathbb{R}^{n(n-1)/2}$ and get a flat metric from this identification.

Recall the *ideal* compactification of N denoted by $\varepsilon N = N \cup \partial N$ from §2.1 with the added points at infinity corresponding to the geodesic rays through the identity $I \in N$. In particular, we can consider εU_n.

Proposition 6.2.2. *The ideal compactification εU_n is equivariant with respect to both the left and the right multiplication actions of U_n on itself.*

Proof. The lines in $\mathfrak{u}_n$ can be described as sets of matrices $\big(l_{ij}(t)\big) \in \mathfrak{u}_n$, $t \in \mathbb{R}$, where for $1 \leq i < j \leq n$ the entries are linear:

$$l_{ij}(t) = x'_{ij} + x''_{ij} t.$$

So the line (l_{ij}) passes through the point (x'_{ij}), and (x''_{ij}) is a directional vector. Now it is clear that for $(a_{ij}) \in \mathfrak{u}_n$ each entry in either $(a_{ij}) \cdot \big(l_{ij}(t)\big)$ or $\big(l_{ij}(t)\big) \cdot (a_{ij})$ is linear in t. In fact, the free terms depend only on the point (x'_{ij}), and the coefficients at t are degree one polynomials in the coordinates of the directional vector (x''_{ij}).

Given any $(g_{ij}) \in N$ and any line $L \subseteq N$, let $(a_{ij}) = \log(g_{ij})$ and $\big(l_{ij}(t)\big) = \log L$. From the calculation above, the set $(g_{ij}) \cdot L$ is the exponential image of the line

$$(a_{ij}) + \big(l_{ij}(t)\big) + \frac{1}{2}\big((a_{ij}) \cdot (l_{ij}(t)) - (l_{ij}(t)) \cdot (a_{ij})\big) + \dots$$

The directional coefficients in this expression are polynomials in the entries of (a_{ij}) and the directional coefficients of $\big(l_{ij}(t)\big)$. So the left multiplication action on U_n extends continuously to the space of normalized directional vectors, or, equivalently, the space of all parallelism classes of rays which is precisely the ideal boundary of the flat space U_n.

Writing out a similar formula for the right multiplication or noticing that $L \cdot g = g(g^{-1}Lg)$, where $g^{-1}Lg$ is another one-parameter subgroup, proves the claim for the right action. $\square$

Remark 6.2.3. Our motive for Proposition 6.2.2 is, of course, that now the restricted actions of any lattice Γ in U_n extend to ∂U_n. What makes this non-trivial is the fact that the left and right Γ-actions on N *cannot be by isometries*. Indeed, Γ would then be a crystallographic group which by a theorem of Bieberbach would intersect the translation subgroup in $\mathrm{Isom}(\mathbb{R}^{n(n-1)/2})$ in a normal free abelian subgroup of finite index. This would contradict the possible non-triviality of the semi-direct product structure on Γ (as in the following Example 6.2.4). As we have just seen, the ideal compactification of $\mathbb{R}^3$ as a non-positively curved space is still Γ-equivariant. Accountable for this is the fact that it is a Γ-equivariant quotient of a cross of the ideal and polyhedral ([4, 82]) compactifications.

Example 6.2.4. Let Γ be the discrete Heisenberg group, the group $U_3(\mathbb{Z})$ of 3×3 strict upper triangular integral matrices under matrix multiplication. It has the well-known presentation

$$\Pi = \langle a, b, c \mid aca^{-1}c^{-1},\ bcb^{-1}c^{-1},\ caba^{-1}b^{-1} \rangle.$$

This is the simplest example of a discrete nilpotent but non-abelian group. Each element of Π can be written uniquely in the form $a^m b^k c^l$, and the identification of Π with the subgroup $\Gamma \subseteq N = U_3(\mathbb{R})$ is given by the mapping

$$a^m b^k c^l \longmapsto \begin{pmatrix} 1 & m & l \\ & 1 & k \\ & & 1 \end{pmatrix}.$$

Γ contains two subgroups

$$M = \left\{ \begin{pmatrix} 1 & m & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\} \cong \mathbb{Z} \quad \text{and} \quad L = \left\{ \begin{pmatrix} 1 & 0 & l \\ & 1 & k \\ & & 1 \end{pmatrix} \right\} \cong \mathbb{Z}^2.$$

M acts on L by

$$m \cdot z = mzm^{-1} = (k, l + mk)$$

for $m \in M$ and $z = (k, l) \in L$ making Γ into the semi-direct product of M and L with multiplication

$$(m, z) \cdot (m', z') = (m + m', z + m \cdot z').$$

Systematizing the convenient notation, let us denote the matrix

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \in N$$

by the ordered triple (x, y, z). In this notation the action of Γ on N is given by

$$(m, k, l) \cdot (x, y, z) = (m + x, k + y, l + z + my).$$

A fundamental domain for this action is

$$\mathcal{D} = \{(x, y, z) \in N : 0 \leq x, y, z \leq 1\}.$$

To illustrate Proposition 6.2.2, a straight line in the flat space N_0 can be expressed in terms of our coordinates as $(x_1+x_2t, y_1+y_2t, z_1+z_2t)$ with the parameter $t \in \mathbb{R}$. Now Γ_0 acts on the set of such lines:

$$(m, k, l) \cdot (x_1 + x_2t, y_1 + y_2t, z_1 + z_2t) =$$
$$(m + x_1 + x_2t, k + y_1 + y_2t, l + my_1 + z_1 + (my_2 + z_2)t).$$

From this equation we see that the parallelism class of lines with $x_2 = y_2 = 0$ is invariant under Γ_0. The same is true for the class of lines with $y_2 = z_2 = 0$. Proposition 6.2.2 shows that this left action extends to the ideal boundary of N_0—the points in ∂N_0 corresponding to the two opposite directions of the line $(0, 0, t)$ are fixed by Γ_0, and the open meridian semi-circles connecting the two points are orbits. The action is continuous since the coefficients at t are degree one polynomials in the coordinates of the directional vector (x_2, y_2, z_2).

The right action of Γ_0 on lines is given by

$$(x_1 + x_2t, y_1 + y_2t, z_1 + z_2t) \cdot (m, k, l) =$$
$$(m + x_1 + x_2t, k + y_1 + y_2t, l + kx_1 + z_1 + (kx_2 + z_2)t).$$

Notice that the same poles with $x_2 = y_2 = 0$ get fixed. Also, the sets of equivalence classes with either $x_2 = 0$ or $y_2 = 0$ are invariant under the right action. In fact, all right Γ_0-orbits in the former set are points.

Let $\phi\colon N \to U_n$ be a faithful unipotent representation of N as in Corollary 6.1.1 for an appropriate choice of $n > 0$. The corresponding map of Lie algebras with the flat metrics induced as above for any choice of the basis in $\mathfrak{n}$ is a totally geodesic embedding. Thus so is ϕ. Notice that in the chosen metric the geodesic lines through e or I are precisely the one-parameter subgroups of N or U_n respectively. We get an extension $\bar{\phi}\colon \varepsilon N \to \varepsilon U_n$. The image $\bar{\phi}(\varepsilon N)$ is the closure of $\phi(N)$ which is invariant under both N-actions. We conclude that the compactification εN of N itself is left and right equivariant in a way which makes $\bar{\phi}$ equivariant.

6.3. **Right Orbits.** We would like to study the right action of N on ∂N in more detail. Since the coordinates of the first kind in each N_i can be derived from subsets of the same canonical basis in Γ, the normal subgroups N_i of N form a flag of totally geodesic flat subspaces. It is also true that each Γ_i is a lattice in the corresponding N_i. The following theorem provides information sufficient for our purposes but is certainly much less precise than the description of right orbits in Example 6.2.4.

Theorem 6.3.1. *Let $l(t)$ be an oriented one-parameter subgroup of N_i and $g \in N$, then the corresponding ray in $l(t) \cdot g$ represents a point in ∂N_i. In other words, ∂N_i is N-(right-)invariant in ∂N.*

Proof. The claim is equivalent to saying that for every line $l(t)$ as in the statement, $l(t) \cdot g$ is a line parallel to N_i. This follows from the Campbell–Hausdorff formula since N_i is normal in N. Indeed, $\mathfrak{n}_i$ is an ideal in $\mathfrak{n}$, so for any $a \in N$ the Poisson bracket $[a, e_i] \in N_{i+1}$. Let us denote the coordinates of $l(t)$ and g by $\xi_i(t)$ and η_i respectively, then the coordinates of $l(t) \cdot g$, $\zeta_i(t)$, satisfy

$$\zeta_i = \xi_i + \eta_i + p_i(\xi_1, \ldots, \xi_{i-1}, \eta_1, \ldots, \eta_{i-1}),$$

where p_i are formal polynomials determined by the Campbell–Hausdorff formula. This shows that if $l(t) \subseteq N_j$ then $\xi_1 = \cdots = \xi_{j-1} = 0$ and ζ_k, $k < j$, are independent of $\xi_j(t), \ldots, \xi_r(t)$. We can conclude that $l(t) \cdot g$ lies in the hyperplane $(\zeta_1, \ldots, \zeta_{j-1}, *, \ldots, *)$ parallel to N_j. □

A consequence of Theorem 6.3.1 is that each ∂N_i consists of right N-orbits in ∂N. Since the right action of N on ∂N_{i-1} is continuous, N is connected, and ∂N_i disconnects ∂N_{i-1}, the two complementary hemispheres are themselves unions of right N-orbits and, hence, are N-invariant.

Corollary 6.3.2. *The inductive disjoint decomposition of ∂N into open hemispheres of all dimensions $0 \leq i \leq r$ with ∂N_i consisting of hemispheres of dimensions $\leq i - 1$ is invariant under the right action of N.*

7. Bounded Saturation in the Boundary Sphere

7.1. **General Properties of Boundedly Saturated Sets.** For any subset K of a metric space (X, d) let $K[D]$ denote the set $\{x \in X : d(x, K) \leq D\}$ which we call the *D-neighborhood* of K.

Definition 7.1.1. Given a metric space (X, d) embedded in a topological space $\hat{X}$ as an open dense subset, a set $A \subseteq Y = \hat{X} - X$ is *boundedly saturated* if for every closed subset C of $\hat{X}$ with $C \cap Y \subseteq A$, the closure of each D-neighborhood of $C \backslash Y$ for $D \geq 0$ satisfies $\overline{(C \backslash Y)[D]} \cap Y \subseteq A$. Clearly, it is enough to consider only those C with $C \cap Y = \overline{C \backslash Y} \cap Y$.

Convention . All of the spaces we consider in this paper have the property that if x is a cluster point of some sequence $\{x_i\}$ then there is a subsequence $\{x_j\}$ so that x is the only cluster point of $\{x_j\}$. For example, this is satisfied by any metrizable space. That the spaces from §9 and Appendices A and C satisfy this condition follows immediately from the definition of basic neighborhoods. When we say that a sequence $\{x_i\}$ *converges* to x and write $x = \lim\{x_i\}$, we understand that the original sequence has been replaced by a converging subsequence.

Lemma 7.1.2. *Let S be a subset of Y which is not boundedly saturated. Then there exists a point $y \in Y \backslash S$ and a sequence $\{y_i\} \subseteq X$ converging to y so that $\overline{\{y_i\}[D]} \cap S \neq \emptyset$ for some $D > 0$.*

Proof. By the hypothesis there is a closed subset $K \subseteq \hat{X}$ with $K \cap Y \subseteq S$ and $\overline{(K \cap X)[D]} \backslash S \neq \emptyset$ for some $D > 0$. Let $y \in \overline{(K \cap X)[D]} \backslash S$. Then there exists a sequence $\{y_i\} \subseteq X$ converging to y with $d(y_i, K \cap X) \leq D$. Consider $K \cap \{y_i\}[D]$; if this set is bounded then $\{y_i\}$ is contained in the bounded set $(K \cap \{y_i\}[D])[D] \subseteq X$ which would make $y \in X$. So there is a sequence $\{z_i\} \subseteq K \cap X$ with $z_i \in K \cap \{y_i\}[D]$ and $\lim_{i \to \infty}\{z_i\} \in K \cap Y \subseteq S$. □

Theorem 7.1.3. *A subset $S \subseteq Y$ is boundedly saturated if for any closed set $C \subseteq \hat{X}$ with $C \cap S = \emptyset$ and any $D > 0$, $S \cap \overline{(C \cap X)[D]} = \emptyset$.*

Proof. Apply Lemma 7.1.2 to the contrapositive statement. □

Notice that the hypothesis of Theorem 7.1.3 is precisely that the complement of S in Y is boundedly saturated. So we get

Corollary 7.1.4. *The collection of boundedly saturated subsets of Y is closed with respect to taking complements, finite intersections and unions. In other words, it is a Boolean algebra of sets.*

7.2. **The Metric in N.** We must begin by identifying the metric in N with respect to which the bounded saturation property of sets in ∂N will be defined. It will be not the Euclidean metric used to construct the boundary but the left invariant Riemannian metric obtained by introducing a suitable inner product in $\mathfrak{n}$. In this situation the diameter of a chosen fundamental domain F is bounded by some number D as is also the diameter of any Γ-translate of the domain.

Definition 7.2.1. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A *quasi-isometry* is a (not necessarily continuous) map $f\colon X_1 \to X_2$ for which there exist constants λ, ϵ and C such that

- for every $x_2 \in X_2$ there exists $x_1 \in X_1$ with $d_2\big(f(x_1), x_2\big) \leq C$,
- $\frac{1}{\lambda} d_1(x_1, y_1) - \epsilon \leq d_2\big(f(x_1), f(y_1)\big) \leq \lambda d_1(x_1, y_1) + \epsilon$ for all x_1, y_1 in X_1.

The crucial property of our metric is that the group Γ with the word metric is embedded quasi-isometrically when viewed as a subgroup of N.

Our choice for F will be the parallelogram spanned by the basis $\{\gamma_i\}$.

7.3. **Bounded Saturation: the Search.** We develop a systematic method of looking for boundedly saturated subsets of Y. Let Z be a left Γ-space with a Γ-invariant open dense complete locally compact (so that bounded closed sets are compact) metric subspace Z^0 on which Γ acts freely, cocompactly, properly discontinuously by isometries. Then according to Milnor for $x_0 \in Z^0$ the embedding $\epsilon\colon \gamma \mapsto \gamma \cdot x_0$ of Γ with the word metric into Z^0 is a quasi-isometry. In the course of the proof one constructs a compact subset $B \subseteq Z^0$ such that $\Gamma \cdot B = \bigcup_{\gamma \in \Gamma} \gamma B = Z^0$. Suppose that in addition there is a right Γ-action on Z which (1) leaves Z^0 invariant, (2) commutes with the left action: $(\gamma_1 \cdot z) \cdot \gamma_2 = \gamma_1 \cdot (z \cdot \gamma_2)$ for any $z \in Z^0$, and (3) restricts to the right translation action on $\epsilon(\Gamma)$, i.e., $(\gamma_2 \cdot x_0) \cdot \gamma_1 = (\gamma_2 \gamma_1) \cdot x_0$ for all γ_1, $\gamma_2 \in \Gamma$.

Theorem 7.3.1. *Let L be a boundedly saturated subset of $Z - Z^0$. Then*

1. *there is a point $z \in L$ which is a limit of $\epsilon(\Gamma)$,*
2. *the right orbit $z \cdot \Gamma \subseteq Z - Z^0$ is contained entirely in L.*

Proof. (1) Take an arbitrary $z' \in L$, and let $\{z_i\}$ be a sequence of points in Z^0 with $\lim_{i \to \infty} \{z_i\} = z'$. Let $B \subseteq Z^0$ be a ball of radius R centered at x_0 with the property that $\Gamma \cdot \operatorname{int} B = Z^0$. So the quasi-isometry constant $C \leq 2R$. Then $\{z_i\}[2R]$ contains all translates of B which contain some z_i, $i \geq 1$. In general, for a set $S \subseteq Z^0$, let $\mathcal{E}(S)$ be the union of all $\gamma \cdot \Omega_\Gamma$, $\gamma \in \Gamma$, such that $s \in \gamma \cdot \Omega_\Gamma$ for some $s \in S$. An quasi-isometry inverse to ϵ can be constructed by sending $z \mapsto \gamma(z)$ if

$z \in \gamma(z) \cdot B$. So $\{\gamma(z_i) \cdot x_0\} \subseteq \{z_i\}[2R]$. The sequence $\{\gamma(z_i)\}$ is unbounded, hence there is

$$z = \lim_{i\to\infty} \{\gamma(z_i) \cdot x_0\} \subseteq \overline{\{z_i\}[2R]} \cap (Z - Z^0) \subseteq L.$$

(2) Take the word metric k-ball B_k in Γ centered at e and act by it on $\{\gamma(z_i) \cdot x_0\}$ from the right. If $b \in B_k$ then $d(e, b) \leq k$, so $d(x_0, x_0 \cdot b) = d(x_0, b \cdot x_0) \leq \lambda k + \epsilon$, so

$$\begin{aligned} & d\big(\gamma(z_i) \cdot x_0, (\gamma(z_i) \cdot x_0) \cdot b\big) \\ = {} & d\big(\gamma(z_i) \cdot x_0, \gamma(z_i) \cdot (x_0 \cdot b)\big) \\ = {} & d(x_0, x_0 \cdot b) \leq \lambda k + \epsilon \end{aligned}$$

for any $\gamma \in \Gamma$. So $\{\gamma(z_i) \cdot x_0\} \cdot B_k \subseteq \{\gamma(z_i) \cdot x_0\}[\lambda k + \epsilon]$. Since $\lim_{i\to\infty}\{z_i \cdot \gamma\} = z \cdot \gamma$ by continuity,

$$z \cdot B_k \subseteq \overline{\{\gamma(z_i) \cdot x_0\}[\lambda k + \epsilon]} \cap (Z - Z^0) \subseteq L.$$

Letting k increase, we see that $z \cdot \Gamma \subseteq L$. □

This theorem indicates outlines of sets which must be very close to being boundedly saturated, and in many cases they are such. An example might be our own application which comes next or the case of a uniform lattice Γ acting on the symmetric space compactified by the ideal boundary. The theorem correctly suggests that each ideal point fixed by the trivial extension of the right action of Γ is also boundedly saturated.

Let us now return to the situation with $Z = \varepsilon N$ where $Z^0 = N$ is given the Γ-invariant metric defined above. Theorem 7.3.1 suggests that the open hemispheres from Corollary 6.3.2 might be good candidates for boundedly saturated subsets of ∂N.

Theorem 7.3.2. *Each connected component of the complement of ∂N_i in ∂N_{i-1} is boundedly saturated for all $1 \leq i \leq r$.*

7.4. **Bounded Saturation: the Proof.** Now we formally confirm the guess we made in §7.3. In the case Z^0 is a Lie group which acts on itself by left multiplication and the chosen metric is left invariant, Theorem 7.3.1 has a much stronger analogue.

Theorem 7.4.1. *Each right Z^0-orbit in $Z - Z^0$ is boundedly saturated.*

Proof. Let $z \in \partial Z^0$ and $C \subseteq Z$ be a closed subset such that $C \cap (Z - Z^0) \subseteq z \cdot Z^0$. Suppose there exists a number D with the property that

$$\overline{(C \cap Z^0)[D]} \backslash z \cdot Z^0 \neq \emptyset.$$

Then there is a sequence $\{y_i\} \subseteq (C \cap Z^0)[D]$ with the limit $\lim_{i\to\infty}\{y_i\} = y \notin z \cdot Z^0$. For each y_i choose $z'_i \in \{z_i\}$ such that $d(y_i, z'_i) \leq D$. Then $\lim_{i\to\infty}\{z'_i\} = \lim_{i\to\infty}\{z_i\}$. Also there are elements $b_i \in Z^0$ such that $z'_i = y_i \cdot b_i$, they satisfy $d(I, b_i) = d(y_i, y_i \cdot b_i) = d(y_i, z'_i) \leq D$. This infinite sequence has a cluster b in the D-ball $B_D \subseteq Z^0$. From the continuity of the right action we have

$$z \cdot b = \lim_{i\to\infty} \{z_i\} \cdot b = \lim_{i\to\infty} \{z_i \cdot b_i\} = \lim_{i\to\infty} \{y_i\} = y$$

which contradicts the assumption. □

This implies Theorem 7.3.2 since by Corollary 6.3.2 every subset of ∂N in question is right N-invariant.

7.5. **Cubical Cellular Decompositions.** Let $I^n = [-1,1]^n$ be the n-dimensional cube embedded in $\mathbb{R}^n$. It has 2^n vertices indexed by various n-tuples with entries either 1 or -1. Let us denote this set by $V_{(-1)}$. We also say that $V_{(-1)}$ is derived from $I_{(-1)} = \{\pm 1\}$ and write this as $V_{(-1)} = I^n_{(-1)}$. Now define the following subsets of I:

$$I_{(0)} = \{-1, 0, 1\}, \quad I_{(1)} = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}, \quad \ldots$$

where

$$I_{(i)} = \left\{-1, \ldots, \frac{k}{2^i}, \frac{k+1}{2^i}, \ldots, 1\right\}, \quad k \in \mathbb{Z}, \quad -2^i \leq k \leq 2^i,$$

for $i \in \mathbb{N}$. We also get the corresponding derived subsets of I^n:

$$V_{(0)}, V_{(1)}, \ldots, V_{(i)} = \{v_i(s_1, \ldots, s_n)\} = I^n_{(i)}, \ldots$$

where

$$v_i(s_1, \ldots, s_n) \stackrel{\text{def}}{=} \left(\frac{s_1}{2^i}, \ldots, \frac{s_n}{2^i}\right), \quad s_j \in \mathbb{Z}, \quad -2^i \leq s_j \leq 2^i.$$

At each stage $V_{(i)}$ is the set of vertices of the obvious cellular decomposition of I^n, where the top dimensional cells are n-dimensional cubes with the j-th coordinate projection being an interval

$$\left[\frac{k_j}{2^i}, \frac{k_j+1}{2^i}\right] \subseteq I, \quad 1 \leq j \leq i.$$

These cells can be indexed by the n-tuples $\{(k_1, \ldots, k_j, \ldots, k_n) : -2^i \leq k_j < 2^i\}$, the coordinates of the lexicographically smallest vertex, $2^{(i+1)n}$ of the n-tuples at all.

These decompositions behave well with respect to a sequence of certain collapses and induce cellular decompositions of the result. The collapses we have in mind are performed in the boundary of the cube I^n and its successive quotients. The first collapse contracts

$$\{(x_1, \ldots, x_{n-1}, *) \in I^n : \exists 1 \leq i \leq n-1 \text{ with } x_i = \pm 1\} \longrightarrow \text{point}.$$

We give this point the projective coordinates $(x_1, \ldots, x_{n-1}, \flat)$. The set

$$\{(x_1, \ldots, x_{n-1}, \flat) : \exists 1 \leq i \leq n-1 \text{ with } x_i = \pm 1\}$$

is the boundary of I^{n-1}. Now we induct on the dimension of the cube. For example, the collapse at the m-th stage can be described as

$$\{(x_1, \ldots, x_{n-m}, *, \flat, \ldots, \flat) \in I^{n-m+1} : \exists 1 \leq i \leq n-m \text{ with } x_i = \pm 1\} \\ \longrightarrow (x_1, \ldots, x_{n-m-1}, \flat, \ldots, \flat).$$

The process stops after $n-1$ stages when the points $(\pm 1, \flat, \ldots, \flat)$ do not get identified. The end result is a topological ball B^n with the CW-structure consisting of two cells of each dimension $0, 1, \ldots, n-1$ and one n-dimensional cell and a continuous composition of collapses $\rho\colon I^n \to B^n$. Each lower dimensional cell is the quotient of the appropriate face in ∂I^n: if the face F was defined by $x_i = \pm 1$ then $\dim \rho(F) = i$. Notice that every old derived cubical CW-structure in I^n induces a CW-decomposition of the image in the obvious way.

Consider the (-1)-st derived decomposition of I^r and the corresponding CW-structure in B^r. Notice that the cells in ∂B^r are in bijective correspondence with the open hemispheres from Theorem 7.3.2. Indeed, if we reverse the order in the definition of the diffeomorphism σ in §6.2 so that $te_i \subseteq \mathfrak{n}$ is mapped onto the

$(r-k+1)$-st coordinate axis in $\mathbb{R}^r$, the sequence of collapses in $\partial I^n \subseteq \mathbb{R}^r$ follows the order from the "youngest" central extension by C_1 to the "oldest" by C_r in N. We will refer to this isomorphism of CW-structures as $\Upsilon\colon \partial B^r \to \partial N$.

There are cubical analogues of links and stars of the usual simplicial notions. Thus the *star* of a vertex is the union of all cells which contain the vertex in the boundary. The *open star* is the interior of the star. For the i-th derived cubical decomposition, the open star of the vertex $v_i(s_1,\dots,s_n)$ will be denoted by $\mathrm{Star}^{\mathrm{o}}\big(v_i(s_1,\dots,s_n)\big)$. These sets form the *open star covering* of I^r.

By *vertices* in ∂N we mean the image $\Upsilon\rho\big(V_{(n)} \cap \partial I^r\big)$. Let $v \in \Upsilon\rho\big(V_{(n)} \cap \partial I^r\big)$ then

$$\mathrm{Star}^{\mathrm{o}}\big((\Upsilon\rho)^{-1}(v) \cap V_{(n)}\big) = \bigcup_{\substack{v_n \in V_{(n)}\\ \Upsilon\rho(v_n)=v}} \mathrm{Star}^{\mathrm{o}}(v_n)$$

is an open neighborhood (the open star) of $(\Upsilon\rho)^{-1}(v)$, and, in fact,

$$\mathrm{Star}^{\mathrm{o}}_n(v) \overset{\mathrm{def}}{=} \Upsilon\rho(\mathrm{Star}^{\mathrm{o}}\big(\rho^{-1}\Upsilon^{-1}(v) \cap V_{(n)})\big)$$

is an open neighborhood of v which we call the *open star* of v. The map $\Upsilon\rho$ is bijective in the interior of I^r, so $\mathrm{Star}^{\mathrm{o}}_n(v)$ can be defined by the same formula for $v \in \Upsilon\rho\big(V_{(n)} \cap \mathrm{int} I^r\big)$.

Remark 7.5.1. The analysis in this section could now be followed by a proof of the Novikov conjecture for the groups of class $\mathcal{N}\imath\ell$. This has been done, however, using various reductions to a situation with small action at infinity (e.g., in [17, 35, 70]). The construction of $\hat{N}$ here can be viewed as a generalization of the application of Theorem 2 to finite products of groups for which Novikov conjecture can be verified using Theorem 1 ([65]). The product argument fails to enlarge the class of groups covered by Theorem 1 because the classifying space functor preserves products, and, therefore, inductive application of this theorem suffices.

Part 4. **Arithmetic Groups of Rank One**

8. The Borel–Serre Enlargement

Let $G = G(\mathbb{Q})$ be a semi-simple algebraic subgroup of $GL_n(\mathbb{Q})$. Recall from §2.3 that a subgroup Γ of G is *arithmetic* if Γ and $G \cap GL_n(\mathbb{Z})$ are commensurable (that is, $\Gamma \cap G \cap GL_n(\mathbb{Z})$ has finite index in both Γ and $G \cap GL_n(\mathbb{Z})$). Γ is a discrete subgroup of the real Lie group of real points $G(\mathbb{R})$ and acts (non-cocompactly) on the symmetric space of maximal compact subgroups $X = G(\mathbb{R})/K$ so that X is a model for $E\Gamma$ if Γ is torsion-free. Borel and Serre ([11]) form a contractible enlargement $\bar{X} \supseteq X$ which depends only on the $\mathbb{Q}$-structure of G so that the action of Γ extends to $\bar{X}$. $\bar{X}$ is another model for $E\Gamma$ but now the action is cocompact.

We will make the preliminary discussion of their construction more general than needed for the proof of Theorem 3 proper. This material will be referred to in the appendix and in certain claims that we also show in their natural generality exceeding the requirements of the proof.

8.1. **Levi Reduction.**

Let H be a linear algebraic group defined over the subfield k of the complex numbers. The connected component of the identity is denoted by H^0. The Zariski topology is always understood in $H(k)$ when $k \neq \mathbb{R}$, and the classical Lie group topology when $k = \mathbb{R}$. If H is connected, put

$$^0H \stackrel{\text{def}}{=} \bigcap_{\chi \in X^*(H)} \ker(\chi^2),$$

where $X^*(H)$ is the group of rational characters. The group ^0H is normal in H and is defined over k. Let S be a maximal k-split torus of the radical RH. Then $H(\mathbb{R}) = A \ltimes {}^0H(\mathbb{R})$, a semi-direct product, where $A = S(\mathbb{R})^0$, and $^0H(\mathbb{R})$ contains every compact subgroup of $H(\mathbb{R})$, and also, if $k = \mathbb{Q}$, every arithmetic subgroup of H. If R_uH denotes the unipotent radical of H, then $\hat{L}_H = H/R_uH$ is the canonical reductive Levi quotient. It is also defined over k.

Notation . An object associated to the reductive Levi quotient $\hat{L}_H$ rather than the group H itself will usually indicate this by wearing a "hat".

Let $\hat{M}_H = {}^0\hat{L}_H$. Denote by $\mathcal{P}_k(H)$ the set of parabolic k-subgroups of H. The projection $\pi_H \colon H \to \hat{L}_H$ induces a bijection $\mathcal{P}_k(H) \leftrightarrow \mathcal{P}_k(\hat{L}_H)$ preserving conjugacy classes over k, and likewise $\mathcal{P}_k(\hat{L}_H) \leftrightarrow \mathcal{P}_k(\hat{L}_H/\hat{C}_H)$, where $\hat{C}_H$ is the center of $\hat{L}_H$.

8.2. **Standard Parabolic Subgroups.** A reference for the following material is [79, §9, §10]. The purpose of this section is to introduce the notation used in §8.3.

Let $\hat{T}_H$ be a maximal k-split torus of $\hat{L}_H/\hat{C}_H$. $\hat{L}_H/\hat{C}_H$ is generated by the centralizers $Z(\hat{S}_H)$, where $\hat{S}_H$ runs through the codimension one subtori of $\hat{T}_H$ ([79], 9.1.2). A codimension one torus is called *regular* if its centralizer is solvable and *singular* otherwise. If S is any torus acting linearly on a vector space V, the *weights* of S in V are the characters $\chi \in X^*(S)$ such that the space

$$V_\chi = \{v \in V : s \cdot v = \chi(s)v\} \neq 0.$$

There are non-zero weights of $\hat{T}_H$ determined uniquely up to sign by regular codimension one subtori (see [79], 9.1.3). They are called *roots* of $\hat{L}_H/\hat{C}_H$ with respect to $\hat{T}_H$. Evidently, they are related to the roots of the Lie algebra of $\hat{L}_H/\hat{C}_H$ determined by the Lie subalgebra of $\hat{T}_H$ from §2.1.2. Let s_χ be the *reflection associated to* χ. Call two Borel subgroups B_1 and B_2 containing $\hat{T}_H$ *adjacent* if

$$\dim B_1 \cap B_2 = \dim B_1 - 1 = \dim B_2 - 1.$$

The positive roots α relative to B such that $s_\alpha \cdot B$ and B are adjacent form a system of *positive simple roots* with respect to $\hat{T}_H$. If P is any (*standard*) parabolic subgroup containing B, it is determined by a set of positive simple roots ([79], 10.3.2).

If $\hat{\Delta}_H$ is a system of positive simple roots with respect to $\hat{T}_H$, let $\hat{P}_\Theta$ (resp. P_Θ) denote the standard k-parabolic subgroup of $\hat{L}_H/\hat{C}_H$ (resp. of H) relative to $\hat{T}_H$ and $\hat{\Delta}_H$ corresponding to the choice of $\Theta \subseteq \hat{\Delta}_H$. This correspondence $\Theta \mapsto P_\Theta$ defines a lattice isomorphism between the power set of $\hat{\Delta}_H$ and the set of standard parabolic k-subgroups of H. Moreover, each $P \in \mathcal{P}_k(H)$ can be written as $hP_\Theta h^{-1}$ for some $h \in H(k)$ and a uniquely determined $\Theta(P) \subseteq \hat{\Delta}_H$.

We discuss the two cases $k = \mathbb{Q}$ or $\mathbb{R}$ simultaneously. As before, X is the symmetric space of maximal compact subgroups of $G(\mathbb{R})$.

8.3. **Geodesic Action.** The geodesic action and the associated enlargements are introduced in the classical paper [11]. We will again recall just enough of that material to establish notation.

Let $P \in \mathcal{P}_k(G)$, and let $\hat{S}_P$ denote the maximal k-split torus of $\hat{C}_P$, and $\hat{A}_P = \hat{S}_P(\mathbb{R})^0$. The dimension of $\hat{A}_P$ is called the parabolic k-rank of P. To each $x \in X$ is associated the Cartan involution θ_x of G that acts trivially on the corresponding maximal compact subgroup. There is a unique θ_x-stable lift $\tau_x \colon \hat{L}_P(\mathbb{R}) \to P(\mathbb{R})$ which gives θ_x-stable liftings $A_{P,x} = \tau_x(\hat{A}_P)$, $S_{P,x} = \tau_x\big(\hat{S}_P(\mathbb{R})\big)$, and $M_{P,x} = \tau_x\big(\hat{M}_P(\mathbb{R})\big)$ of the subgroups $\hat{A}_P$, $\hat{S}_P(\mathbb{R})$, and $\hat{M}_P(\mathbb{R})$.

Definition 8.3.1. The *geodesic action* of $\hat{A}_P$ on X is given by $a \circ x = a_x \cdot x$, where $a_x = \tau_x(a) \in A_{P,x}$ is the lifting of $a \in \hat{A}_P$.

X can be viewed as the total space of a principal $\hat{A}_P$-bundle under the geodesic action. $\hat{A}_P$ can be openly embedded into $\mathbb{R}^{\mathrm{card}(\hat{\Delta}-\Theta(P))}$ via

$$\hat{A}_P \longmapsto (\mathbb{R}_+^*)^{\mathrm{card}(\hat{\Delta}-\Theta(P))}.$$

Let $\bar{A}_P$ be the "corner" consisting of $\hat{A}_P$ together with positive card$(\hat{\Delta} - \Theta(P))$-tuples where the entry ∞ is allowed with the obvious topology making it diffeomorphic to $(0,\infty]^{\mathrm{card}(\hat{\Delta}-\Theta(P))}$. $\hat{A}_P$ acts on $\bar{A}_P$, and the *corner* $X(P)$ associated to P is the total space of the associated bundle $X \times_{\hat{A}_P} \bar{A}_P$ with fiber $\bar{A}_P$. Denote the common base of these two bundles by $e(P) = \hat{A}_P \backslash X$. In particular, $e(G^0) = X$.

Definition 8.3.2. The *Borel–Serre enlargement*

$$\bar{X}_k = \bigsqcup_{P \in \mathcal{P}_k(G)} e(P)$$

has a natural structure of a manifold with corners in which each corner $X(P) = \bigsqcup_{Q \supseteq P} e(Q)$ is an open submanifold with corners. The action of $Q(k)$ on X extends to the enlargement $\bar{X}_k$. The faces $e(P)$, $P \in \mathcal{P}_k(G)$, are permuted under this action.

Example 8.3.3 ($\bar{\boldsymbol{X}}(\boldsymbol{SL_2})$)**.** The hyperbolic plane X can be thought of as the open unit disk $\mathbb{E}$ in $\mathbb{C}$ or as the upper half-plane $\mathbb{H}$. Elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Q})$ act on $\mathbb{H}$ from the left as Möbius transformations: $z \mapsto \frac{az+b}{cz+d}$. The action extends to the hyperbolic boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. $\mathbb{E}$ and $\mathbb{H}$ are related via the biholomorphic Cayley mappings $\mathbb{H} \to \mathbb{E}, z \mapsto \frac{z-i}{z+i}$ and $\mathbb{E} \to \mathbb{H}, z \mapsto i\frac{1+z}{1-z}$. The *rational points* on the unit circle $\partial\mathbb{E}$ are the image of $\mathbb{Q} \subseteq \mathbb{R} \subseteq \partial\mathbb{H}$. The proper $\mathbb{Q}$-parabolic subgroups P are the stabilizers of the rational points p in $\partial\mathbb{E}$. All of them are Borel subgroups.

For each P the positive reals $\lambda \in \mathbb{R}_+$ act on X by translations of magnitude $\log\lambda$ along hyperbolic geodesics in the direction of the cusp p. This is the geodesic action. Let $X \to e(P)$ be the quotient mapping. It is a principal fibration with hyperbolic geodesics as fibers and the structure group $\mathbb{R}_+$. Each geodesic γ can be completed to a half-line by adding a limit point e_γ in the positive direction of the $\mathbb{R}_+$-action. Extend the action of $\mathbb{R}_+$ trivially to e_γ. The corner $X(P)$ associated to P is the total space of the associated fiber bundle with typical fiber $\gamma \cup \{e_\gamma\}$. Now $X(P) = X \cup e(P)$ where $e(P)$ is a copy of $\mathbb{R}$ "over" p which parametrizes the geodesics converging to p. Take $\bar{X}$ to be $\bigcup_P X(P)$ where P ranges over all proper $\mathbb{Q}$-parabolic subgroups.

Given a point and an open interval $y \in V \subseteq e(P)$, the restriction of a cross-section of the principal bundle $X \to e(P)$ to V determines an open neighborhood W of y in $X(P)$ *defined by geodesic influx from* V *into* X, i.e., W consists of all points on geodesics connecting the image of the cross-section to V (including the latter but not the former). This description makes it clear that $\bar{X}$ is a Hausdorff space. Every $g \in G$ acts as a Möbius transformation on X and sends a geodesic converging to a rational point to another hyperbolic geodesic. If $g \in \Gamma \subseteq SL_2(\mathbb{Q})$ then the new geodesic converges to a rational point and thus defines $g \cdot y \in \partial\bar{X}$.

Example 8.3.4 ($\boldsymbol{\bar{X}(SL_3)}$)**.** Make the choice of a maximal compact subgroup $K = SO_3(\mathbb{R})$ in $G = SL_3(\mathbb{R})$. Let P_0 be the Borel $\mathbb{Q}$-subgroup of G consisting of the upper triangular matrices, and let T_0 be the torus of diagonal matrices denoted by $\mathrm{diag}(t_i)$. Now $A_0 = \{\mathrm{diag}(t_i) \in T_0 : t_i > 0\}$ is the split component of T_0 which is stable with respect to the Cartan involution θ_K. Let Φ be the set of roots of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G determined by the Cartan subalgebra $\mathfrak{a}_{\mathbb{C}}$. Since $G = SL_3$ is split over $\mathbb{Q}$ by applying Theorem 18.7 of [10], we may identify Φ and $\Phi_{\mathbb{R}}$. Choose an ordering on Φ so that the weights of $\mathfrak{a}$ are positive. The set of simple roots with respect to this ordering is $\Delta = \{\alpha_1,\ \alpha_2\}$, where α_i denotes the usual mapping t_i/t_{i+1} on T_0.

The conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G are parametrized by subsets J of Δ. In particular, if Q is a maximal parabolic $\mathbb{Q}$-subgroup, then it is conjugate to a standard maximal parabolic $\mathbb{Q}$-subgroup P_j given by

$$P_j = P_{\Delta - \{\alpha_j\}} = \{(a_{ij}) \in G : a_{ik} = 0,\ k \le j < i\}, \quad j = 1, 2.$$

So the standard parabolic subgroups in G are either P_0 or one of

$$P_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in G \right\} \quad \text{or} \quad P_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \in G \right\}.$$

We say that Q is *of type* j. If we put

$$T_{\Delta - \{\alpha_j\}} = \left(\bigcap_{\alpha_i \in \Delta, i \ne j} \ker \alpha_i \right)^0,$$

we have $P_j = Z(T_{\Delta - \{\alpha_j\}}) \cdot N_{P_j}$. If A_j is the θ_K-stable split component of P_j in the radical of P_j, and $M_j = {}^0L_j$, where $L_j = Z(A_j)$ is the Levi subgroup, then we get the Levi decomposition $P_j = M_j \cdot A_j \cdot N_j$. Explicitly, for the standard Borel subgroup P_0 we have

$$M_0 = \{\mathrm{diag}(t_i) :\ t_i = \pm 1\}, \quad N_0 = \left\{ \begin{pmatrix} 1 & n_{12} & n_{13} \\ & 1 & n_{23} \\ & & 1 \end{pmatrix} \in G \right\}.$$

For the maximal standard parabolic subgroups

$$M_1 = \left\{ \begin{pmatrix} F & \\ & \epsilon \end{pmatrix} \in G :\ F \in SL_2^{\pm}(\mathbb{R}),\ \epsilon = \pm 1 \right\},$$

where

$$SL_2^{\pm}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} :\ a, b, c, d \in \mathbb{R},\ \det = \pm 1 \right\},$$

and

$$M_2 = \left\{ \begin{pmatrix} \epsilon & \\ & F \end{pmatrix} \in G : \ F \in SL_2^{\pm}(\mathbb{R}), \ \epsilon = \pm 1 \right\}.$$

The other groups in the corresponding decompositions are:

$$A_1 = \left\{ \begin{pmatrix} a^{-1} & & \\ & a^{-1} & \\ & & a^2 \end{pmatrix} \in G : \ a \in (\mathbb{R}^*)^+ \right\}, \quad N_1 = \left\{ \begin{pmatrix} 1 & 0 & n_3 \\ & 1 & n_2 \\ & & 1 \end{pmatrix} \in G \right\},$$

$$A_2 = \left\{ \begin{pmatrix} b^2 & & \\ & b^{-1} & \\ & & b^{-1} \end{pmatrix} \in G : \ b \in (\mathbb{R}^*)^+ \right\}, \quad N_2 = \left\{ \begin{pmatrix} 1 & n_1 & n_3 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in G \right\}.$$

Denote by I_3 the 3×3 identity matrix and put

$$d_1 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix},$$

then, for $i = 1, 2$, M_i consists of two connected components: $M_1 = \{d_2, I_3\} \times M_1^0$ and $M_2 = \{d_1, I_3\} \times M_2^0$ with $M_i^0 \cong SL_2(\mathbb{R})$.

In this case $\operatorname{rank}_{\mathbb{Q}} G = 2$, so the Tits building is a graph whose vertices are the points and lines of the projective plane over $\mathbb{Q}$, and edges are the set of incidence relations among them. If the orbit of A_0 is identified with the first plane quadrant, then $\bar{A}_0$ is the corner with interior A_0. The orbits of A_1 and A_2 are the horizontal and vertical lines in A_0. The space $e(P_0)$ intersects each $\bar{A}_0$ at its vertex, while $e(P_i)$, $i = 1, 2$, intersect $\bar{A}_0$ in the vertical and horizontal lines of its boundary. Fixing a vertex $W(P)$ let C range over all chambers having $W(P)$ as an endpoint, i.e., over all $W(B)$ for Borel $\mathbb{Q}$-subgroups $B < P$. These are indexed by the projective line over $\mathbb{Q}$. The corresponding 3-dimensional strata $e(B)$ are disjoint in the boundary of the 4-dimensional space $\overline{e(P)}$.

Now everywhere in this paragraph the field $\mathbb{Q}$ may be replaced by $\mathbb{R}$. The resulting combinatorial object $T_3(\mathbb{R})$ is dual to the combinatorics of the Satake compactification of X rather than the non-compact rational enlargements of Borel–Serre and Satake.

8.4. **Geodesic Influx Neighborhoods.** We borrow a definition from [90]. Let $q_P \colon X \to e(P)$ denote the bundle map. For any open subset $V \subseteq e(P)$ a cross-section σ of q_P over V determines a translation of V from the boundary of $\bar{X}_k$ into the interior X. For any $t \in \hat{A}_P$ put

$$\hat{A}_P(t) = \{a \in \hat{A}_P : \ a^\alpha > t^\alpha \text{ for all } \alpha \in \Delta_P\},$$

where Δ_P is the set of those simple roots with respect to a lifting of $\hat{T}_P$ that occur in $R_u P$ (transported back to $\hat{A}_P$). It is complementary to $\Theta(P)$ in $\hat{\Delta}$.

Definition 8.4.1. For any cross-section $\sigma(V)$, a set of the form $\hat{W}(V, \sigma, t) = \hat{A}_P(t) \circ \sigma(V)$ will be called an *open set defined by geodesic influx from* V *into* X.

There is a natural isomorphism

$$\mu_\sigma \colon \hat{A}_P(t) \times V \xrightarrow{\simeq} \hat{W}(V, \sigma, t)$$

which extends to a diffeomorphism

$$\bar{\mu}_\sigma \colon \bar{A}_P(t) \times V \xrightarrow{\simeq} W(V, \sigma, t).$$

Now $W(V,\sigma,t)$ is a neighborhood of V in $\bar{X}$ with $\bar{\mu}_\sigma(\{(\infty,\dots,\infty)\}\times V)=V$. We will call it an *open neighborhood defined by geodesic influx from V into X*.

8.5. **The Construction of $E\Gamma$.** All of that done so far works for more general homogeneous H-spaces than symmetric spaces for semi-simple H. Borel and Serre call them spaces of type $S{-}k$. For each $Q\in\mathcal{P}_k(G)$, $e(Q)$ is such a space. So

$$\overline{e(Q)}(k)=\bigsqcup_{P\in\mathcal{P}_k(Q)} e(P)=\bigsqcup_{Q\supseteq P\in\mathcal{P}_k(G)} e(P)$$

can be formed; it is diffeomorphic to the closure $\overline{e(Q)}$ of $e(Q)$ in $\bar{X}_k$. In fact, whenever $P\subseteq Q$, $\hat{A}_Q$ is canonically a subgroup of $\hat{A}_P$ so that the geodesic actions are compatible. $\hat{A}_P$ acts geodesically on $e(Q)$ through $\hat{A}_P/\hat{A}_Q$ with quotient $e(P)$. The stratum $e(P)\subseteq\overline{e(Q)}$ is the set of limit points of this geodesic action.

Recall that the parabolic $\mathbb{Q}$-subgroups index the simplices $W(P)$ of the Tits building $T(\mathbb{Q})$ of G. For $G=SL_n$ the Tits building $T_n(\mathbb{Q})$ is the simplicial complex with one vertex for each non-trivial subspace of $\mathbb{Q}^n$ and a set of vertices spanning a simplex if and only if the corresponding subspaces can be arranged into a flag. The dimension of the strata $e(P)$ and the incidence relations among their closures reflect the structure of this building as follows:

$$\dim e(P)+\dim\sigma_P=\dim X-1,$$
$$e(P)\cap\overline{e(Q)}\neq\emptyset\iff e(P)\subseteq\overline{e(Q)}\iff W(Q)\subseteq W(P)\iff P\subseteq Q.$$

The minimal parabolic (Borel) $\mathbb{Q}$-subgroups correspond to the strata $e(P)$ of dimension $\dim X-\operatorname{rank}_{\mathbb{Q}}G$, and to the maximal simplices of the building.

Remark 8.5.1. When B is a Borel $\mathbb{R}$-subgroup of G, we have the Iwasawa decomposition $G(\mathbb{R})=K\cdot A_B\cdot N_B(\mathbb{R})$ (see [68, Theorem 3.9]), the same as the one corresponding to the Weyl chamber σ_B. Then $X\approx A_B\cdot N_B(\mathbb{R})$, and the geodesic action of A_B on X coincides with multiplication. The quotient $e(B)$ can be viewed as the underlying space of the nilpotent group $N_B(\mathbb{R})$.

The main result of Borel and Serre about this construction is that $\bar{X}_{\mathbb{Q}}$ is contractible, the action of Γ on $\bar{X}_{\mathbb{Q}}$ is proper, and the quotient $\Gamma\backslash\bar{X}_{\mathbb{Q}}$ is compact. So, indeed, $\bar{X}_{\mathbb{Q}}$ is the new $E\Gamma$ we can use. The space $\hat{X}$ to be constructed in §9 will compactify $\bar{X}_{\mathbb{Q}}$.

8.6. **Satake Compactifications.** Workers in different fields mean different objects when they speak of Satake compactifications. The earlier constructions are compactifications of a globally symmetric space ([74]) which were later compared to Martin and Furstenberg compactifications and have applications in analysis; the later ones are compactifications of (locally symmetric) arithmetic quotients of bounded symmetric domains ([75]) which are the quotients of certain "rational" portions of the first construction with a properly redefined topology. We are interested in the first construction and the techniques used to study the second. The references for this material are [42, 74, 75, 89].

Let G be as before and $\tau\colon G(\mathbb{R})\to\mathbb{SL}(\mathbb{V})$ be a finite-dimensional representation with finite kernel. For an admissible inner product on V, let v^* denote the adjoint of v. The admissibility of the inner product means that $\tau(g)\tau\big(\theta_K(g)\big)^*=I$. So the mapping $\tau_0(g):=\tau(g)\tau(g)^*$ descends to X. Each $\tau_0(g)$ is a self-adjoint endomorphism of V. Factoring out the action of the scalars, we get $\tau_0\colon X\to PS(V)$ which is an equivariant embedding. Taking the closure of the image, one gets the *Satake compactification* $X^{\mathrm{S}}(\tau) = \overline{\tau_0(X)}$.

It turns out that the G-action on X extends to $X^{\mathrm{S}}(\tau)$ and the boundary $X^{\mathrm{S}}(\tau) - X$ decomposes into orbits of certain subgroups of G called *boundary components*. The subgroups are the parabolic subgroups which are τ*-connected* (see [89, p. 322]). They also correspond to Θ*-connected* subsets of Δ for some $\Theta \subseteq \Delta$. This choice in $\Delta(SL_3) = \{\alpha_1, \alpha_2\}$ gives one of the two different compactifications.

The spaces X^{S}_{Θ} are certainly compact and Hausdorff as the closures of bounded domains in $PS(V)$.

8.6.1. *Maximal Satake Compactification.* This is X^{S}_{Δ} corresponding to $\Theta = \Delta$. It coincides with the reductive Borel–Serre enlargement (cf. [91, Remark 7.12]), so we do not need to go into fine details at this point.

8.6.2. *Minimal Satake Compactification.* The case we are interested in—of the group $G = SL_n$ with the identical representation of $\mathfrak{sl}_n(\mathbb{R})$—one of the two *minimal Satake compactifications* of $\mathcal{P}_n$ is the example which Satake himself computed in [74, §5.1]. It corresponds to the choice $\Theta = \{\alpha_1\}$. The result is the space

$$X^{\mathrm{S}}_1 = \overline{\mathcal{P}_n} = \bigcup_{i=1}^{n} G \cdot \mathcal{P}_i.$$

For $n = 3$ the boundary of X^{S} is a disjoint union of hyperbolic disks and points. The stabilizers of hyperbolic disks are conjugates of the standard maximal parabolic subgroup P_1; the stabilizers of the points are the conjugates of P_2. In general, the boundary strata of minimal Satake enlargements $X^{\mathrm{S}}_{1,k}$ are in bijective correspondence with the maximal parabolic k-subgroups. It is easy to see that the two minimal compactifications X^{S}_1 and X^{S}_2 are equivalent.

Our interest in Satake compactifications is rather technical. According to Zucker, they can be viewed as targets for surjective maps from the Borel–Serre enlargements (see §9.3).

9. Compactification of $E\Gamma$

The construction performed here can be compared to other compactifications of a symmetric space X, namely those of Martin ([62]), Satake and Furstenberg ([74, 37]), Karpelevič ([52]), and the conical compactification ([4]) by the ideal boundary, which have been used in the study of geometry and functional analysis on X. Our $\hat{X}$ also contains $\bar{X}$ as an open dense subspace. This gives it more algebraic flavor than is present in (at least the original formulations of) the other constructions.

The corner $X(P)$ can be constructed for any parabolic subgroup of G defined over $\mathbb{R}$ (see [11]). This means that instead of $\bar{X} = \bar{X}_{\mathbb{Q}}$ we can obtain a (larger) space $\bar{X}_{\mathbb{R}} = \bigcup_P X(P)$ where P ranges over all proper $\mathbb{R}$-parabolic subgroups. In general, there may appear complications in the way $\bar{X}_{\mathbb{Q}}$ and $\bar{X}_{\mathbb{R}}$ fit together arising, for example, from the fact that the $\mathbb{Q}$-rank of G may not be equal to the $\mathbb{R}$-rank. Restricting our attention to the case of $\mathrm{rank}_{\mathbb{R}} G = 1$ (which we assume from now on with the exception of §9.2) avoids such phenomena.

9.1. **Proper Borel–Serre Strata.** For an arithmetic subgroup Γ of $G(\mathbb{Q})$ and any Borel subgroup $B \in \mathcal{B}_{\mathbb{Q}}$, $\Gamma_B = \Gamma \cap B(\mathbb{Q})$ is the largest subgroup which acts in $e(B)$. If we write the Langlands decomposition as $B(\mathbb{R}) = \mathbb{M}(\mathbb{R}) \cdot \mathbb{A} \cdot \mathbb{N}(\mathbb{R})$ then $\Gamma_B \subseteq {}^0B(\mathbb{R}) = M(\mathbb{R}) \cdot \mathbb{N}(\mathbb{R})$. In fact, $\Gamma_B = \Gamma_M \ltimes \Gamma_N$ for $\Gamma_M = \Gamma \cap M(\mathbb{R})$, $\Gamma_N = \Gamma \cap N(\mathbb{R})$. Since in our case Γ is torsion-free and $M = Z(A) \cap K$, $\Gamma_B = \Gamma_N$.

Proposition 9.1.1. $\Gamma_N = \Gamma \cap N$ *is a uniform nilpotent lattice in* N.

This is precisely the property called *admissibility* in [39], and our Γ are proved to be always admissible in [39, Theorem 5.3].

We start by compactifying each $e(B)$, $B \in \mathcal{B}_{\mathbb{R}}$, $\Gamma \cap B(\mathbb{R})$-equivariantly, then provide the new points with certain neighborhoods which will form a part of the basis for the topology on $\hat{X}$. Recall Remark 8.5.1. The lattice $\Gamma \cap N$ acts on N via left multiplication. We will see that this is, in fact, the action of Γ_B on the stratum $e(B)$.

9.2. **The Action on a Stratum.** To discuss the action on a stratum, we return to the generality and notation of §8.1 and 8.2. Let P be a parabolic $\mathbb{R}$-subgroup of G. Recall that R_uP is the unipotent radical of P then $\hat{L}_P = P/R_uP$ is the reductive Levi quotient, $\pi_P \colon P \to \hat{L}_P$ is the projection, and $\hat{M}_P = {}^0\hat{L}_P$. The real points of the Levi quotient split as a direct product

$$\hat{L}_P(\mathbb{R}) = \hat{M}_P(\mathbb{R}) \times \hat{A}_P,$$

and there is the Langlands decomposition

$$P(\mathbb{R}) = M_{P,x} \cdot A_{P,x} \cdot L_{P,x}.$$

Recall that K_x is the stabilizer of x in $G(\mathbb{R})$ acting on X. Then $K_{P,x} = K_x \cap P(\mathbb{R})$ is the stabilizer of x in $P(\mathbb{R})$. The Borel–Serre stratum $e(P) = P(\mathbb{R})/K_{P,x} \cdot A_{P,x}$ is a space of type S for P, but it is not a symmetric space in the usual sense. Notice that it is acted upon from the left by $R_uP(\mathbb{R})$.

Definition 9.2.1. The quotient $\hat{e}(P)$ is called the *reductive Borel–Serre stratum.*

Denote the quotient map by $\mu_P \colon e(P) \to \hat{e}(P)$. Let $\hat{K}_P = \pi_P(K_{P,x})$, then $\hat{K}_P$ is a maximal compact subgroup of $\hat{M}_P(\mathbb{R})$ and is lifted to $K_{P,x}$ by τ_x. From the Langlands decomposition,

$$\hat{e}(P) = R_uP(\mathbb{R})\backslash P(\mathbb{R})/K_{P,x} \cdot A_{P,x} = \hat{L}_P(\mathbb{R})/\hat{K}_P \cdot \hat{A}_P \cong \hat{M}_P(\mathbb{R})/\hat{K}_P$$

is the "generalized" symmetric space associated to the reductive group $\hat{L}_P$: in general, it may not be connected, and it may have trivial $\mathbb{R}^*$ factors, because the radical RL_P may still be non-trivial (see [42, §7, 8]).

Proposition 9.2.2. *For each* $P \in \mathcal{P}_{\mathbb{R}}(G)$*, the principal* $R_uP(\mathbb{R})$*-fibration* μ_P *extends to a principal fibration*

$$\bar{\mu}_P \colon \overline{e(P)} \longrightarrow \overline{\hat{e}(P)}.$$

Proof. This can be extracted from [11, §§2.8, 3.10, 5.2, 7.2(iii)]. Let $Q \subseteq P$ be proper parabolic subgroups with unipotent radicals $R_uQ \supseteq R_uP$, then Q determines a parabolic subgroup

$$Q^P = \pi_P(Q) = Q/R_uP \subseteq \hat{L}_P = P/R_uP$$

with unipotent radical $R_uQ^P = R_uQ/R_uP$. Now A_{Q^P} is canonically identified with $A_{P,B}$ (in the notation of Borel and Serre), and the geodesic actions of A_Q

on $e(P)$ and $\hat{e}(P)$ commute with μ_P. So $X_P(Q)$ is a principal $R_uP(\mathbb{R})$-bundle over $X_{\hat{L}_P}(Q^P)$, and the projection $\tau_Q\colon X_P(Q) \to X_{\hat{L}_P}(Q^P)$ extends μ_P. These fibrations τ_* are compatible with the order in the lattice $\mathcal{P}(P)$ in the sense that for each pair $Q_1 \subseteq Q_2 \subseteq P$ the restriction of τ_{Q_1} to $e(Q_2)$ is the projection of a principal $R_uP(\mathbb{R})$-fibration with base $e(Q_2^P)$. So the principal fibrations τ_* are also compatible with the inclusions $X(Q_2) \hookrightarrow X(Q_1)$ and match up to give a principal fibration for $\overline{e(P)}$ over $\overline{\hat{e}(P)}$. □

Proposition 9.2.3 ([88]). *There is a diffeomorphism*

$$F\colon R_uP(\mathbb{R}) \times \hat{e}(P) \longrightarrow e(P)$$

given by

$$F(u, z\hat{K}_P\hat{A}_P) = u \cdot \tau_x(z)K_{P,x}A_{P,x} \in e(P) = P(\mathbb{R})/K_{P,x}A_{P,x}.$$

Here, $z\hat{K}_P\hat{A}_P \in \hat{e}(P) = \hat{L}_P(\mathbb{R})/\hat{K}_P\hat{A}_P$. The map F depends on the choice of the basepoint x which determines the lift τ_x.

Lemma (7.8) of [42] gives a very convenient formula for the action of $P(\mathbb{R})$ on $e(P)$ in terms of the coordinates that F provides. Notice that for any $g \in P(\mathbb{R})$, $g \cdot \tau_x\mu_P(g^{-1}) \in \ker(\mu_P) = R_uP(\mathbb{R})$, so

$$g \cdot u \cdot \tau_x\mu_P(g^{-1}) = gug^{-1} \cdot g\tau_x\mu_P(g^{-1}) \in R_uP(\mathbb{R})$$

for all $g \in P(\mathbb{R})$, $u \in R_uP(\mathbb{R})$.

Lemma 9.2.4. *The action of $P(\mathbb{R})$ on $R_uP(\mathbb{R}) \times \hat{e}(P)$ is given by*

$$g \cdot (u, z\hat{K}_P\hat{A}_P) = (g \cdot u \cdot \tau_x\mu_P(g^{-1}), \mu_P(g) \cdot z\hat{K}_P\hat{A}_P).$$

This formula shows that $R_uP(\mathbb{R})$ acts only on the first factor by translation. Specializing it to the action of the discrete subgroup Γ_P shows that in the case of the standard (as well as any) Borel subgroup B_0, when $R_uB_0(\mathbb{R}) = N \approx e(B_0)$, the action is precisely the left multiplication action of Γ_{B_0} as a subgroup of N. Thus the discussion in §6 becomes relevant, and $e(B_0)$ may, indeed, be compactified Γ_{B_0}-equivariantly by ∂N.

It also follows from the formula that there are other equivariant enlargements where the strata are reductive Borel–Serre strata.

Definition 9.2.5. The *reductive Borel–Serre enlargement* $\bar{X}_k^\rho$ ($k = \mathbb{Q}$ or $\mathbb{R}$) of X is the topological space obtained from the corresponding Borel–Serre enlargement $\bar{X}_k$ by collapsing each nilmanifold fiber of the projection $\pi_P\colon e(P) \to \hat{e}(P)$ to a point. These projections combine to give a quotient map $\pi\colon \bar{X}_k \to \bar{X}_k^\rho$.

The quotient $\bar{X}_{\mathbb{Q}}^\rho/\Gamma$ is usually called the *reductive Borel–Serre compactification.*

9.3. **Comparison with Satake Compactifications.** Assume that X is a Hermitian symmetric space. An example is the space $\mathcal{P}_n$ of positive definite bilinear forms, and this represents it as a bounded symmetric domain. We denote by $f(Q)$ the boundary strata of the maximal Satake compactification X^{S} corresponding to maximal parabolic subgroups $Q \subseteq G(\mathbb{R}) = \mathbb{I}_{\not{k}}(\mathbb{X})$. The Levi factor of Q decomposes as an almost direct product $M_Q = Q_hQ_l$, i.e., the factors Q_h, Q_l commute and the intersection $Q_h \cap Q_l$ is finite, where Q_h is the centralizer of the center $\mathcal{W}_Q = Z(R_uQ)$ in M_Q. Now $f(Q) \cong Q_h/(K \cap Q_h)$ which is again a Hermitian symmetric space. The symmetric space for the other factor Q_l is an open self-adjoint homogeneous cone in a vector space.

Let B_0 be the standard Borel subgroup of G. The maximal standard parabolic subgroups $Q_1, \ldots, Q_r$ of G contain B_0. They can be totally ordered as follows:

$$Q_i \prec Q_j \iff \mathcal{W}_{Q_i} \subseteq \mathcal{W}_{Q_j} \iff \overline{f(Q_i)} \supseteq f(Q_j).$$

Enumerate the maximal standard parabolic subgroups so that $Q_1 \prec Q_2 \prec \cdots \prec Q_r$. Recall that any standard parabolic subgroup P can be written uniquely as $P = Q_{i_1} \cap Q_{i_2} \cap \cdots \cap Q_{i_s}$, where $i_1 < i_2 < \cdots < i_s$. Define $P^\dagger = Q_{i_s}$. Notice that if $P_1 \subseteq P_2$ then $P_1^\dagger \supseteq P_2^\dagger$.

The generalized symmetric space $X(M_Q) = M_Q/K_Q$ factors as a direct product $Q_h/(Q_h \cap K_Q) \times Q_l/(Q_l \cap K_Q)$. This determines the projection

$$p_Q \colon X(M_Q) \longrightarrow Q_h/(Q_h \cap K_Q).$$

Now let P be an arbitrary parabolic subgroup of G and $Q = P^\dagger$. The choice of basepoint determines a map of generalized symmetric spaces

$$\Theta_{P,Q} \colon X(M_P) \longrightarrow X(M_Q).$$

Theorem 9.3.1 (Zucker). *The composition $\Phi_P = p_Q \circ \Theta_{P,Q}$ is the restriction of a map $\Phi_\alpha \colon \bar{X}^\rho_\mathbb{R} \to X^\mathrm{S}_\alpha$, where X^S_α is the (minimal) Baily–Borel Satake compactification of X. This map factors through other $\Phi_\Theta \colon \bar{X}^\rho_\mathbb{R} \to X^\mathrm{S}_\Theta$ for $\Theta \ni \alpha$. Composing Φ_Θ with $\pi_\mathbb{R}$ from Definition 9.2.5 gives $\Phi \colon \bar{X}_\mathbb{R} \to X^\mathrm{S}_\Theta$. In particular, there is a continuous map $\Phi \colon \bar{X}_\mathbb{R} \to X^\mathrm{S}_\Delta$.*

This theorem is essentially contained in [89, §2, §3]. Zucker is more interested in the restriction of Φ_Θ to $\bar{X}^\rho_\mathbb{Q}$ but his §2 is very general and §3 works $\mathbb{R}$-rationally in our situation.

For $G = PSL_3$ the map Φ is the identity on $X(SL_3)$, collapses the closure of each $e(gP_2g^{-1})$, and projects each $e(gP_1g^{-1}) \cong \hat{e}(gP_1g^{-1}) \times R_u(gP_1g^{-1})(\mathbb{R})$ onto the first factor.

9.4. **The Boundary Set.** The conjugation action of $G(\mathbb{R})$ permutes the Borel–Serre strata $e(B)$ associated to the proper standard parabolic $\mathbb{R}$-subgroup B_0. Each such stratum may be compactified as in §6.2. The lines in $e(B)$, $B \in \mathcal{P}(G)$, are the images of lines in $e(B_0)$ under these automorphisms, so $G(\mathbb{R})$ also acts on the disjoint union of the ideal compactifications $\varepsilon(B)$, i.e., on

$$\delta X \overset{\mathrm{def}}{=} \varepsilon(B_0) \underset{B_0(\mathbb{R})}{\times} G(\mathbb{R}).$$

Warning . δX comes with the identification topology which we are going to use in the ensuing construction, but it will not be the subspace topology induced from the resulting topology on $\hat{X}$.

Definition 9.4.1. $\hat{X} = \bar{X}_\mathbb{R} \cup \delta X = X \sqcup \delta X$.

The topology in $\hat{X}$ will be introduced à la Bourbaki. We are referring to

Proposition 9.4.2 ([12, Proposition 1.2.2]). *Let X be a set. If to each $x \in X$ there corresponds a set $\mathcal{N}(x)$ of subsets of X such that*

1. *every subset of X containing one from $\mathcal{N}(x)$ itself belongs to $\mathcal{N}(x)$,*
2. *a finite intersection of sets from $\mathcal{N}(x)$ belongs to $\mathcal{N}(x)$,*
3. *the element x belongs to every set in $\mathcal{N}(x)$,*
4. *for any $N \in \mathcal{N}(x)$ there is $W \in \mathcal{N}(x)$ such that $N \in \mathcal{N}(y)$ for every $y \in W$,*

then there is a unique topology on X such that, for each $x \in X$, $\mathcal{N}(x)$ is the set of neighborhoods of x.

By a *neighborhood* of a subset A in a topological space they understand any subset which contains an open set containing A.

The space $\bar{X}_{\mathbb{R}}$ is the $\mathbb{R}$-Borel–Serre construct and has the topology in which each corner $X(B)$ is open. For $y \in \bar{X}_{\mathbb{R}}$ let $\mathcal{N}(y) = \{\mathcal{O} \subseteq \hat{X} : \mathcal{O}$ contains an open neighborhood of y in $\bar{X}_{\mathbb{R}}\}$.

Notation . Given an open subset $U \subseteq \varepsilon(B)$, let $\mathcal{O}(U) = q_B^{-1}(V)$, the total space of the restriction to $V = U \cap e(B)$ of the trivial bundle q_B over $e(B)$ with fiber A_B. If U is any open subset of δX, let

$$\mathcal{O}(U) = \bigcup_{B \in \mathcal{P}_{\mathbb{R}}} \mathcal{O}\big(U \cap e(B)\big).$$

In either case define $\mathcal{C}(U) = \{z \in \bar{X}_{\mathbb{R}} :$ there is $\mathcal{O} \in \mathcal{N}(z)$ such that $\mathcal{O} \cap X \subseteq \mathcal{O}(U)\} \cup \{z \in \delta X \backslash \bar{X}_{\mathbb{R}} :$ there is an open $U' \subseteq \delta X$ such that $z \in U'$ and $\mathcal{O}(U') \subseteq \mathcal{O}(U)\}$.

Now for $y \in \delta X \backslash \bar{X}_{\mathbb{R}}$, let $\mathcal{N}(y) = \{\mathcal{O} \subseteq \hat{X}$: there is an open set $U \subseteq \delta X$ containing y with $\mathcal{C}(U) \subseteq \mathcal{O}\}$. This defines a system of neighborhoods $\mathcal{N}(y)$ for any $y \in \hat{X}$. For a subset $\mathcal{S} \subseteq \hat{X}$ let $\mathcal{N}(\mathcal{S}) = \{\mathcal{O} \subseteq \hat{X} : \mathcal{O} \in \mathcal{N}(y)$ for every $y \in \mathcal{S}\}$ and call $\mathcal{S}$ *open* if $\mathcal{S} \in \mathcal{N}(\mathcal{S})$.

Proposition 9.4.3. *If $B \in \mathcal{B}_{\mathbb{R}}$, and $U \subseteq \varepsilon(B)$ is an open subset, then $\mathcal{C}(U)$ is open in $\hat{X}$.*

Proof. If $y \in \mathcal{C}(U) \cap \bar{X}_{\mathbb{R}}$ then there is $\mathcal{O} \in \mathcal{N}(y)$ with $\mathcal{O} \cap X \subseteq \mathcal{O}(U)$. So there is an open neighborhood N of y in $\bar{X}_{\mathbb{R}}$ contained in $\mathcal{O}$. For any $z \in N$, $\mathcal{O} \in \mathcal{N}(z)$, hence $z \in \mathcal{C}(U)$. So N is an open neighborhood of y contained in $\mathcal{C}(U)$, and $\mathcal{C}(U) \in \mathcal{N}(y)$.

If $y \in \mathcal{C}(U) \cap \delta X$, say $y \in \varepsilon(B')$, then there exists an open subset $U' \subseteq \varepsilon(B')$ such that $z \in U'$ and $\mathcal{O}(U') \subseteq \mathcal{O}(U)$. Since $\mathcal{O}(U) \subseteq \mathcal{C}(U)$, $\mathcal{O}(U') \subseteq \mathcal{C}(U)$. Also $z \in \mathcal{C}(U') \cap \bar{X}_{\mathbb{R}}$ means there is $\mathcal{O} \in \mathcal{N}(z)$ such that $\mathcal{O} \cap X \subseteq \mathcal{O}(U')$, but $\mathcal{O}(U') \subseteq \mathcal{O}(U)$, so $z \in \mathcal{C}(U) \cap \bar{X}_{\mathbb{R}}$. Also $z \in \mathcal{C}(U') \cap \delta X$, say $z \in \varepsilon(B'')$, means there is an open subset $U'' \subseteq \varepsilon(B'')$ such that $z \in U''$ and $\mathcal{O}(U'') \subseteq \mathcal{O}(U')$, but $\mathcal{O}(U') \subseteq \mathcal{O}(U)$, so $z \in \mathcal{C}(U) \cap \delta X$. This shows that $\mathcal{C}(U') \subseteq \mathcal{C}(U)$. By definition, $\mathcal{C}(U) \in \mathcal{N}(y)$.

We conclude that $\mathcal{C}(U)$ is a neighborhood of each $y \in \mathcal{C}(U)$. □

Proposition 9.4.4. *If $B \in \mathcal{B}_{\mathbb{R}}$, and U_1, $U_2 \subseteq \varepsilon(B)$ are open subsets, then*

$$\mathcal{C}(U_1) \cap \mathcal{C}(U_2) = \mathcal{C}(U_1 \cap U_2).$$

Proof. The proof of Proposition 9.4.3 shows that if $U' \subseteq \varepsilon(B')$ with $\mathcal{O}(U') \subseteq \mathcal{O}(U_i)$ then $\mathcal{C}(U') \subseteq \mathcal{C}(U_i)$, $i = 1, 2$. Apply this to $U' = U_1 \cap U_2$, then $\mathcal{C}(U_1 \cap U_2) \subseteq \mathcal{C}(U_1) \cap \mathcal{C}(U_2)$. On the other hand, let $z \in \mathcal{C}(U_1) \cap \mathcal{C}(U_2)$. For $z \in \bar{X}_{\mathbb{R}}$ there is an open neighborhood N of z in $\bar{X}_{\mathbb{R}}$ such that $N \cap X \subseteq \mathcal{O}(U_i)$. So $N \cap X \subseteq \mathcal{O}(U_1) \cap \mathcal{O}(U_2) = \mathcal{O}(U_1 \cap U_2)$, and $z \in \mathcal{C}(U_1 \cap U_2)$. For $z \in \delta X$, say $z \in \varepsilon(B')$, there are $U_i{}' \subseteq \varepsilon(B')$ so that $\mathcal{O}(U_i{}') \subseteq \mathcal{O}(U_i)$, $i = 1, 2$. So $\mathcal{O}(U_1{}' \cap U_2{}') = \mathcal{O}(U_1{}') \cap \mathcal{O}(U_2{}') \subseteq \mathcal{O}(U_1) \cap \mathcal{O}(U_2) = \mathcal{O}(U_1 \cap U_2)$, $\mathcal{C}(U_1) \cap \mathcal{C}(U_2) \subseteq \mathcal{C}(U_1 \cap U_2)$, and $z \in \mathcal{C}(U_1 \cap U_2)$. □

Theorem 9.4.5. *The open subsets of $\hat{X}$ form a well-defined topology in $\hat{X}$.*

Proof. We need to check that the four characteristic properties from Proposition 9.4.2 are satisfied by the system of neighborhoods $\mathcal{N}(x)$, $x \in \hat{X}$. (1) and (3) are clear from definitions. (2) follows from Proposition 9.4.4. Given any $N \in \mathcal{N}(x)$, $x \in \partial e(B)$, there is $U \in \varepsilon(B)$ such that $\mathcal{C}(U) \subseteq N$. Take $W = \mathcal{C}(U)$. By Proposition 9.4.3 $N \in \mathcal{N}(y)$ for any $y \in W$. Thus (4) is also satisfied. □

Remark 9.4.6. If Γ is an arithmetic subgroup of $G(\mathbb{Q})$, it is immediate from the construction that this compactification is Γ-equivariant. In fact, the action of $G(\mathbb{R})$ on X extends to $\hat{X}$ which is in contrast to the fact that this action does not extend to $\bar{X}$.

Example 9.4.7 (Arithmetic Fuchsian Groups)**.** Consider an arbitrary proper parabolic $\mathbb{R}$-subgroup P of $G = SL_2$. It acts on X just as the $\mathbb{Q}$-subgroups stabilizing a point $p(P)$ in $\partial\mathbb{E}$, i.e., P permutes geodesics abutting to $p(P)$. Attach a line at $p(P)$ parametrizing these geodesics; this is the general construction of corners $X(P)$ from ([11, §5]) in the case $k = \mathbb{R}$. If P fixes a rational point then $X(P) = \bar{X}_{\mathbb{Q}}$. Complete each stratum: now $E(P) = e(P) \cup \{-\infty, +\infty\}$. The resulting set is $\hat{X}$ in which every $X(P)$ is declared to be open. So typical open neighborhoods of $z \in e(P)$ in $\hat{X}$ are the open neighborhoods of z in $X(P)$. Given a line $e(P)$ and one of its endpoints y, a typical open neighborhood of y consists of

- y itself and an open ray in $e(P)$ asymptotic to y,
- an open (Euclidean) set U in $\mathbb{E}$ bounded by the hyperbolic geodesic γ abutting to $p(P)$ representing the origin of the ray in $e(P)$—the one which is the union of geodesics representing other points of the ray,
- points in various $e(B)$, $B \in \mathcal{P}_{\mathbb{R}}$, such that $p(B)$ is on the arc in $\partial\mathbb{E}$ connecting $p(P)$ with $p(R)$, the opposite end of γ, which are represented by geodesics with a subray inside U,
- each endpoint of the corresponding $E(B)$ if $B \neq P$, R, and
- the endpoint of $E(R)$ which is the limit of a ray in $e(R)$ contained in the set from (3).

This choice generates a well-defined topology consisting of subsets which contain a neighborhood of each of its members. For example, the intersection of finitely many rays $\{\rho_i\} \subseteq E(P)$ converging to the same end y of $E(P)$ is the smallest ray, and the corresponding neighborhood of y is the intersection of the neighborhoods determined by $\{\rho_i\}$. In other words, the totality of all neighborhoods constructed above forms a base.

With the topology on $\hat{X}$ generated as above, the subspace $X \subseteq \hat{X}$ has the hyperbolic metric topology, and $\delta X = \hat{X} - X$ is simply $S^1 \times I$ with an analogue of the lexicographic order topology ([80, Exercise 48]). In terms of the description of the lexicographic ordering on the unit square $I \times I$ given in [80], the analogue we refer to is the quotient topology on $S^1 \times I$ associated to the obvious identification $(0, y) \sim (1, y)$ for all $y \in I$. In particular, δX is compact but not separable and, therefore, not metrizable.

At this point the reader may want to look at Appendix B which analyses the compactification of $\bar{X}(SL_2)$. That case allows several technical simplifications while still providing a good model for the general construction.

10. Geometric Properties of $\hat{X}$

We are going to see that $\hat{X}$ is not metrizable and, therefore, has no geometry in the usual sense. On the other hand, the geometry of the non-positively curved non-compact symmetric spaces X is well understood, and one of the tools used in their study is the ideal boundary ∂X attached to X so that $X \cup \partial X$ with the so called cone topology is compact and contractible. The goal of this section is to relate $\mathcal{C}\mathrm{ov}^{\mathrm{s}} \hat{X}$ to $\mathcal{C}\mathrm{ov}^{\mathrm{s}}(X \cup \partial X)$. *

10.1. **General Influx Engulfing.** Let G be a semi-simple linear algebraic $\mathbb{Q}$-group with equal $\mathbb{Q}$- and $\mathbb{R}$-ranks.

Lemma 10.1.1. *Let $q_B \colon X \to e(B)$ be the Borel–Serre bundle associated to a minimal parabolic subgroup $B \in \mathcal{B}_{\mathbb{R}}(G)$. Consider a compact set $C \subseteq e(B)$ and the restriction of q_B to C with the total space $q_B|C$. Then the closure of $q_B|C$ in $X \cup \partial X$ consists of the union of $q_B|C$ with apartments at infinity ∂A, where A is a flat such that $A = q_B^{-1}(c)$, $c \in C$.*

Proof. Let $y \in \mathrm{cl}(q_B|C)$ and $\{y_i\}$ be a sequence of points in $q_B|C$ converging to y. If $y \in X$ then $y \in q_B|C$ by compactness of C.

Consider $y \in \mathrm{cl}(q_B|C) \backslash X$, pick a section σ of q_B, and introduce the following notation: $\bar{y}_i = \sigma\big(q_B(y_i)\big)$, $z = \lim_{i\to\infty} q_B(y_i)$, $\bar{z} = \sigma(z)$. Let γ be the unique unit speed geodesic ray from $\bar{z}$ asymptotic to y, γ_i be the unit speed geodesic from $\bar{z}$ to y_i, and ρ_i be the geodesic from $\bar{y}_i$ to y_i.

If λ_i is a sequence of geodesic rays in X, it is said to *converge* to a geodesic ray λ if $\lambda(t) = \lim_{i\to\infty} \lambda_i(t)$ for every $t \in [0, \infty)$. The geodesic segments γ_i (or their extensions) do converge to γ (see [5], 3.2), i.e., $\lim_{i\to\infty} d\big(\gamma_i(t), \gamma(t)\big) = 0$. On the other hand, we can also

Claim . $\lim\limits_{i\to\infty} d\big(\rho_i(t), \gamma_i(t)\big) = 0.$

Proof of claim. Let ρ_i^- and γ_i^- be the unit speed geodesics from y_i to $\bar{y}_i$ and $\bar{z}$ respectively, then

$$\begin{aligned} \rho_i(t) &= \rho_i^-\big(d(y_i, \rho_i(t))\big), \\ \gamma_i(t) &= \gamma_i^-\big(d(y_i, \gamma_i(t))\big), \\ &= \gamma_i^-\big(d(y_i, \rho_i(t)) + \delta_i\big), \quad \delta_i \in \mathbb{R}. \end{aligned}$$

Let $M = \min\{d(y_i, \bar{y}_i), d(y_i, \bar{z})\}$. Without any loss of generality we can assume that $M = d(y_i, \bar{y}_i)$ so that $\delta_i \geq 0$. Using the terminology of [3], the geodesic bicombing of X is bounded, so there are constants k_1 and k_2 such that

$$d\big(\rho_i^-(s), \gamma_i^-(s)\big) \leq k_1 d(\bar{y}_i, \bar{z}) + k_2$$

for any $0 \leq s \leq M$. In fact, we can do better and use $k_1 = 1$, $k_2 = 0$. Indeed, recall Toponogov's definition of non-positive curvature ([5]). By Theorem 2.1 of [ ]

$$\begin{aligned} d\big(\rho_i(t), \gamma_i(t + \delta_i)\big) &\leq d\big(\bar{y}_i, \gamma_i^-(M)\big) \\ &\leq d\big(\bar{y}_i', \gamma_i^-(M)'\big) \\ &\leq d(\bar{y}_i', \bar{z}') = d(\bar{y}_i, \bar{z}), \end{aligned}$$

*It is more natural (and generalizable) to compare δX with the maximal Satake boundary (see §8.6) rather than with the ideal sphere. In the case at hand, however, the two spaces are homeomorphic.

where the primes denote the corresponding points in the comparison triangles. By the triangle inequality

$$d\big(\gamma_i(t+\delta_i), \gamma_i(t)\big) = |\delta_i| = |d(y_i, \bar{y}_i) - d(y_i, \bar{z})| \leq d(\bar{y}_i, \bar{z}),$$

thus

$$d\big(\rho_i(t), \gamma_i(t)\big) \leq 2d(\bar{y}_i, \bar{z}),$$

and

$$\lim_{i\to\infty} d\big(\rho_i(t), \gamma_i(t)\big) \leq 2 \lim_{i\to\infty} d(\bar{y}_i, \bar{z}) = 0.$$

This proves the claim. □

We finally get $\lim_{i\to\infty} d\big(\rho_i(t), \gamma(t)\big) = 0$. A direct way to see that $\rho_i \to \gamma$ would be to construct the limit geodesic ρ from $\bar{z}$, where each $\rho(t)$ is the limit of the Cauchy sequence $\rho_i(t)$. By Lemma 2.4.1 of [52], ρ is asymptotic to y, hence coincides with the geodesic γ.

Since the fibers $q_B^{-1}(c)$, $c \in C$, are totally geodesic, $q_B(\rho_i(t)) = q_B(y_i)$ for all times t. So $q_B(\gamma(t)) = \lim_{i\to\infty} q_B(y_i) = z$. We conclude that $\gamma(t) \in q_B^{-1}(z)$ for all t, thus $y \in \partial q_B^{-1}(z)$, $z \in C$.

The reverse inclusion of the sets is obvious. □

Let $\bar{X}_{\mathrm{b}} = X \cup \bigcup_{B\in\mathcal{B}_{\mathbb{R}}} e(B)$, where all of $X(B)$ are open. The construction from §9.4 can be performed with the standard Borel subgroup in the present generality. So the corresponding stratum $e(B)$ gets ideally compactified to $\varepsilon(B) \cong \varepsilon N \subseteq \delta_{\mathrm{b}} X$. Denote $\bar{X}_{\mathrm{b}} \cup \delta_{\mathrm{b}} X$ by $\hat{X}_{\mathrm{b}}$.

Recall that $e(B)$ can be interpreted as the underlying space of a nilpotent Lie group $N = N_B$ (Remark 8.5.1). Derived cubical cellular decompositions of I^r with vertices $V_{(n)}$ induce cellular decompositions and open coverings of εN. The open coverings are composed of the images of open stars of vertex inverses in I^r. The map we have in mind is $\Upsilon\rho\colon I^r \to N$, where Υ is the extension of the map from §7.5. We fix the choice of $\Upsilon\rho$ made for B from now on. The vertices in εN are defined to be the set $\Upsilon\rho(V_{(n)})$. The stars of ∂N in this family of cellular decompositions form a nested sequence of regular neighborhoods of the boundary denoted by

$$\mathrm{Reg}_n(B) = \mathrm{Reg}_n(\partial N_B) \stackrel{\mathrm{def}}{=} \bigcup_{\substack{v\in\Upsilon\rho(V_{(n)})\\ v\in\partial N_B}} \mathrm{Star}_n^{\circ}(v).$$

Recall also the notion of geodesic influx neighborhood $W(V, \sigma, t)$ from §8.4. Given a point $x_0 \in X$, there is a horocycle (also called *horosphere* in the $\mathbb{R}$-rank one situation) $N_B \cdot x_0$ passing through x_0 parametrizing the orbits of $\hat{A}_B$. This defines a section $\sigma\colon e(B) \to X$ of q_B with $(\sigma \circ q_B)(x_0) = x_0$. Let us denote the corresponding geodesic influx neighborhood $W(e(B), \sigma, 0)$ by $W_B(x_0)$.

Lemma 10.1.2. *Given any minimal parabolic subgroup $B \in \mathcal{B}_{\mathbb{R}}(G)$ and an open neighborhood U of $\varepsilon(B)$ in $\hat{X}$, then U contains the restriction to X of an open neighborhood $V(B)$ in $X \cup \partial X$ of the corresponding Weyl chamber at infinity $W(B) \subseteq \partial X$.*

Proof. Since $\varepsilon(B)$ is compact, there is a neighborhood $W_B(x_0)$ of $e(B)$ contained in U. For the same reason there is an integer n large enough so that $\mathcal{C}\big(\mathrm{Star}_n^{\circ}(v)\big) \subseteq U$

for every vertex $v \in \Upsilon\rho\big(V_{(n)} \cap \partial I^r\big)$. Thus

$$\partial e(B) \subseteq \bigcup_v \mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_n(v)\big) = \mathcal{C}\big(\mathrm{Reg}_n(B)\big) \subseteq U.$$

It is obvious directly from the definition that $\mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_n(v)\big)$ is an open neighborhood of $\mathrm{Star}^{\mathrm{o}}_n(v) \cap \partial e(B)$. We obtain a new open neighborhood of $\varepsilon(B)$ in $\hat{X}$ by taking the union $W_B(x_0) \cup \mathcal{C}\big(\mathrm{Reg}_n(B)\big) \subseteq U$ which we denote by $\mathcal{V}_n(B, x_0)$ or simply $\mathcal{V}_n(B)$ when the choice of $W_B(x_0)$ is not important.

Notice that $\mathcal{O}\big(\mathrm{Reg}_n(B)\big) = \mathcal{C}\big(\mathrm{Reg}_n(B)\big) \cap X$ is the union of all chambers and walls in $q_B^{-1}(z)$, $z \in \mathrm{Reg}_n(B) \cap e(B)$, based at $\sigma(z)$. Similarly, $W_B(x_0)$ is the union of all chambers based at $\sigma(z)$, $z \in e(B)$, and asymptotic to $W(B) \subseteq \partial X$. So $X \backslash \mathcal{V}_n(B)$ consists of chambers and walls based at $\sigma(z)$ in the flats $q_B^{-1}(z)$, $z \in \varepsilon(B) \backslash \mathrm{Reg}_n(B)$, and *not* asymptotic to $W(B) \subseteq \partial X$. This is the difference $q_B^{-1}\big(\varepsilon(B) \backslash \mathrm{Reg}_n(B)\big) \backslash W_B(x_0)$. By Lemma 10.1.1 the closure of this set in $X \cup \partial X$ consists of Weyl chambers and walls in the flats $q_B^{-1}(z)$, $z \in \varepsilon(B) \backslash \mathrm{Reg}_n(B)$, and the corresponding apartments in ∂X excluding $W(B)$ and the chambers asymptotic to it. So, the open complement $V(B) = V_n(B)$ of this set contains $W(B)$. Finally, $V_n(B) \cap X = \mathcal{V}_n(B) \cap X \subseteq U$. □

It was possible to state these lemmas without presenting the construction of $\hat{X}$ for a semi-simple group G. There is no point in continuing in this generality here but the following statements can be adjusted to provide the connection between $\mathfrak{Cov}^{\mathrm{s}} \hat{X}$ and $\mathfrak{Cov}^{\mathrm{s}}\, \varepsilon X$ in the more general situation.

10.2. **Back to $\mathbb{R}$-rank 1.** Assume again that $\mathrm{rank}_{\mathbb{R}} G = 1$.

Lemma 10.2.1. *Consider a subset of $X \cup \partial X$ of the form $V_n(B)$. Then $V_n(B) \cap \partial X$ consists of chambers at infinity $W(P)$ such that $\varepsilon(P)$ has a neighborhood $N \subseteq \hat{X}$ whose restriction $N \cap X \subseteq \mathcal{O}\big(\mathrm{Reg}_n(B)\big)$.*

Proof. First, let $P \in \mathcal{B}_{\mathbb{R}}$ have the property that $W(P) \in V_n(B) \cap \partial X$. In this case the claim reduces to finding some $\mathrm{Reg}_m(P)$ so that

$$\mathcal{O}\big(\mathrm{Reg}_m(P)\big) = \mathcal{C}\big(\mathrm{Reg}_m(P)\big) \cap X \subseteq \mathcal{O}\big(\mathrm{Reg}_n(B)\big),$$

which follows from the fact that the map of power sets $\phi_{B,P} \colon \mathcal{P}\big(e(B)\big) \to \mathcal{P}\big(e(P)\big)$ defined by $\phi_{B,P}(\mathcal{S}) = q_P\big(q_P^{-1}(\mathcal{S})\big)$ is relatively proper. This means that the image of a compact set is relatively compact and follows from the case $\mathcal{S} =$ point, which is equivalent to the fact that the closure of any flat lies entirely in $\bar{X}_{\mathbb{R}}$. In general, this last fact relies on the Hausdorff property, but the claim can be seen directly in the rank one situation as follows. Let $K \subseteq e(B)$ be a compact subset and $\{y_i\} \subseteq e(P)$ a sequence such that $L = \lim_{i \to \infty} \{y_i\} \in \partial e(P)$ and $q_P^{-1}(y_i) \backslash \mathcal{O}(\complement K) \neq \emptyset$ for all y_i. The endpoints of each geodesic $q_P^{-1}(y_i)$ are $W(P)$ and another point $z_i \in \complement V_n(B) \subseteq \partial X$. Since $V_n(B)$ is open, $z = \lim_{i \to \infty} \{z_i\} \in \complement V_n(B)$. This represents L as the class of the geodesic asymptotic to z and $W(P)$ which contradicts the original hypothesis. The union of $\mathcal{O}\big(\mathrm{Reg}_m(P)\big)$ and a suitable geodesic influx neighborhood of $e(P)$ is a required neighborhood N of $\varepsilon(P)$.

Now suppose that $\varepsilon(P)$ has a neighborhood N described in the statement. Then there is a section σ of q_P such that all Weyl chambers based at $\sigma(\xi)$, $\xi \in e(P)$, and asymptotic to $W(P)$ are contained in the neighborhood, and, therefore, miss $X \backslash V_n(B)$ completely. It is now clear that no Weyl camber in $X \backslash V_n(B)$ is asymptotic to $W(P)$, so $W(P) \subseteq \mathcal{O}\big(\mathrm{Reg}_n(B)\big)$. □

Corollary 10.2.2. *Using the notation above, if the parabolic subgroup $P \in \mathcal{P}_{\mathbb{R}}$ has the property that $W(P) \subseteq V_n(B)$ then $\varepsilon(P) \subseteq \mathcal{V}_n(B)$.*

Proof. This is a corollary of the proof above. One can see immediately that $e(P) \subseteq \mathcal{V}_n(B)$. For $y \in \partial e(P)$, $y \in \mathcal{C}\big(\mathrm{Reg}_m(P)\big)$ which is of the form constructed in Lemma 10.2.1. In other words, for $P' \in \mathcal{P}_{\mathbb{R}}$ with $W(P') \subseteq V_m(P)$ we get $\mathcal{O}(\mathrm{Reg}_k(P')) \subseteq \mathcal{O}(\mathrm{Reg}_m(P))$. So $\mathcal{C}\big(\mathrm{Reg}_m(P)\big) \subseteq \mathcal{C}\big(\mathrm{Reg}_n(B)\big)$, hence $\partial e(P) \subseteq \mathcal{V}_n(B)$, and, finally, $\varepsilon(P) \subseteq \mathcal{V}_n(B)$. □

Lemma 10.2.3. *If $\{U_P\}$, $P \in \mathcal{P}_{\mathbb{R}}$, is a collection of open sets in $\hat{X}$ each U_P containing $\varepsilon(P)$ whose restrictions cover $\hat{X} - X$, then the sets $V(U_P)$ can be chosen so that they cover ∂X.*

Proof. Clear, since every point of ∂X belongs to a well defined Weyl chamber. □

11. Topological Properties of $\hat{X}$

11.1. **Hausdorff Property.** $\bar{X}_{\mathbb{R}}$ is an open subspace of $\hat{X}$, so it suffices to check the Hausdorff property for $x, y \in \delta X \backslash \bar{X}_{\mathbb{R}}$. If $x, y \in \varepsilon(B)$ for some $B \in \mathcal{B}_{\mathbb{R}}$ then they can be separated by open neighborhoods U_x, $U_y \subseteq \varepsilon(B)$ with $\mathcal{O}(U_x) \cap \mathcal{O}(U_y) = \emptyset$ which get completed to open neighborhoods $\mathcal{C}(U_x) \cap \mathcal{C}(U_y) = \emptyset$. So suppose $x \in \varepsilon(B_1)$, $y \in \varepsilon(B_2)$, $B_1 \neq B_2$. The points $W(B_1)$ and $W(B_2)$ are limit points of a unique apartment which projects to $x' \in e(B_1)$, $y' \in e(B_2)$. Choose $n \in \mathbb{N}$ large enough so that $x' \notin \mathrm{Reg}_n(B_1)$, then $\mathcal{C}\big(\mathrm{Reg}_n(B_1)\big) \cap \varepsilon(B_2) = \emptyset$. Now choose $m \in \mathbb{N}$ large enough so that

$$\mathrm{Reg}_m(B_2) \cap q_{B_2}\big(\mathcal{O}(\mathrm{Reg}_n(B_1)\big) = \emptyset.$$

The existence of such m follows from the same argument as in the proof of Lemma 10.2.1. Now $\mathcal{C}\big(\mathrm{Reg}_n(B_1)\big)$ and $\mathcal{C}\big(\mathrm{Reg}_m(B_2)\big)$ are disjoint open neighborhoods of x and y respectively.

Remark 11.1.1. It is not true that the auxiliary construct $\hat{X}_{\mathrm{b}}$ from §10 is always Hausdorff. This has to do with rank, and the simplest example is $\hat{X}_{\mathrm{b}}(SL_3)$. Here each maximal 2-dimensional flat consists of six chambers and six walls. Pick two walls which are in opposition: they lie on a geodesic γ through the base point and determine two walls $W(P_1)$, $W(P_2)$ at infinity. If $z_1 = q_{P_1}(\gamma) \in e(P_1)$ then let $z_1^u \in R_u P_1(\mathbb{R})$ be the first coordinate projection of $F^{-1}(z)$ (in the notation of Proposition 9.2.3). The point $z_2^u \in R_u P_2(\mathbb{R})$ is defined similarly. The two points are the limits of γ in $\bar{X}$. It turns out that the points of $\{z_1^u\} \times \hat{e}(P_1)$ and $\{z_2^u\} \times \hat{e}(P_2)$ match bijectively in this manner.

By the real analogue of Proposition 9.2.2, for any $P \in \mathcal{P}_{\mathbb{R}}$ the principal $R_u P(\mathbb{R})$-fibration μ_P extends to a principal fibration

$$\bar{\mu}_{P,\mathbb{R}} \colon \overline{e(P)}_{\mathbb{R}} \longrightarrow \overline{\hat{e}(P)}_{\mathbb{R}}$$

Since $\overline{\hat{e}(P)}_{\mathbb{R}} = \bar{X}_{\mathbb{R}}$ for $X = SL_2(\mathbb{R})/SO_2(\mathbb{R})$, each level gets compactified as in Example 9.4.7. In particular, $\{z_i^u\} \times \overline{\hat{e}(P_i)}$, $i = 1, 2$, embed in the closures of the corresponding strata. It is now easy to see that the bijective correspondence described above extends to these enlargements and to find points $y_i \in \{z_i^u\} \times \big(\overline{\hat{e}(P_i)} - \hat{e}(P_i)\big)$ so that any two neighborhoods of y_1 and y_2 in the respective enlargements contain some points $x_i \in \{z_i^u\} \times \hat{e}(P_i)$ which are matched. Equivalently, y_1 and y_2 are inseparable in $\hat{X}_{\mathrm{b}}$.

11.2. **Compactness.** Given any open subset U of $\hat{X}$ containing $\hat{X} - X$, since such a subset would contain $\varepsilon(P)$ for every $P \in \mathcal{P}_{\mathbb{R}}$, its restriction to X would also contain an open neighborhood in $X \cup \partial X$ of the corresponding Weyl chamber at infinity $W(P)$ according to Lemma 10.1.2. As before, this says that $U \cap X$ is the restriction of an open subset of $X \cup \partial X$ containing ∂X. By compactness of ∂X, $U \cap X$ contains a collar on ∂X.

Now given any open covering $\mathcal{U}$ of $\hat{X}$, let $\{U_{1,P}, \ldots, U_{k_P,P}\} \subseteq \mathcal{U}$ be any finite subcollection which covers the compact subspace $\varepsilon(P)$ for $P \in \mathcal{P}_{\mathbb{R}}$. The sets $\varepsilon(P)$ cover $\hat{X} - X$. Since the unions $U_P = \bigcup_i U_{i,P}$ contain the corresponding $\varepsilon(P)$ individually, they together cover $\hat{X} - X$. We now apply Lemma 10.1.2 to find open neighborhoods $Y(P)$ of $\varepsilon(P)$ inside U_P and open neighborhoods $V(P)$ of $W(P)$ in $X \cup \partial X$ which have $Y(P) \cap X = V(P) \cap X$. By Lemma 10.2.3 the sets $V(P)$, $P \in \mathcal{P}_{\mathbb{R}}$, cover ∂X.

Choose a finite subcollection $\{P_i\}$, $P_i \in \mathcal{P}_{\mathbb{R}}$, $i = 1, \ldots, m$, such that $\{V(P_i)\}$ still cover ∂X. The first paragraph shows that their union must contain a collar on ∂X. The complement of this collar in $X \cup \partial X$ is closed and contained in X, hence is compact. Let $U_{m+1}, \ldots, U_n$ be a finite collection of sets from $\mathcal{U}$ such that $U_{m+1} \cap X, \ldots, U_n \cap X$ cover the complement of the collar. Each Weyl chamber $W(P)$ is contained in at least one set $V(P_i)$. By Lemma 10.2.1 the corresponding set U_{P_i} in $\hat{X}$ contains $\varepsilon(P)$. This means that $\{U_{P_i}\}$, $i = 1, \ldots, m$, cover $\hat{X} - X$. Since $\bigcup_i (U_{P_i} \cap X) \supseteq \bigcup_i (V(P_i) \cap X)$, the sets $U_{P_1}, \ldots, U_{P_m}, U_{m+1}, \ldots, U_n$ cover $\hat{X}$. In other words,

$$\{U_{1,P_1}, \ldots, U_{k_{P_1},P_1}, \ldots, U_{1,P_m}, \ldots, U_{k_{P_m},P_m}, U_{m+1}, \ldots, U_n\}$$

is a finite subcovering of $\mathcal{U}$.

Corollary 11.2.1. *The space $\hat{X}$ is a compactification of $\bar{X}$, i.e., a compact Hausdorff space containing $\bar{X}$ as an open dense subset. In fact, the combination of the Hausdorff property and compactness makes $\hat{X}$ normal.*

11.3. **Čech-acyclicity.** Here $\hat{X}$ being *Čech-acyclic* is equivalent to the weak triviality of the homotopy inverse limit

$$\check{h}(\hat{X}; KR) = \underset{\mathcal{U} \in \mathcal{C}ov\hat{X}}{\underleftarrow{\operatorname{holim}}} \, (N\mathcal{U} \wedge KR),$$

where $\mathcal{C}ov\hat{X}$ is the category of finite rigid coverings of $\hat{X}$. This property is the weakening of the contractibility assumption in Theorem 1.

Since the continuous map $f \colon \hat{X} \to \varepsilon X$ defined by

$$x \longmapsto \begin{cases} x & \text{if } x \in X, \\ W(P) & \text{if } x \in \varepsilon(P), \end{cases}$$

has contractible image and point inverses, it would be desirable to have an analogue of the Vietoris–Begle theorem for the modified Čech theory. We prove a weaker

Theorem 11.3.1. *If $f \colon X \to Y$ is a surjective continuous map with $f^{-1}(y)$ contractible for each $y \in Y$, and Y is Chogoshvili-acyclic for any abelian coefficient group, then*

$$\check{f} \colon \check{h}(X; KR) \longrightarrow \check{h}(Y; KR)$$

is a weak homotopy equivalence. So both X and Y are Čech-acyclic.

The proof is an amalgam of results from [7, 8, 23, 50, 73]. The construction of Chogoshvili is the one we have sketched in §4.3; it extends the Steenrod homology theory on the subcategory of compacta. Berikashvili ([7, Theorem 2]) proved the uniqueness of such an extension

$$h_*(\_\,,\_)\colon \textsc{CompHaus}^2 \longrightarrow \textsc{AbGroups}$$

when it satisfies the following three axioms.

Axiom A . If (X, K) is a compact Hausdorff pair then the projection $(X, K) \to (X/K, \text{point})$ induces an isomorphism $h(X, K) \to h(X/K, \text{point})$.

Axiom B . For the diagram $\{(S^n_\alpha, \text{point}), \pi_{\alpha\beta}\}$, where S^n_α is a finite bouquet of n-dimensional spheres and $\pi_{\alpha\beta}\colon S^n_\alpha \to S^n_\beta$ is a mapping sending each sphere of the bouquet either to the distinguished point or homeomorphically onto a sphere in the target, there are isomorphisms

$$h_i\left(\lim_{\overleftarrow{\alpha}} \{(S^n_\alpha, \text{point}), \pi_{\alpha\beta}\}\right) \cong \lim_{\overleftarrow{\alpha}} \{h_i(S^n_\alpha, \text{point}), \pi_{\alpha\beta}\}.$$

Let $\mathcal{E} = (E_1, \ldots, E_k) \in \mathcal{A}$ be a finite decomposition as in §4.3. Let $N_{\mathcal{E}}$ denote the nerve of the finite closed covering $\overline{\mathcal{E}} = (\overline{E_1}, \ldots, \overline{E_k})$. Then

$$N(X) \overset{\text{def}}{=} \lim_{\overleftarrow{\mathcal{E}\in\mathcal{A}}} N_{\mathcal{E}}, \quad N_p(X) \overset{\text{def}}{=} \lim_{\overleftarrow{\mathcal{E}\in\mathcal{A}}} N^p_{\mathcal{E}},$$

where K^p denotes the p-th skeleton of the simplicial complex K. There is a unique continuous map

$$\omega\colon N(X) \longrightarrow X$$

determined by the condition that if $y = \{y_{\mathcal{E}}\} \in N(X)$, $y_{\mathcal{E}} \in N_{\mathcal{E}}$, and $\sigma_{y,\mathcal{E}} = (E_1, \ldots, E_i)$ is the minimal simplex in $N_{\mathcal{E}}$ containing $y_{\mathcal{E}}$, the *carrier* of $y_{\mathcal{E}}$, then $\omega(y) \in \overline{E_1} \cap \cdots \cap \overline{E_i}$. Indeed, $\bigcap_{\sigma_{y,\mathcal{E}}} \overline{E_1} \cap \cdots \cap \overline{E_i} \neq \emptyset$, and uniqueness follows from the Hausdorff property of X.

Axiom C . The natural homomorphism

$$\operatorname*{colim}_{\overrightarrow{p\geq 0}} h_*\big(N_p(X)\big) \longrightarrow h_*(X)$$

induced by ω is an isomorphism.

In [50] Inassaridze derives the Vietoris–Begle theorem for such a theory with coefficients in the category of abelian groups *Ab*. His theorem requires point inverses to be homologically trivial. Applying the theorem to the map $f\colon X \to Y$, we get an isomorphism

$$H_*(f; A)\colon H_*(X; A) \cong H_*(Y; A)$$

of Chogoshvili homology groups for any $A \in Ab$. So X itself is Chogoshvili-acyclic for any abelian group of coefficients.

Now the main tool of Berikashvili in [7, 8] is the following characterization ([8, Theorems 3.1, 3.4]).

Theorem 11.3.2. *A generalized homology theory k_* on the category of compact Hausdorff spaces satisfies Axioms A, B, and C, if and only if there exists a functorial convergent Atiyah–Hirzebruch spectral sequence with*

$$E^2_{p,q} = H_p\big(X; k_q(\text{point})\big) \Longrightarrow k_{p+q}(X).$$

When X is Chogoshvili-acyclic for all abelian coefficient groups, this sequence collapses at the E_2-term with just the right entries in the 0-th column to make X k-acyclic. Axiom A is satisfied by any Steenrod theory. To complete the proof of Čech-acyclicity of $\hat{X}$, it suffices to verify that Axioms B and C hold for the modified Čech theory with coefficient spectrum $K(R)$ (cf. [73]).

Lemma 11.3.3. *Let $\{X_\alpha\}$ be an inverse system of compact Hausdorff spaces with $X = \varprojlim_\alpha X_\alpha$. Then there is a spectral sequence with*

$$E^2_{p,q} = \varprojlim_\alpha{}^p \ \check{h}_q(X_\alpha; KR)$$

converging to $\check{h}_*(X; KR)$.

Proof. This is identical to the proof of Theorem 8.5.1 from [26]. Observe that

$$N(\mathfrak{Cov}^{\mathrm{s}} X) \cong \{N(\mathfrak{Cov}^{\mathrm{s}} X_\alpha)\} \in \text{PRO}-\text{S}-\text{SETS}.$$

The lemma follows from the Bousfield–Kan spectral sequence (Theorem 1.2.8) applied to $\{N(\mathfrak{Cov}^{\mathrm{s}} X_\alpha)\}$ viewed as an object in the category PRO$-$(PRO$-$S$-$SETS). □

Apply the lemma to the system of wedges of spheres and notice that

$$\varprojlim_\alpha{}^p \ \check{h}_q(S^n_\alpha; KR) = 0 \text{ for } p > 0$$

by Corollary 1.2 of [8] or [51]. Since also

$$\varprojlim_\alpha{}^0 \ \check{h}_q(S^n_\alpha; KR) = \varprojlim_\alpha \ \check{h}_q(S^n_\alpha; KR),$$

the isomorphism between the E^2- and E^∞-terms is the one required in Axiom B.

Lemma 11.3.4. *For each integer $\varpi \leq 0$ there is a spectrum $K^\varpi(R)$ with*

$$K^\varpi_i(R) = \begin{cases} \pi_i(K^\varpi R) = 0 & \text{for } i < \varpi, \\ K^\varpi_i(R) = K_i(R) & \text{for } i > \varpi. \end{cases}$$

Proof. $K^\varpi R = \Omega^{-\varpi} \operatorname{Spt}\big(i\mathcal{C}_{-\varpi}(R)\big)$. See §1.4 for notation. □

Apply Lemma 11.3.3 to each $N_s(X)$, $s \geq 0$, and homology theory $\check{h}_*(\_; K^\varpi R)$, getting spectral sequences with

$$E^2_{p,q} = \varprojlim_{\mathcal{E}\in\mathcal{A}}{}^p \ \check{h}_q(N^s_{\mathcal{E}}; K^\varpi R).$$

Now, for any finite complex C

$$\check{h}_n(C^p; K^\varpi R) = \check{h}_n(C; K^\varpi R)$$

for $p > n - \varpi$. So each entry in the E^2-term associated with $N_s(X)$ with q-coordinate $< s+\varpi$ coincides with the corresponding entry in the E^2-term associated with $N(X)$. Passing to the limit as $s \to \infty$ we see that the natural map

$$t(X)\colon \operatorname*{colim}_{\overrightarrow{p\geq 0}} \check{h}_*\big(N_p(X); K^\varpi R\big) \longrightarrow \check{h}_*\big(N(X); K^\varpi R\big)$$

is an isomorphism.

Remark 11.3.5. When X is finite-dimensional so that, in particular, it has finite covering dimension, the sequence of spectral sequences converges after $\dim X$ terms, so this argument is valid with the non-connective coefficient spectrum $K(R)$ itself.

Notice that the natural homomorphism from Axiom C factors as

$$\underset{\substack{\longrightarrow \\ p\geq 0}}{\operatorname{colim}}\ \check{h}_*\big(N_p(X); K^{\varpi}R\big) \xrightarrow{t(X)} \check{h}_*\big(N(X); K^{\varpi}R\big) \xrightarrow{\omega_*} \check{h}_*(X; K^{\varpi}R).$$

It remains to show that ω_* is an isomorphism.

Recall a construction due to Eldon Dyer. Let $\mathcal{U} \in CovX$ and set

$$L_{\mathcal{U}}(X) = \{(y, x) \in N\mathcal{U} \times X : x \in \operatorname{cl}(U_1 \cap \cdots \cap U_k)\},$$

where $\sigma_y = (U_1, \ldots, U_k)$ is the carrier of y. This is a closed subspace of $N\mathcal{U} \times X$. The second coordinate projection $\omega_{\mathcal{E}} \colon L_{\mathcal{U}}(X) \to X$ is a homotopy equivalence because X is normal. The homotopy inverse is $\psi_{\mathcal{U}} \colon X \to L_{\mathcal{U}}(X)$, $x \mapsto (g_{\mathcal{U}}(x), x)$, where $g_{\mathcal{U}} \colon X \to N\mathcal{U}$ is the canonical map associated to a partition of unity. Let $\pi^{\mathcal{V}}_{\mathcal{U}} \colon L_{\mathcal{V}}(X) \to L_{\mathcal{U}}(X)$ be the restriction of the map $\pi^{\mathcal{V}}_{\mathcal{U}} \times \mathrm{id} \colon N_{\mathcal{V}}(X) \times X \to N_{\mathcal{U}}(X) \times X$. Then

$$N(X) = \underset{\substack{\longleftarrow \\ \mathcal{U}\in \mathcal{C}\mathrm{ov}^{\mathrm{s}} X}}{\lim} \{L_{\mathcal{U}}(X), \pi^{\mathcal{V}}_{\mathcal{U}}\},$$

$\omega_{\mathcal{U}} \circ \pi^{\mathcal{V}}_{\mathcal{U}} = \omega_{\mathcal{V}}$, and the map $N(X) \to X$ induced by $\{\omega_{\mathcal{U}}\}$ coincides with ω. Since each $\omega_{\mathcal{U}}$ is a homotopy equivalence,

$$\omega_* \colon \check{h}\left(\underset{\substack{\longleftarrow \\ \mathcal{U}\in \mathcal{C}\mathrm{ov}^{\mathrm{s}} X}}{\lim} L_{\mathcal{U}}(X); K^{\varpi}R\right) \longrightarrow \check{h}(X; K^{\varpi}R)$$

is an isomorphism (e.g., once more using Lemma 11.3.3). Passing to another (homotopy) colimit, one gets the result for $K(R)$ instead of semi-connective $K^{\varpi}(R)$. This verifies Axiom C.

Remark 11.3.6. Since $\hat{X}$ is finite-dimensional, we could avoid using the spectra $K^{\varpi}(R)$ as explained in Remark 11.3.5. In other words, Theorem 11.3.1 is true for any coefficient spectrum with the additional hypothesis that X is finite-dimensional.

11.4. **Remarks about Topological Properties.**

Remark 11.4.1. $\hat{X}$ is non-metrizable for the same reasons as $\hat{\mathbb{E}} - \mathbb{E}$ in Example 9.4.7 with the lexicographic order topology: both are compact but not separable. Note also that the action of Γ at infinity is large, and although $\hat{X}$ happens to be Čech-acyclic, it is unlikely to be contractible. These three features of $\hat{X}$ make Theorem 1 inapplicable.

Remark 11.4.2. This is related to the previous remark. Observe that in the case $G = SL_2$ (discussed in detail in Example 9.4.7) the identification map $\hat{X} \to \bar{X}^+$ can be factored through another compactification of $\bar{X}$ where all irrational strata are collapsed to points. All of our arguments can be done for that space. The matters can be simplified even further by noticing that the action of Γ on the resulting space is small at infinity, and the space $\hat{X}$ itself is metrizable. Note, however, that this cannot be arranged in our more general situation, because the action of Γ_{B_0} on $\varepsilon(B_0)$ is already not small.

11.5. **Properties of Satake Compactifications.** It is clear that Satake compactifications from §8.6 are compact and Hausdorff. We will also need to use their homological triviality, and this seems to be a good place to establish that.

Theorem 11.5.1. *Each space X^{S}_{Θ} is Chogoshvili-acyclic.*

Proof. Recall that the Chogoshvili homology theory is the unique extension of the Steenrod–Sitnikov homology to compact Hausdorff spaces from the category of compacta satisfying the three axioms of Berikashvili. So X^{S}_{Θ} needs to be Steenrod-acyclic. We denote the Steenrod–Sitnikov homology by $H_*(\_)$. Our main tool is the following version of the Vietoris–Begle theorem.

Theorem 11.5.2 (Nguen Le Ahn [56])**.** *Let $f\colon X \to Y$ be a continuous surjective map of metrizable compacta so that*

$$\widetilde{H}_i(f^{-1}(y); G) = 0$$

for all $y \in Y$, $i \le n$. Then if G is a countable group, the induced homomorphism

$$H_q(f)\colon H_q(X; G) \to H_q(Y; G)$$

is an isomorphism for $0 \le q \le n$ and an epimorphism for $q = n + 1$.

According to [46, Theorem 1], X^{S}_{Δ} is homeomorphic to the Martin compactification $X^{\mathrm{M}}(\lambda_0)$ of X at the bottom of the positive spectrum λ_0 (cf. [81]).

There is also the Karpelevič compactification X^{K} which is defined inductively ([52]) and maps equivariantly onto $X^{\mathrm{M}}(\lambda_0)$ (see [46, Theorem 4]). Theorem 11.5.2 applies to this map $f\colon X^{\mathrm{K}} \to X^{\mathrm{M}}(\lambda_0)$ because the fibers of f are easily seen to be genuinely contractible using the result of Kushner [54] that X^{K} is homeomorphic to a ball. The same result applied to X^{K} itself shows that all of the spaces in

$$D^n \cong X^{\mathrm{K}} \xrightarrow{f} X^{\mathrm{M}}(\lambda_0) \cong X^{\mathrm{S}}_{\Delta} \xrightarrow{\Phi_1} X^{\mathrm{S}}_{1}$$

are Chogoshvili-acyclic. □

12. Bounded Saturation in the Boundary

12.1. **The Metric in $\bar{X}$.** The space $\hat{X}$ contains $\bar{X}$ as an open dense Γ-subset, in particular Γ acts continuously on $\bar{X}$ as before. The metric that we are going to use in $\bar{X}$ is a transported Γ-invariant metric. It can be obtained by first introducing any bounded metric in the compact $\bar{X}/\Gamma$ and then taking the metric in $\bar{X}$ to be the induced path metric where the measured path-lengths are the lengths of the images in $\bar{X}/\Gamma$ under the covering projection. In this situation the diameter of a chosen fundamental domain Δ is bounded by some number D as is also the diameter of any Γ-translate of the domain. Notice that this metric is very different from the one Borel and Serre used in [11, §8.3]. The general metrization theorems of Palais they used produce metrics which are bounded at infinity.

The crucial property of our metric is that by choosing a base point x_0 in Δ and taking its orbit under the Γ-action we can embed the group Γ with the word metric quasi-isometrically in $\bar{X}$. In this sense, the metric is similar to the left invariant metric in a nilpotent Lie group used in §7.2.

12.2. **Fundamental Domains and Sets.** Precise information about the geometry of a fundamental domain is important in analytic number theory and is hard to obtain. In our situation, the choice of the fundamental domain for the action in $\bar{X}$ is not that important since of any two domains each is contained in the L-neighborhood of the other for some $L > 0$. This makes the difference between domains indistinguishable in the boundedly controlled situation. In order to compute the action, however, we have to pick a particular domain or a slight generalization of the notion, a fundamental set.

Definition 12.2.1. If Γ acts as a discrete transformation group on a space X then a *fundamental domain* of Γ is a subset $\Delta \subseteq X$ such that

(1) $\Gamma \cdot \bar{\Delta} = X$ where $\bar{\Delta}$ is the closure of Δ,
(2) $\Delta \cap \gamma\Delta = \emptyset$ if $\gamma \neq e$.

If X is the symmetric space G/K for a linear semi-simple Lie group G, $\pi\colon G \to G/K$ is the natural projection, and $\Gamma < G$ is a discrete subgroup, G and Γ act on X from the left. Reembed Γ in G by conjugating by an element of K so that $\pi(e) \neq \gamma \cdot \pi(e)$ for any $\gamma \in \Gamma$, $\gamma \neq e$. Recall that X has a left G-invariant metric ds^2, and there is the corresponding distance function $d\colon X^2 \to \mathbb{R}_{\geq \nvdash}$. Define

$$\mathfrak{E} = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\pi(\mathfrak{e}), \gamma \cdot \mathfrak{x}) \geq \mathfrak{d}(\pi(\mathfrak{e}), \mathfrak{x}),\ \gamma \in \Gamma\}.$$

This set is called the *Poincaré fundamental domain*; it is the closure of a fundamental domain in our narrower sense.

Definition 12.2.2. Let G be a reductive $\mathbb{Q}$-group, and let $\Gamma \subseteq G(\mathbb{Q})$ be an arithmetic subgroup. Then $\Omega \subseteq G(\mathbb{R})$ is a *fundamental set* for Γ if

(1) $K \cdot \Omega = \Omega$ for a suitable maximal compact subgroup $K \subseteq G(\mathbb{R})$,
(2) $\Gamma \cdot \Omega = G(\mathbb{R})$,
(3) $\Omega^{-1}\Omega \cap (xG(\mathbb{Z})y)$ is finite for all x, y in $G(\mathbb{Q})$.

Remark 12.2.3. Property (1) implies that the image of Ω in $X = G(\mathbb{R})/K$ is a fundamental set for the induced action of Γ on X. If Ω is a fundamental set for $\Gamma = G(\mathbb{Z})$ then the property (3) allows to construct a fundamental set for any subgroup Γ' commensurable with Γ by taking $\Omega' = \bigcup_{\sigma \in \Sigma} \sigma \cdot \Omega$, where Σ is a set of representatives of $\Gamma'/(\Gamma \cap \Gamma')$. The classical constructions of Siegel fundamental sets can be seen in [68, §§4.2, 4.3].

Let P_0 be the standard minimal parabolic $\mathbb{Q}$-subgroup of G, let A be the maximal $\mathbb{Q}$-split torus of G contained in P_0, and K be the maximal compact subgroup in $G(\mathbb{R})$ whose Lie algebra is orthogonal (relative to the Killing form) to the Lie algebra of $A(\mathbb{R})$. Let

$$A_t = \{a \in A(\mathbb{R})^0 : \alpha(a) \leq t, \forall \alpha \in \Delta\}.$$

Recall that $P_0 = Z_G(A) \cdot R_u(P_0)$. Furthermore, $Z_G(A) \approx A \cdot F$ where F is the largest connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $Z_G(A)$. From the Iwasawa decomposition, $G(\mathbb{R}) = K \cdot P(\mathbb{R})$. This yields the following decomposition:

$$G(\mathbb{R}) = K \cdot A(\mathbb{R})^0 \cdot F(\mathbb{R}) \cdot R_u P_0(\mathbb{R}).$$

Definition 12.2.4. A *Siegel set* in $G(\mathbb{R})$ is a set of the form

$$\Sigma_{t,\eta,\omega} = K \cdot A_t \cdot \eta \cdot \omega,$$

where η and ω are compact subsets of $F(\mathbb{R})$ and $R_u P_0(\mathbb{R})$ respectively. Sometimes, as in [11], instead of $\eta\cdot\omega$ a compact or relatively compact subset of $M_P(\mathbb{R})\cdot R_u P_0(\mathbb{R})$ is taken.

Theorem 12.2.5 (Borel)**.** *There are a Siegel set $\Sigma = \Sigma_{t,\eta,\omega}$ and a finite set $C \subseteq G(\mathbb{Q})$ such that $\Omega = C \cdot \Sigma$ is a fundamental set for Γ.*

The results of Garland and Raghunathan from [39] determine the form of the Poincaré fundamental domains which look like the sets described in Theorem 12.2.5. They work with non-uniform lattices Γ in a rank one linear algebraic semi-simple Lie group G.

Fix some Iwasawa decomposition $G = K \cdot A_0 \cdot N_0$. There is a parabolic subgroup $P_0 < G$ with Langlands decomposition $P_0 = M_0 \cdot A_0 \cdot N_0$.

Theorem 12.2.6 (Selberg, Garland–Raghunathan)**.** (1) *The total number of geodesic rays $r(t)$, $t \in \mathbb{R}_{\geq\varkappa}$, such that $r(0) = \pi(e)$, $r(\mathbb{R}_{\geq\varkappa}) \subseteq \mathfrak{E}$, is finite. Denote the minimal such number by M and choose $\mathfrak{E}$ with this number of cusps.*

(2) *If $r_i = \lim_{t\to\infty} r_i(t) \in \partial X$, $1 \leq i \leq M$, let $\Gamma_i = \mathrm{stab}(r_i)$. Then $\mathfrak{E} = \bigcup_{\mathfrak{i}=\mathfrak{o}}^{\mathfrak{M}} \mathfrak{E}_\mathfrak{i}$, where $\mathfrak{E}_\mathfrak{o}$ is a compact set, and there is $t_0 > 0$ such that*

$$r_i(\mathbb{R}_{\geq\approx\varkappa}) \subseteq \mathfrak{E}_\mathfrak{i},\ \ \mathfrak{E}_\mathfrak{i} \cap \mathfrak{E}_\mathfrak{j} = \emptyset \text{ for } \mathfrak{o} \neq \mathfrak{i} \neq \mathfrak{j} \neq \mathfrak{o},\ \ \mathfrak{E}_\mathfrak{i} = \mathfrak{g}_\mathfrak{i}\Sigma_\mathfrak{i},$$

where $g_i \in G$, $g_i \notin \Gamma\backslash e$, and $\Sigma_i = \{x \in X : x = r(\mathbb{R}_{\geq\lessdot})$ for geodesic rays $r\colon y \to W(P_0)$, $y \in \omega_i$, $\omega_i \subseteq X$ are compact$\}$.

(3) *One has $g_i\Gamma_i g_i^{-1} < M_0 \cdot N_0$, where N_0 is the maximal nilpotent subgroup of the stabilizer of the standard cusp, and ω_i is the closure of a fundamental domain for $g_i\Gamma_i g_i^{-1}$ in the horocycle $N_0 \cdot r(l)$.*

When Γ is an arithmetic subgroup, the cusps r_i are rational, i.e., $g_i \in G(\mathbb{Q})$. If Γ is torsion-free then $g_i\Gamma_i g_i^{-1}$ acts freely in $N \cdot r(l)$. Also, there is $\omega = \omega_i$, $1 \leq i \leq M$. Consider $\Gamma' = \langle \Gamma, g_1, \ldots, g_M\rangle < G(\mathbb{Q})$. This subgroup has a fundamental domain with unique cusp which is contained in a Siegel set $\Sigma_{t,\omega}$, ω being the closure of a fundamental domain of $\Gamma' \cap N_0$ in $N_0(\mathbb{R}) \cong (\mathbb{P}_\varkappa)$. So $\Delta \subseteq q_{P_0}^{-1}(\omega)$ and $(\Gamma'/\Gamma' \cap N_0) \cdot \mathrm{cl}(\Delta) = q_{P_0}^{-1}(\omega)$. According to part (3) of Theorem 12.2.6, Δ can be completed to the fundamental domain $\bar{\Delta}$ of Γ in $\bar{X}$ so that $\mathrm{cl}_{\bar{X}}(\bar{\Delta}) = \mathrm{cl}_X(\Delta) \cup \omega$.

12.3. **Quasi-isometry Invariance.** Every two arithmetic subgroups Γ_1, Γ_2 in G are commensurable, hence their Cayley graphs are quasi-isometric. This also implies that if d_i are Γ_i-invariant metrics in $\bar{X}$ transported from $\bar{X}/\Gamma_i$, $i = 1, 2$, then $(\bar{X}, d_1)$ and $(\bar{X}, d_2)$ are quasi-isometric.

Proposition 12.3.1. *The system of boundedly saturated sets in $\hat{X} - \bar{X}$ is a quasi-isometry invariant of $(\bar{X}, d)$.*

Proof. Let $(\bar{X}, d_1)$, $(\bar{X}, d_2)$ be quasi-isometric structures on $\bar{X}$. It suffices to show that for a subset $\Omega \subseteq \bar{X}$ and a large $D_1 \gg 0$, the enlargement $\Omega[D_1]_1$ with respect to d_1 is contained in $\Omega[D_2]_2$ for some $D_2 > 0$. If λ, ϵ are the constants associated to the quasi-isometry $\mathrm{id}\colon (\bar{X}, d_2) \to (\bar{X}, d_1)$, let $D_2 = \frac{D_1 - \epsilon}{\lambda}$. Then $x \in \Omega[D_2]_2 \Rightarrow d_2(x, o) \leq D_2$, $o \in \Omega \Rightarrow d_1(x, 0) \leq \lambda d_2(x, o) + \epsilon = D_1$. $\square$

Recall $\Gamma' < G(\mathbb{Q})$ constructed in §12.2. Since $g_i\Sigma_{l,\omega}g_i^{-1}$ are precisely the parabolic vertices of $\mathfrak{E}$, the complement $\mathfrak{E}\setminus\bigcup_{\mathfrak{i}=1}^{\mathfrak{M}} \mathfrak{g}_\mathfrak{i}\Delta\mathfrak{g}_\mathfrak{i}^{-1}$ is compact, so a Γ-domain is contained in $\bar{\Delta}[D]_{\Gamma'}$ for some $D > 0$. This implies even more directly that the boundedly saturated sets determined by Γ and Γ' coincide. Now we can study the bounded saturation using the simpler domain $\bar{\Delta}$.

12.4. **Saturation in Rational Strata.** Fix the coordinate map $\varrho \overset{\text{def}}{=} \sigma^{-1} \colon \mathbb{R}^{\diagdown} \to \mathbb{N}_{\nvdash} \cong (\mathbb{P}_{\nvdash})$ defined in §6.2. Let $\mathcal{O} = \{(x_i) \in \mathbb{R}^{\diagdown} : \nvdash \leq \curvearrowleft\sqsupset \leq \nVdash,\ \forall \nVdash \leq \sqsupset \leq \diagdown\}$, then $\varrho(\mathcal{O})$ is a domain for $\Gamma' \cap N_0$ in $e(P_0)$. The translates form a cellular decomposition of $e(P_0)$. The induced decompositions of $e(P)$, $P = gP_0g^{-1} \in \mathcal{P}_{\mathbb{Q}}$, are invariant under $\Gamma' \cap N$, hence are well-defined.

Let $Z_i \overset{\text{def}}{=} \langle \gamma_i \rangle = G_i \cap \Gamma'_P$, where $\Gamma'_P = \Gamma' \cap P(\mathbb{Q})$. The computation in the proof of Theorem 6.3.1 shows that the union of translates of the fundamental cube $g\varrho(\mathcal{O})g^{-1}$ in $e(P)$ by the coset Γ'_P/Z_i disconnects $e(P)$. If $\chi_i \colon \mathbb{Z} \to \mathbb{Z}_{\sqsupset}$ is the obvious isomorphism then $\chi_i(n) \cdot \Gamma'_P/Z_i$ also disconnect $e(P)$. We will call these unions of cells *walls* in $e(P)$ and denote them by $\mathcal{W}_{i,n}$.

Proposition 12.4.1. *Each cell in the decomposition of $\partial e(P)$ from Theorem 7.3.2 for $P \in \mathcal{P}_{\mathbb{Q}}$ is boundedly saturated in Y.*

Proof. The closures of walls in $e(P)$ disconnect $\varepsilon(P)$. The complements of $\mathcal{W}_{i,n}$ are denoted by $\mathcal{R}^{\pm}_{i,n}$. Note that the cell in $\partial e(P)$ corresponding to the i-th coordinate and the positive or negative direction is the inverse limit of $\mathcal{R}^{\pm}_{i,n}$, $n \in \mathbb{Z}$, ordered by inclusion. Choose a cell σ by fixing i and $+$, loosing no generality. If $y \in \delta X \backslash \varepsilon(P)$, say $y \in \varepsilon(P')$, then the geodesic asymptotic to both $W(P)$ and $W(P')$ projects to $\bar{y} \in e(P)$. Then $\bar{y} \in \mathcal{W}_{i,n}$ for some $n \in \mathbb{Z}$. Denote $\mathcal{C}\big(\mathrm{int}(\mathcal{W}_{i,n+1} \cup \mathcal{W}_{i,n+2} \cup \mathcal{W}_{i,n+3})\big)$ by $\mathcal{B}_{i,n+2}$. If the subset $\Xi \subseteq \Gamma' \cap N_P$ makes $\mathcal{W}_{i,n+2} = \Xi \cdot \omega$ then $(\Gamma'/\Gamma' \cap N)\,\Xi \cdot \bar{\Delta} \subseteq \mathcal{B}_{i,n+2}$, so $\mathcal{B}_{i,n+2}$ is a *barrier* separating y and σ into the different connected components of $\delta X \backslash \mathcal{B}_{n,m+2}$: $y \in \mathcal{H}^-_{i,n+2}$, $\sigma \subseteq \mathcal{H}^+_{i,n+2}$. If $\{y_s\} \subseteq \mathcal{H}^-_{i,n+2}$ is a sequence converging to y then

$$\overline{\{y_s\}[1]} \cap \mathcal{H}^+_{i,n+2} = \emptyset.$$

Inductively

$$\overline{\{y_s\}[D]} \cap \mathcal{H}^+_{i,n+3D} = \emptyset,$$

therefore, $\overline{\{y_s\}[D]} \cap \sigma = \emptyset$. By Lemma 7.1.2, σ is boundedly saturated. $\square$

12.5. **Interstratous Saturation.** Two sequences $\{x^1_i\}$, $\{x^2_i\}$ in a metric space (X, d) are called *fellow travelers* if there is $K > 0$ such that $d(x^1_i, x^2_i) \leq K$ for every $i \in \mathbb{N}$.

Lemma 12.5.1. *Let $\{y_i\}$ and $\{z_i\}$ be sequences in $(\bar{X}, d)$ converging to $y \in \varepsilon(P_y) \cap Y$, $z \in \varepsilon(P_z) \cap Y$. If $P_y \neq P_z$ then the sequences do not fellow-travel.*

Proof. Let $\bar{y}_i$, $\bar{z}_i$ be the points in the image of the imbedding $\imath \colon \Gamma' \hookrightarrow X$, $\gamma \mapsto \gamma \cdot x_0$, $x_0 \in \Delta \subseteq \bar{\Delta}$, in the same translate of the domain $\bar{\Delta}$ as y_i and z_i. Since $d(y_i, \bar{y}_i) \leq D$, $d(z_i, \bar{z}_i) \leq D$, it suffices to show that $\{\bar{y}_i\}$, $\{\bar{z}_i\}$ do not fellow-travel.

Suppose that $\bar{y}_i = \imath(\gamma'_i)$, $\bar{z}_i = \imath(\gamma''_i)$. Observe that if the sequences $\{\bar{y}_i\}$, $\{\bar{z}_i\}$ fellow-travel in the Γ-invariant metric d_Γ then they also fellow-travel in the Riemannian metric d_G. Indeed, in the Γ-invariant metric

$$d(\bar{y}_i, \bar{z}_i) = d\big((\gamma'_i)^{-1}(y_i), (\gamma'_i)^{-1}(z_i)\big) = d\big(\imath(I), (\gamma'_i)^{-1}(z_i)\big).$$

Since there is a constant M such that $d(\bar{y}_i, \bar{z}_i) \leq M$, all of $(\gamma'_i)^{-1}(z_i)$ are contained in a word-metric ball in $\imath(\Gamma)$ of radius M centered at $\imath(I)$. They form a finite set which is, therefore, bounded in the Riemannian metric d_G in X which is G-invariant. So there is a constant N such that

$$d_G(\bar{y}_i, \bar{z}_i) = d_G\big(\imath(I), (\gamma_i')^{-1}(z_i)\big) \leq N.$$

Now each translate $\gamma \cdot \bar{\Delta}$ contains at most a finite number of points $\{y_i\}$ for otherwise $y \in \gamma \cdot \omega \subseteq \gamma \cdot \bar{\Delta} \subseteq \complement Y$. Thus the sequence $\{\bar{y}_i\}$ takes on infinitely many values. By inspection of projections into $e(P_y)$, $\lim_{i\to\infty}\{\bar{y}_i\} = W(P_y) \in \partial X$. Same argument shows that the limit of $\{\bar{z}_i\}$ in εX is $W(P_z)$. This shows that the sequences $\{\bar{y}_i\}$, $\{\bar{z}_i\}$ do not fellow-travel in the Riemannian metric. Neither do they in our metric d_Γ by the observation above. □

Corollary 12.5.2. *Each stratum-component $\varsigma(P) = \varepsilon(P) \cap Y$, $P \in \mathcal{P}_\mathbb{R}$, of Y is boundedly saturated.*

Proof. If $\varsigma(P)$ is not boundedly saturated then there are fellow-traveling sequences $\{y_i\}$, $\{z_i\}$ converging to $y \in \varsigma(P)$ and $z \notin \varsigma(P)$ (see the proof of Lemma 7.1.2). This is impossible by Lemma 12.5.1. □

That the boundaries of rational strata are boundedly saturated is not new: this follows from Proposition 12.4.1 and Corollary 7.1.4. But Corollary 12.5.2 also says that each $\varepsilon(P)$, $P = \mathcal{P}_\mathbb{R}\backslash\mathcal{P}_\mathbb{Q}$, is boundedly saturated.

Remark 12.5.3. It is impossible to use the theorems of §7 about right actions here: the right action of $G(\mathbb{R})$ on X does not extend to $\bar{X}$. For example, in the SL_2-situation, in the upper-half plane model, the image of the y-axis

$$iy \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = y \left(\frac{\beta\delta + \alpha\gamma + i}{\delta^2 + \gamma^2} \right)$$

is a straight Euclidean line with slope $\frac{1}{\beta\delta+\alpha\gamma}$, not a geodesic. It is interesting to note, however, that there are constructions of Mumford et al. ([4], [53, IV, §2]) to which the right action naturally extends. Compare the pictures on page 179 in [53]. The lines in the second picture, if extended, should converge at the other endpoint of the horizontal geodesic. They are precisely the images of the right action computed above.

13. Proof of Theorem 3

13.1. **Orderly Coverings.** Given a covering $\mathcal{U} \in \mathrm{Cov}^{\mathrm{s}}\hat{X}$, the proof of compactness of $\hat{X}$ (§11.2) required a construction of a finite covering $\{V(P_i)\}$ of ∂X associated to some regular neighborhoods $\mathrm{Reg}_{n_i}(P_i)$ of $\partial e(P_i)$. We want a variation on that construction which, given a covering $\mathcal{U} \in \mathrm{Cov}^{\mathrm{s}}Y$, refines $\mathcal{U}$. Recall from §7.5 that $\mathrm{Star}^{\mathrm{o}}_n(v)$ is the open star of $v \in \Upsilon\rho\big(V_{(n)}\big) \subseteq \varepsilon(P)$, $P \in \mathcal{P}_\mathbb{R}$, in the projection of the n-th derived cubical decomposition of I^r.

Definition 13.1.1. Let us alter the notation $\mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_n(v)\big)$ for the moment to mean $\mathrm{Star}^{\mathrm{o}}_n(v)$ when $v \in e(P)$. Define

$$\mathrm{Ord}_n(v) = Y \cap \mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_n(v)\big) \setminus \bigcup_{P' \in \mathcal{F}} \varepsilon(P'),$$

where $\mathcal{F}$ is the set of all P' with $q_P\big(W(P')\big) \in \partial\mathrm{Star}^{\mathrm{o}}_n(v)$.

It is clear that $\mathrm{Ord}_n(v)$ is an open neighborhood of v in Y. For any covering $\mathcal{U} \in \mathcal{C}\mathrm{ov}^{\mathrm{s}}Y$ there is an order n such that $\{\mathrm{Ord}_n(v) : v \in \Upsilon\rho\big(V_{(n)}\big) \cap Y\}$ refines $\{U \in \mathcal{U} : U \cap \varepsilon(P) \neq \emptyset\}$. Now it is clear from compactness of Y that there is a finite set $\{P_k : k \in \Lambda\} \subseteq \mathcal{P}_{\mathbb{R}}$ and integers n_k so that $\bigcup_k \{\mathrm{Ord}_{n_k}(v) : v \in \Upsilon\rho\big(V_{(n_k)}\big) \subseteq \varepsilon(P_k)\}$ refines the given $\mathcal{U} \in \mathcal{C}\mathrm{ov}^{\mathrm{s}}Y$. The full cofinal subcategory of $\mathcal{C}\mathrm{ov}^{\mathrm{s}}Y$ consisting of such *orderly* refinements will be denoted by $\mathcal{O}\mathrm{rd}^{\mathrm{s}}Y$.

In order to create manageable rigid coverings, consider the "excised" versions of the sets $\mathrm{Ord}_n(v)$:

$$\mathrm{ExcOrd}_n(v) = \mathrm{Ord}_n(v) \backslash \mathrm{Star}^{\mathrm{o}}_n(v).$$

Now $\mathcal{E}\mathrm{xc}\mathcal{O}\mathrm{rd}^{\mathrm{s}}Y$ is the category of open coverings $\mathcal{V}$ which contain some $\mathcal{U} \in \mathcal{O}\mathrm{rd}^{\mathrm{s}}Y$ as a subset and may contain $\mathrm{ExcOrd}_n(v)$ if $\mathrm{Ord}_n(v) \in \mathcal{U}$. The cofinality property mentioned above is certainly not affected.

Definition 13.1.2. Let $\textsc{PreOrd}Y$ be the full subcategory of $\mathcal{C}ov Y$ with objects $\beta \in \textsc{PreOrd}Y$ satisfying

- $\mathrm{im}\,\beta \in \mathcal{E}\mathrm{xc}\mathcal{O}\mathrm{rd}^{\mathrm{s}}Y$,
- $y \in \varepsilon(P_k)$ for some $k \Longleftrightarrow \beta(y) = \mathrm{Ord}_{n_k}(v)$ for some $v \in \varepsilon(P_k)$.

It is implicit in the second condition that for $y \notin \varepsilon(P_k)$ for all k, there exists $\ell \in \Lambda$ and n_ℓ with $\beta(y) = \mathrm{ExcOrd}_{n_\ell}(v)$ for some $v \in \varepsilon(P_\ell)$. Define $\mathcal{O}rd\,Y$ to be the full subcategory of $\mathcal{C}ov Y$ closed under $\times$-operation generated by $\textsc{PreOrd}Y$.

It is easy to see that $\mathcal{O}rd\,Y$ is not cofinal in $\mathcal{C}ov Y$ but satisfies the hypotheses on the category $\mathcal{C}$ in §4.4. Recall that the conclusion of that section was that the map

$$\jmath^* \colon \check{h}(Y; KR) \longrightarrow \underset{\substack{\longleftarrow \\ \mathcal{O}rd\,Y}}{\mathrm{holim}}\,(N\_ \wedge KR)$$

induced by the inclusion $\jmath \colon \mathcal{O}rd\,Y \hookrightarrow \mathcal{C}ov Y$ is a weak homotopy equivalence.

13.2. **Definition of $\{\alpha\}$.** The idea is to define finite rigid coverings of Y by boundedly saturated sets which can be naturally "piecewisely" approximated by coverings from $\mathcal{O}rd\,Y$. Let $N_0 = N_{P_0}$ for the standard $P_0 \in \mathcal{P}_{\mathbb{Q}}$. Consider the covering of ∂I^r by the 2^r open stars of $V_{(-1)}$ in the (-1)-st derived decomposition. The images $\Upsilon_0\rho\big(\mathrm{Star}^{\mathrm{o}}(v_{-1}(s_1, \ldots, s_r))\big)$ cover the sphere ∂N_0. The sets are no longer open but they are boundedly saturated with respect to the Γ-invariant metric as a consequence of Proposition 12.4.1 and Corollary 7.1.4.

Definition 13.2.1. The covering $\mathcal{A}_0$ of ∂N_0 by the sets

$$\Upsilon_0\rho\big(\mathrm{Star}^{\mathrm{o}}(v_{-1}(s_1, \ldots, s_r))\big)$$

is finite but not open. This choice generates the category $\{\alpha_0\}$ of finite rigid coverings α_0 of ∂N_0. Notice that it follows from property (2) of finite rigid coverings that $\mathrm{im}\,\alpha_0 = \mathcal{A}_0$.

Notice that the homotopy type of $N\mathcal{A}_0$ is, in fact, that of the $(r-1)$-dimensional sphere: the nerve of $\mathcal{A}_0$ is the same as the nerve of the open star covering of ∂I^r with respect to the (-1)-st derived decomposition, and that can be easily seen to be homotopy equivalent to S^{r-1}.

The choice of $\mathcal{A}_0$ provides well-defined coverings $\mathcal{A}_P$ of $\varepsilon(P)$, $P \in \mathcal{P}_{\mathbb{Q}}$, by $G(\mathbb{Z})$-translates of $\mathcal{A}_0$. There are also associated rigid coverings $\{\alpha_P\}$ with $\mathrm{im}\,\alpha_P = \mathcal{A}_P$.

Definition 13.2.2. Given a covering $\omega \in \mathcal{O}rd\,Y$, $\omega = \pi_1 \times \cdots \times \pi_m$, where each $\pi_i \in \textsc{PreOrd}Y$. Let $\{P_k : k \in \Lambda\}$ be the finite collection of parabolic $\mathbb{R}$-subgroups associated to $\pi_1, \ldots, \pi_m$. Collect the following data:

(1) for each $P \in \mathcal{P}_{\mathbb{Q}}$ pick an arbitrary $\alpha_P \in \{\alpha_P\}$—in particular, for each $P_k \in \mathcal{P}_{\mathbb{Q}}$ there is $\alpha_k \in \{\alpha_{P_k}\}$,
(2) for each $P \notin \mathcal{P}_{\mathbb{Q}}$ take α_P to be the constant rigid covering with $\operatorname{im} \alpha_P = \varepsilon(P)$.

Define the following finite rigid covering $\alpha(\omega, \alpha_P)$:

$$\alpha(y) = \begin{cases} \alpha_k(y) \cup \big(\omega(y) \backslash \varepsilon(P_k)\big) & \text{if } y \in \varepsilon(P_k) \text{ for some } k \in \Lambda, \\ \omega(y) & \text{otherwise.} \end{cases}$$

Since each $\omega(y)$, $y \notin \varepsilon(P_k)$ for $k \in \Lambda$, is a union of closed strata $\varepsilon(P) \cap Y$, it is boundedly saturated by Corollary 12.5.2. Same is true about $\omega(y) \backslash \varepsilon(P_k)$, $y \in \varepsilon(P_k)$. Also, all $\operatorname{im} \alpha_k$, $k \in \Lambda$, are boundedly saturated by Proposition 12.4.1 and Corollary 7.1.4. Thus all $\alpha(y)$, $y \in Y$, are boundedly saturated subsets of Y.

13.3. **Proof of Theorem 3.** Now we can return to the argument started in §5.4 and use $\mathcal{C} = \mathcal{O}rd\,Y$ and $\alpha' = \alpha(\omega, \alpha_P)$ for $\omega \in \mathcal{O}rd\,Y$. We have seen that $\jmath^*$ is a weak equivalence. It remains to see that the orderly sets are nice enough for all inclusions $\mathfrak{sat}_* \colon \mathfrak{N}\beta \hookrightarrow \mathfrak{N}\alpha(\beta)$ to be homotopy equivalences.

This is the way one would proceed if there were no need to make the construction of $\hat{X}$ equivariant. Instead of $\varepsilon(B_0)$ we would use non-equivariant but simpler compactifications by cubes of appropriate dimensions. With the obvious choices of cubical derived decompositions (induced by Υ which is now a cellular homeomorphism) and the other constructions repeated literally, the saturation process in the boundaries of rational strata would produce sets which are stars of lower-dimensional sides. It would be enough to consider the stars of the vertices.

Now notice that there is a projection of this hypothetical situation to the real equivariant Y. This projection induces an equivalence on the Čech homology level by our weak Vietoris–Begle theorem 11.3.1. Also, the images of the saturations in the hypothetical boundary Y^{hyp} project to precisely the boundedly saturated sets we construct in Y. There is a well-defined functorial "lift" from our α's to the saturations in Y^{hyp} with the same combinatorics. The induced maps form a commutative diagram:

$$\begin{array}{ccc} \underset{\mathcal{O}rd\,Y^{\mathrm{hyp}}}{\underleftarrow{\operatorname{holim}}}\, (N\_ \wedge KR) & \xrightarrow{\mathfrak{sat}_*} & \underset{\mathcal{O}rd\,Y^{\mathrm{hyp}}}{\underleftarrow{\operatorname{holim}}}\, N\alpha(\_) \wedge KR \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ \underset{\mathcal{O}rd\,Y}{\underleftarrow{\operatorname{holim}}}\, (N\_ \wedge KR) & \xrightarrow{\alpha_*} & \underset{\mathcal{O}rd\,Y}{\underleftarrow{\operatorname{holim}}}\, N\alpha(\_) \wedge KR. \end{array}$$

So our constructions induce precisely the needed map. Now inclusions of nerves $N\beta \hookrightarrow N\alpha(\beta)$ induce natural weak equivalences $N\beta \wedge KR \simeq N\alpha(\beta) \wedge KR$. This follows from the fact that factoring out a contractible subcomplex generated by a subset of vertices factors through the inclusion into the complex where the same subset generates a simplex. This is precisely what happens with finitely many disjoint subcomplexes associated to sets covering the special strata. We can conclude that α_* is a weak homotopy equivalence by Theorem 1.2.2.

Remark . It is easy to see that the obvious reconstruction of ω from $\alpha(\omega, \alpha_P)$ which "forgets" about the choices of α_k in $\{\alpha_{P_k}\}$ defines a functor $R\colon \{\alpha\} \to \mathcal{O}rd\, Y$. This is the exact inverse to α. The natural induced map

$$R^*\colon \underset{\overleftarrow{\mathcal{O}rd\, Y}}{\operatorname{holim}} (N\alpha\_ \wedge KR) \longrightarrow \underset{\overleftarrow{\{\alpha\}}}{\operatorname{holim}} (N\alpha R\_ \wedge KR) = \underset{\overleftarrow{\{\alpha\}}}{\operatorname{holim}} (N\_ \wedge KR)$$

is a weak equivalence of spectra by Theorem 1.2.3. Now the composition

$$R^*\alpha_* \jmath^*\colon \check{h}(Y; KR) \xrightarrow{\simeq} \underset{\overleftarrow{\alpha \in \{\alpha\}}}{\operatorname{holim}} (N\alpha \wedge KR)$$

is a weak homotopy equivalence with the promised target.

Appendix A. Other Theories. Other Groups. Other Methods

A.1. **Extensions to Other Theories.** The extension of the K-theoretic results to L-theory is very formal using the basic results of [20, §§4, 5]. The statements about the L-theoretic assembly maps are the same as before when the coefficient spectrum is replaced by the non-connective spectrum $L(R)$ for a ring with involution R satisfying $K_{-i}(R) = 0$ for sufficiently large i. The homotopy groups $\pi_i\big(L(R)\big)$ are the surgery obstruction groups $L_i(R)$.

The extension to A-theory is trickier. The necessary details are provided by [22] and earlier papers of W. Vogell.

If $C^*(\Gamma)$ denotes the group C^*-algebra of Γ (the completion of $L^1(\Gamma)$ in the greatest C^*-norm), Kasparov defines

$$\alpha\colon RK_*(B\Gamma) \longrightarrow K_*\big(C^*(\Gamma)\big).$$

The splitting of this map implies the Novikov conjecture for Γ—see the explanation on page 414 of [71] or Corollary 2.10 in [72]. The recent work of Carlsson–Pedersen–Roe will extend the methods used here to work for this C^*-algebraic version of α.

A.2. **Hilbert Modular Groups.** Let F be a totally real algebraic number field of degree n over $\mathbb{Q}$, let $\mathcal{O}_F$ be the ring of integers of F. Consider $G = R_{F/\mathbb{Q}} SL_2$, the $\mathbb{Q}$-group obtained from SL_2/F by restriction of scalars according to Weil ([86, §1.3]). Then $G(\mathbb{Q}) = \mathbb{SL}_{\not\models}(\mathbb{F})$, $G(\mathbb{R}) = \mathbb{SL}_{\not\models}(\mathbb{R})^{\ltimes}$ is a connected semi-simple Lie group, $K = SO(2)^n$ is a maximal compact subgroup, and the associated symmetric space $X(G) = \mathbb{H}^{\ltimes}$ has rank n. Any subgroup of finite index in $G(\mathbb{Z}) = \mathbb{SL}_{\not\models}(\mathcal{O}_{\mathbb{F}})$ is an irreducible lattice in $G(\mathbb{R})$ embedded via the inclusion $SL_2(\mathcal{O}_F) \hookrightarrow SL_2(\mathbb{R})^{\ltimes}$ by using the n distinct $\mathbb{Q}$-homomorphisms $F \to \mathbb{R}$ as coordinate functions.

The *Hilbert modular groups* are $SL_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in the real quadratic field $\mathbb{Q}(\surd)$. Here the two homomorphisms $\mathcal{O}_d \to \mathbb{R}$ are the inclusion and the Galois conjugation. We will assume that Γ is a neat arithmetic subgroup of $SL_2(\mathcal{O}_d)$. (A subgroup Γ is *neat* when the subgroup of $\mathbb{C}^*$ generated by the eigenvalue of any element of Γ is torsion-free. In particular, Γ itself is torsion-free.) The quotients $\Gamma\backslash X$ are called the Hilbert(–Blumenthal) modular surfaces.

Examples of such arithmetic subgroups of G are the *principal congruence subgroups*

$$\Gamma[\ell] = \ker\Big(SL_2(\mathcal{O}_d) \xrightarrow{\text{mod}} SL_2\big(\mathcal{O}_d/(\ell)\big)\Big)$$

for $\ell \geq 3$.

Theorem A.2.1. *If Γ is a neat arithmetic subgroup of $R_{\mathcal{O}_d/\mathbb{Q}}SL_2$ then the assembly map α is a split injection.*

If P_0 is the standard parabolic $\mathbb{Q}$-subgroup of SL_2 then $B_0 = P_0 \times P_0$ is the standard parabolic $\mathbb{Q}$-subgroup of $G = R_{\mathcal{O}_d/\mathbb{Q}}SL_2$. The $\mathbb{Q}$-rank of G is one. The stabilizer of the standard cusp in Γ is a uniform lattice in the solvable Lie group $\mathcal{S}o\ell$, and the associated stratum in the Borel–Serre enlargement can be identified with the underlying space of $\mathcal{S}o\ell$ ([31], [44, §3.H]), where the stabilizer acts by left multiplication.

The group $\mathcal{S}o\ell$ can be expressed as a semi-direct product $\mathbb{R}^{\nvDash} \rtimes \mathbb{R}$: if the elements of the set $\mathcal{S}o\ell = \mathbb{R}^{\nvDash} \times \mathbb{R}$ are (x, y, z), the action of z is the linear transformation given by $(x, y) \mapsto (e^z \cdot x, e^{-z} \cdot y)$. We can transport the flat metric from $\mathbb{R}^{\nVdash}$ into $\mathcal{S}o\ell$ using this identification. The straight lines through the origin in $\mathcal{S}o\ell$ are then given as

$$L = \{(x_1^b + tx_1^d, x_2^b + tx_2^d, x_3^b + tx_3^d) : t \in \mathbb{R}\}.$$

Here is the formula for the left action of $(y_1, y_2, y_3) \in \mathcal{S}o\ell$ on this line:

$$(y_1, y_2, y_3) \circ L = \big(y_1 + e^{y_3}(x_1^b + tx_1^d), y_2 + e^{-y_3}(x_2^b + tx_2^d), y_3 + x_3^b + tx_3^d\big).$$

It shows that $\mathcal{S}o\ell$ acts on oriented parallelism classes of lines in the stratum. This is very similar to the situation we had in §6.2 in a general connected nilpotent group. The difference is that the flat metric does not come from the Lie algebra of $\mathcal{S}o\ell$, so the right multiplication action of $\mathcal{S}o\ell$ on itself may not and, in fact, does not extend to parallelism classes of lines. There is one set of lines, however, invariant under the right action: if $x_3^d = 0$ then

$$L \circ (y_1, y_2, y_3) = \big(x_1^b + tx_1^d + e^{x_3^b} y_1, x_2^b + tx_2^d + e^{-x_3^b} y_2, x_3^b + y_3\big).$$

The formula also shows that each class of lines in this set is actually fixed by the right action.

Now consider the ideal compactification of $\mathcal{S}o\ell$ with the flat metric. Each point in $\partial(\mathcal{S}o\ell)$ with $x_3^d = 0$ can be blown up to a closed segment, the interior points corresponding to subclasses of lines with the common coordinate $-\infty < x_3^b < +\infty$. The result will be called $\mathcal{S}o\ell\,\widehat{\ }$.

The same methods as in §7 apply and show that each open segment is boundedly saturated as well as each of the endpoints and each of the complementary hemispheres. The closed segments above are the elements of a cylinder $\mathcal{S}o\ell \cup [-\infty, +\infty] \times \partial D^2 \subseteq \mathcal{S}o\ell\,\widehat{\ }$. Let us identify $\mathcal{S}o\ell\,\widehat{\ }$ with the closed cylinder $[-\infty, +\infty] \times D^2$ and embed $\mathcal{S}o\ell\,\widehat{\ }$ in $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z) as the set defined by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq 1$. Given a natural number n, define the *n-th standard sectoral decomposition* of $\mathcal{S}o\ell\,\widehat{\ }$ to be the representation of $\mathcal{S}o\ell\,\widehat{\ }$ as the union of sectors $\mathcal{S}_n(i, j, k) = \{(r, \theta, z) \in \mathbb{R}^3 : i/2^n \leq r \leq (i+1)/2^n,\ j\pi/2^{n-1} \leq \theta \leq (j+1)\pi/2^{n-1},\ k/2^n \leq z \leq (k+1)/2^n\}$ for every choice of the integral triple $0 \leq i, j, k \leq 2^n - 1$. The points $v_n(i, j, k) = (i/2^n,\ j\pi/2^{n-1},\ k/2^n)$ for $0 \leq i, j, k \leq 2^n$ will be called *vertices*. A vertex in the N-th subdivision determines star $\mathrm{Star}\big(v_n(i, j, k)\big) = \{(r, \theta, z) \in \mathbb{R}^3 : (i-1)/2^n \leq r \leq (i+1)/2^n,\ (j-1)\pi/2^{n-1} \leq \theta \leq (j+1)\pi/2^{n-1},\ (k-1)/2^n \leq z \leq (k+1)/2^n\}$ with the obvious modifications when i or k equals 0 or 1. Also *links* and *open stars* are defined by direct analogy with their simplicial analogues.

The boundary set δX is the union of the rational strata

$$\delta_{\mathbb{Q}} X \overset{\text{def}}{=} \mathcal{S}o\ell^{\frown} \underset{B_0(\mathbb{Q})}{\times} G(\mathbb{Q})$$

and the irrational points at infinity with the auxiliary topology defined as in §9.4. Now $\hat{X}_{\text{b}}$ can be formed by the obvious analogy with §10. The basic neighborhoods of irrational points at infinity are the "completions" of their neighborhoods in the spherical topology.

Using the argument from §10 it is easy to see that $\hat{X}_{\text{b}}$ is compact but not Hausdorff due to the arrangement of higher dimensional maximal flats in X (see Remark 11.1.1). In order to induce the Hausdorff property, consider the set map $f\colon \hat{X}_{\text{b}} \to \varepsilon X$. The idea is to make this map continuous. Introduce a new topology in $\hat{X}_{\text{b}}$ generated by the intersections of basic neighborhoods $\mathcal{N}(x)$, $x \in \hat{X}_{\text{b}}$, and the preimages of neighborhoods of $f(x) \in \varepsilon X$. Since each fiber of f is Hausdorff, and the analogue of Lemma 10.2.1 holds, the new topology on $\hat{X}_{\text{b}}$ is Hausdorff and makes f a quotient map. Denote the new space by $\hat{X}$. The map $f\colon \hat{X} \to \varepsilon X$ can be used as in §11.3 to show that $\hat{X}$ is Čech-acyclic.

The rest of the argument for the $\mathbb{R}$-rank one case generalizes easily, we only need to indicate the boundedly saturated sets we choose inside the rational boundary strata. It suffices to show the subsets of $\hat{e}(B_0) \cong \mathcal{S}o\ell^{\frown}$. For the chosen $n \in 2\mathbb{N}$ and $\xi \in \{0,1\}$, they are $A(n,j,\xi) = \{(r,\theta,\xi)\} \cup \{(r,\theta,z) : r = 1,\ (2j+\xi-1)\pi/2^{n-1} \leq \theta \leq (2j+\xi+1)\pi/2^{n-1},\ z \in (1-\xi,\xi]\}$, where $0 \leq j \leq (\sqrt{2})^n - 1$ is an integer. These are open stars of certain collections of vertices in the $2n$-th standard sectoral decomposition.

Remarks . (1) The construction of the map f is apparently the correct way to deal with the general case of an arithmetic subgroup of an arbitrary split rank semi-simple group. The target must be the maximal Satake compactification which coincides with εX in the rank one situation.

(2) The ad hoc construction of $\mathcal{S}o\ell^{\frown}$ is designed to be analogous to the $\mathcal{N}\imath\ell$ case. The correct way to deal with $\mathcal{S}o\ell \cong e(B_0)$ is, of course, using the product formula from Lemma 9.2.4.

A.3. **Other Approaches to Novikov Conjectures.** There has been a lot of research done on Novikov conjectures in various forms and related conjectures like Baum–Connes'. The most recent progress known to us is connected with the work of Bökstedt–Hsiang–Madsen, Connes–Gromov–Moscovici, Farrell–Jones, Ferry–Weinberger, Higson–Roe, Julg–Kasparov, Ogle, and others.

The method of S. Ferry and S. Weinberger ([32, 33, 34, 35]) uses a similar "bounded control philosophy". They call an endomorphism of a metric space $f\colon X \to X$ *bounded* if there is $k > 0$ such that $d(f(x),x) < k$ for all $x \in X$.

Theorem A.3.1 (Ferry–Weinberger [32])**.** *If Γ is a discrete group such that $K = K(\Gamma,1)$ is a finite complex and the universal cover $X = \tilde{K}$ has a compactification $\hat{X}$ with the properties that*

(1) *the boundary $\hat{X} - X \subseteq \hat{X}$ is a Z-set, i.e., admits a homotopy $F_t\colon \hat{X} \to \hat{X}$ with $F_0 = \mathrm{id}$, $F_t(\hat{X}) \subseteq X$ for all $t > 0$, and*

(2) *every continuous bounded function $f\colon X \to X$ extends by identity to a continuous function $\hat{f}\colon \hat{X} \to \hat{X}$,*

then the K- and L-theoretic Novikov conjectures for Γ hold.

We wish to describe one difficulty in using our compactification in this approach. The Lie algebra $\mathfrak{n}$ of the nilpotent radical of $P \in \mathcal{P}_{\mathbb{Q}}$ decomposes into a direct sum $\mathfrak{n} = \mathfrak{n}_\lambda \oplus \mathfrak{n}_{2\lambda}$, where λ is the *unique* simple root of (P, A)—a consequence of the $\mathbb{R}$-rank one assumption. The dimensions $\dim(\mathfrak{n}_\lambda)$, $\dim(\mathfrak{n}_{2\lambda})$ equal the multiplicities of λ, 2λ. After exponentiating we get $N = N_\lambda N_{2\lambda}$ with $N_\lambda \cap N_{2\lambda} = \{I\}$ and $N_{2\lambda} = [N, N]$. If $N_{2\lambda} \neq \{I\}$ then N is a non-abelian two-step nilpotent group with center $N_{2\lambda}$.

In the situation when N is non-abelian, there exists an element $g \in N$ such that g acts non-trivially from the right on ∂N. This action $\partial\psi_g \colon \partial N \to \partial N$ is the extension from the action ψ_g on N which in its turn extends to a bounded endomorphism Ψ of $\bar{X}$. The point is that Ψ cannot be extended to an endomorphism of $\hat{X}$ by identity on Y, so even our compactification of $e(P)$ cannot be used here. However, this obstacle disappears in the case when N is abelian, for instance, in the case of $G = SO_0(1, n)$.

Appendix B. Geometry of $\hat{X} = \hat{X}(SL_2)$

This appendix should illustrate the general arguments in the main text. The simple situation of $X = X(SL_2)$ provides good intuitive motivation for the more general statements.

B.1. Topological Properties of $\hat{X}$.

Proposition B.1.1. *For the symmetric space $X = SL_2(\mathbb{R})/SO_2(\mathbb{R})$ the space $\hat{X}$ is compact.*

Proof. For any open covering $\mathcal{U}$ of $\hat{X}$ there is a finite number of connected neighborhoods in $U_1, \ldots, U_k$ which cover $\hat{X} - X$. This is so because the lexicographic order topology on $S^1 \times I$ is compact. The restrictions $U_1|X, \ldots, U_k|X$ are open subsets of $X = \mathbb{E}$ and contain the interior of a collar on $\partial\mathbb{E}$. It follows that there is a closed and bounded subset S of the hyperbolic space X which together with $U_1, \ldots, U_k$ covers $\hat{X}$. If S is covered by $\{U_{k+1}, \ldots, U_n\} \subseteq \mathcal{U}$ then $U_1, \ldots, U_n$ form a finite subcovering of $\mathcal{U}$. $\square$

Remark B.1.2. The proof just given depends on the fact that $\delta X - \{\text{point}\}$ can be (lexicographically) ordered, and the order topology makes δX compact. The analogues of δX in §9.4 are not this easily recognizable. A whole section in the main text is devoted to the geometric analysis of the relation between open coverings of $\hat{X}$ and εX.

Proposition B.1.3. *The space $\hat{X}$ is Hausdorff.*

Proof. Let $x, y \in \hat{X}$, $x \neq y$. If x and y lie in the closure of the same stratum $e(P)$ then, since $E(P)$ is Hausdorff, there are open $U_x \cap U_y = \emptyset$, $x \in U_x$, $y \in U_y$. If $\mathcal{C}(U)$ denotes either the basis open neighborhood of the endpoint based on U or a geodesic influx neighborhood of U, it is clear that $\mathcal{C}(U_x) \cap \mathcal{C}(U_y) = \emptyset$. The case $x, y \in \bar{X}_{\mathbb{R}}$ is easy since $\bar{X}_{\mathbb{R}}$ is Hausdorff ([11, Theorem 7.8]) and open in $\hat{X}$. If $x \in E(P_1)$, $y \in E(P_2)$, $P_1 \neq P_2$, it is clear that there are two geodesics abutting to $p(P_1)$ and $p(P_2)$ respectively, or a geodesic abutting to $p(P_i)$ and a horocircle centered at $p(P_j)$, $j \neq i$, so that the appropriate regions they bound in X are disjoint. These regions generate disjoint open neighborhoods of the given points. $\square$

Theorem B.1.4. *The space $\hat{X}$ is Čech-acyclic.*

Proof. $\check{h}(\_)$ is a Steenrod functor. In particular, if $A \subseteq Z$ is closed then $k(A) \to k(Z) \to k(Z/A)$ is a fibration up to natural weak homotopy equivalence. Fix a proper $\mathbb{R}$-parabolic subgroup P of $G(\mathbb{R})$. Consider $\hat{X} - X(P) \subseteq \hat{X} \to \hat{X}/\hat{X} - X(P)$ and note that $\hat{X}/\hat{X} - X(P)$ is a disk D^1. $X(P)$ is open in $\hat{X}$, so the Steenrod property above produces a homology long exact sequence with every third term being trivial. Now it suffices to show that $\hat{X} - X(P)$ is Čech-acyclic. This will follow from the following geometric

Lemma B.1.5. *For each $\mathcal{U} \in CovS$ there exists a refinement $\mathcal{C}$ such that $N\mathcal{C}$ is contractible.*

Proof. View $S = \hat{X} - X(P) = \hat{X} - X - e(P)$ as lexicographically ordered points of the unit square $[0,1] \times [0,1]$ between $(0,1)$ and $(1,0)$. Make a choice of a function $\beta\colon [0,1] \to \mathbb{N} \cup \{\infty\}$ such that $\beta^{-1}(\infty) = \{0,1\}$, and for each $\xi \in (0,1)$ $\mathcal{U}|\{\xi\} \times [0,1]$ is refined by the open star covering of the $\beta(\xi)$-th barycentric subdivision of $\{\xi\} \times [0,1]$. Let $p\colon S \to [0,1]$ be the set map projecting S onto the first coordinate, and let $\mathcal{U}'$ be the refinement of $\mathcal{U}$ consisting of connected components of members of $\mathcal{U}$ ($\mathcal{U}'$ needs no longer be finite).

Call $U \in \mathcal{U}'$ *long* if $p(U) \subseteq [0,1]$ has non-empty interior in the Euclidean topology on $[0,1]$. By compactness of $[0,1]$ it is possible to select a finite collection of long sets such that $[0,1] = \bigcup_{U \in T} p(U)$. Define $D = \{x \in [0,1] |\ x \notin \mathrm{int}(\sigma)$ for all $\sigma = p(U), U \in T\}$. This is a finite subset of $[0,1]$ with the natural order: $0 = d_0 < d_1 < \cdots < d_n < d_{n+1} = 1$. Each closed set Δ_i of points in S between $(d_i, 1)$ and $(d_{i+1}, 0)$ is covered by long sets, so there exists a number L_i such that if $[\xi, \xi + L_i] \subseteq [d_i, d_{i+1}]$ then $[\xi, \xi + L_i] \subseteq p(U)$ for some long $U \in \mathcal{U}'$. By the choice of the function β, if $\eta \in [d_i, d_{i+1})$ and $\eta + L_i \leq d_{i+1}$, and $L(\eta, L_i)$ is an open subset of points strictly between $\big(\eta, 1 - (\frac{1}{2})^{\beta(\eta)}\big)$ and $\big(\eta + L_i, (\frac{1}{2})^{\beta(\eta + L_i)}\big)$ then $L(\eta, L_i) \subseteq U$ for some $U \in \mathcal{U}'$. So, if we cover Δ_i by finitely many sets of the form $L(\eta, L_i)$ and cover each $\{d_i\} \times (0,1)$, $1 \leq i \leq n$, by the open star covering corresponding to the $\beta(d_i)$-th barycentric subdivision of $\{d_i\} \times [0,1]$, then the resulting covering of S is finite and refines $\mathcal{U}$.

First, note that the nerve of $\{L(\eta, L_i)\}$ for a fixed $0 \leq i \leq n$ is contractible. The sets $L(\eta, L_i)$ are naturally ordered; let us start with the leftmost set. The sets that intersect it also intersect pairwise, so the leftmost set is a vertex of a simplex in the nerve (looked at as a simplicial complex) attached to the rest of the complex by the opposite face. Collapse the simplex onto that face. Now repeat the procedure with the second from the left set and continue inductively. Next, note that the nerve of elements in the open star coverings of a single segment $\{d_i\} \times (0,1)$ is just a simplicial line which connects higher dimensional subcomplexes contractible as above. So the whole nerve is contractible. □

As in the construction of the classical Čech homology theory, because of contiguity of induced maps between the nerves of coverings, although $CovS$ is directed but not filtering, $\{\pi_q(N\_ \wedge KR)\}_{CovS}$ becomes a pro-group.

Lemma B.1.6. *The object $\{\pi_q(N\_ \wedge KR)\}_{CovS}$ is isomorphic to $\pi_q(KR) = K_q(R)$ in* PRO−GROUPS. *In particular, it is stable.*

Proof. For a chosen $\mathcal{U} \in CovS$ let $f = f_{\mathcal{U}}\colon \pi_q(N\mathcal{U} \wedge KR) \to K_q(R)$ be the map induced by $N\mathcal{U} \to$ point. Since $CovS$ is directed, it follows from Lemma B.1.5

that all $f_{\mathcal{U}}$ define the same pro-homomorphisms $\{\pi_q(N\_ \wedge KR)\}_{CovS} \to K_q(R)$. To define the inverse, for a given $\mathcal{U} \in CovS$ pick any refinement $\mathcal{C} > \mathcal{U}$ with contractible nerve and take $g_{\mathcal{U}}$ to be the composition

$$K_q(R) \longrightarrow \pi_q(N\mathcal{C} \wedge KR) \longrightarrow \pi_q(N\mathcal{U} \wedge KR)$$

where the first arrow is the inverse of the isomorphism induced by the constant map. Routine checking shows that $\{g_{\mathcal{U}}\}$ and f are, indeed, mutual inverses. $\square$

Corollary B.1.7. *The object $\{\pi_q(N\_ \wedge KR)\}_{\mathcal{C}ovS}$ is stable.*

Combining Corollary B.1.7 with Theorem C from [26, §4.5], we can conclude that

$$\varprojlim_{\mathcal{C}ovS}{}^{p} \; \pi_q(N\_ \wedge KR) = 0$$

for all $p > 0$ and all $q \geq 0$. For the same reason

$$\varprojlim_{\mathcal{C}ovS}{}^{0} \; \pi_q(N\_ \wedge KR) \cong \varprojlim_{\mathcal{C}ovS} \; \pi_q(N\_ \wedge KR) = \pi_q(KR).$$

Finally, recall Theorem 1.2.8. From the discussion above, the E_2-term in the spectral sequence converging to

$$\pi_* \left(\operatorname*{holim}_{\overleftarrow{\mathcal{C}ovS}} N\_ \wedge KR \right)$$

has only one non-zero column, and the groups in the column are the homology of a point. So S and, therefore, $\hat{X}$ are Čech-acyclic. $\square$

B.2. **Study of Bounded Saturation.** The group Γ acts continuously on $\bar{X}$ and on $Y = \hat{X} - \bar{X}$ since each element of $G(\mathbb{Q})$ maps a geodesic abutting to an irrational point to a geodesic with the same property. Recall that the metric we use in $\bar{X}$ is a transported Γ-invariant metric obtained by introducing any bounded metric in the compact surface $\bar{X}/\Gamma$, then taking the metric in $\bar{X}$ to be the induced path metric so that the measured path-lengths are the lengths of the images in $\bar{X}/\Gamma$ under the covering projection. In this situation the diameter of a fundamental domain F or any Γ-translate of F is bounded by some number D. If we choose a base point x_0 in $F \cap X$ and take its orbit under the Γ-action, we can embed the group Γ with the word metric quasi-isometrically in $\bar{X}$. x_0 also lies in a translate $\gamma \cdot T$ of the classical fundamental hyperbolic triangle T for the $SL_2(\mathbb{Z})/\{\pm 1\}$-action. Let k be a constant such that $\gamma \cdot T \subseteq F[k]$.

Proposition B.2.1. (1) *If $p(P_1)$ and $p(P_2)$ are two distinct points in $\partial\mathbb{E}$ and two sequences in $\bar{X}$ converge to points $y_1 \in E(P_1) \subseteq Y$ and $y_2 \in E(P_2) \subseteq Y$ then they diverge in the Γ-invariant metric on $\bar{X}$.*

(2) *If a sequence s_1 converges to $y_1 \in E(P)$ for some irrational point $p(P)$ in $\partial\mathbb{E}$ and $y_2 \in E(P)$ then there exists a fellow-traveler s_2 of s_1 (in the Γ-invariant metric) which converges to y_2.*

(3) *If a sequence s converges to an endpoint y of a rational stratum $E(P)$ then $\overline{s[d]} \cap Y = \{y\}$ for all d.*

Proof. Part (1) is the specialization of Lemma 12.5.1. It suffices to look at sequences contained in X. There the sequences converge to distinct ideal boundary points, hence diverge in the classical $SL_2(\mathbb{R})$-invariant metric, hence in our metric too. Part

(3) is the specialization of Proposition 12.4.1. Let $\Omega_1 = g_1\Omega$, $\Omega_2 = g_2\Omega$, $\ldots$ be a sequence of translates of the fundamental domain Ω of $\Gamma \cap P(\mathbb{Q})$ in $e(P)$ converging to y. Taking unions with appropriate hyperbolic tringles $g_i T$ gives translates of the fundamental domain $\bar{T}$ for $SL_2(\mathbb{Z})$ in $\bar{X}$. The unions $g_i\bar{X} \cup g_i \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \bar{T}$ disconnect $\bar{X}$. This easily implies that the statement is true for the $SL_2(\mathbb{Z})$-invariant metric. Since $\gamma \cdot T \subseteq F[k]$ and $\overline{F[k]}$ is compact in $\bar{X}$, it is also true for the Γ-invariant metric. This argument is especially clear in the upper-half plane picture with $p(P) = \infty$.

The matrix $\mu = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ maps the y-axis in $\mathbb{H}$ onto the geodesic connecting the rational points b/d and a/c which is thus contained in $\mu \cdot T \cup \mu \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \cdot T$ when T is chosen to be the classical hyperbolic triangle with the y-axis as the core. So any hyperbolic geodesic abutting to an irrational point in $[b/d, a/c]$ eventually starts passing through unions of the form

$$g \cdot T \cup g \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot T \subseteq g\gamma^{-1} \cdot F[k] \cup g \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma^{-1} \cdot F[k]$$

whose diameter is bounded by $2D + 4k$.

The *fellow-traveling* property used in the statement of part (2) is simply the fact that for each sufficiently large time t the hyperbolic geodesic σ converging to y_2 passes through $(g \cdot F)[2D + 3k]$ where $g \in \Gamma$ is such that $s_1(t) \in g \cdot F$. A choice of $s_2(t) \in \sigma \cap (g \cdot F)[2D + 3k]$ creates a desired sequence with the fellow-traveling constant $2D + 4k$. A similar argument deals with the endpoints of $E(P)$. $\square$

The proposition can now be rephrased as follows: (1) Given a closed subset C of $\hat{X}$ such that $C \cap Y \subseteq E(P_1)$ for a proper $\mathbb{R}$-parabolic subgroup and $p(P_1)$ is not a rational point, then for any other point $p(P_2)$ no $y \in E(P_2)$ is in the closure of any d-neighborhood of $C \backslash Y$. (2) Given a closed subset C of $\hat{X}$ such that $C \cap Y \subseteq E(P)$ for a proper $\mathbb{R}$-parabolic subgroup P and $p(P)$ is not a rational point, then every $y \in E(P)$ is in the closure of some d-neighborhood of $C \backslash Y$, and, in fact, all of them are in the closure of the $2D + 3k$-neighborhood of $C \backslash Y$. (3) Given a closed subset C of $\hat{X}$ such that $C \cap Y$ is an endpoint y of a rational stratum $E(P)$ then $\overline{(C \setminus Y)[d]} \cap Y = \{y\}$ for all d.

Corollary B.2.2. *Each point in the boundary of a rational stratum is boundedly saturated. The closure of each irrational stratum is boundedly saturated; moreover, it is a minimal set with this property.*

This determines *all* boundedly saturated subsets of Y. In §12 we obtain similar information sufficient to use in the main argument but much less precise than here.

B.3. **Proof of Theorem 3 for Free Groups (an illustration).** The class of torsion-free arithmetic subgroups Γ of $SL_2(\mathbb{Q})$ coincides with the class of finitely generated free groups which are the fundamental groups of the surfaces with boundary $\bar{X}/\Gamma$, where $X = SL_2(\mathbb{R})/SO_2(\mathbb{R})$. Another name for these groups is torsion-free arithmetic Fuchsian groups because $PSL_2(\mathbb{Q}) = SL_2(\mathbb{Q})/\pm 1$.

Several features in the proof of Theorem 3 (§13) simplify in this case. For a given proper parabolic $\mathbb{R}$-subgroup P and a natural number n_P, neighborhoods of vertices v in the n_P-th derived decomposition of $\varepsilon(P)$ as follows:

$$\mathrm{Ord}_{n_P}(v) = \begin{cases} \mathrm{Star}^{\circ}_{n_P}(v) & \text{if } P \in \mathcal{P}_{\mathbb{R}} \backslash \mathcal{P}_{\mathbb{Q}},\ v \in e(P), \\ Y \cap \mathcal{C}\big(\mathrm{Star}^{\circ}_{n_P}(v)\big) \backslash \varepsilon(P') & \text{if } v \in \partial e(P), \end{cases}$$

where P' is the proper parabolic subgroup with $W(P')$ at the other end of the geodesic $q_P^{-1}\big(\partial\mathrm{Star}^{\mathrm{o}}_{n_P}(v)\big)$. Define also

$$\mathrm{ExcOrd}_{n_P}(v) = \mathrm{Ord}_{n_P}(v) \backslash \mathrm{Star}^{\mathrm{o}}_{n_P}(v).$$

As in Definition 13.1.2, $\mathcal{O}rd\,Y$ is the $\times$-completion of the category $\textsc{PreOrd}Y$ of rigid coverings β such that $\operatorname{im}\beta$ consists of (all) $\mathrm{Ord}_{n_P}(v)$ and $\mathrm{ExcOrd}_{n_P}(v)$, $P \in \mathfrak{P}$, for some finite collection $\mathfrak{P}$ of proper parabolic $\mathbb{R}$-subgroups satisfying the two conditions from the definition. Notice that the second condition implies that each (non-empty) $\mathrm{Ord}_{n_P}(v) \cap Y$ is in $\operatorname{im}\beta$. By §4.4 the map of homotopy limits of nerves induced by the inlusion $\jmath\colon \mathcal{O}rd\,Y \hookrightarrow \mathcal{C}ov Y$ is a weak homotopy equivalence.

Now given a rigid covering $\beta \in \textsc{PreOrd}Y$, let

$$\alpha(y) = \begin{cases} \mathrm{Ord}_{n_P}(v) \cup \varepsilon(P) & \text{if } v \in \varepsilon(P),\ P \in \mathfrak{P}\backslash\mathcal{P}_{\mathbb{Q}}, \\ \beta(y) & \text{otherwise.} \end{cases}$$

Each $\alpha(y)$ is boundedly saturated. All such $\alpha(\beta)$ for $\beta \in \textsc{PreOrd}Y$ form a contractible category $\{\alpha\}$. The rest of the argument coincides with §13.3. The point is that $N\beta \simeq N\alpha(\beta)$ since the disjoint subcomplexes in $N\beta$ spanned by $\{\beta(\beta^{-1}\mathrm{Ord}_{n_P}(v))\}$, $v \in \varepsilon(P)$, $\mathcal{P} \in \mathfrak{P}\backslash\mathcal{P}_{\mathbb{Q}}$, are contractible, and $N\alpha(\beta)$ is constructed by replacing them with the full simplices they generate.

Appendix C. The Case of a Non-Reductive Group

The compactifications we have described can also be used in non-semi-simple situations. In fact, they are necessary in the inductive construction for more general groups of higher rank: there proper parabolic subgroups need not be even reductive. We take the simplest such standard parabolic subgroup

$$P_1(\mathbb{Q}) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq \mathbb{SL}_{\nvDash}(\mathbb{Q})$$

and prove Theorem 3 for its torsion-free arithmetic subgroups. The result is certainly not new for these are semi-direct products $F_2 \rtimes \mathbb{Z}_{\nvDash}$, but we deal with them directly whereas other methods (cf. [17], [35], [70]) always use reduction.

C.1. **Construction of $\hat{X}(P_1)$.** The subgroup $P_1 \subseteq SL_3$ is proper parabolic and is usually taken to be a standard maximal parabolic subgroup (cf. Example 8.3.4). As such it has the Levi decomposition $P_1(\mathbb{R}) = \mathbb{M}_{\nvDash}(\mathbb{R})\mathbb{A}_{\nvDash}\mathbb{N}_{\nvDash}$, where

$$M_1(\mathbb{R}) = \left\{ \begin{pmatrix} F & \\ & \epsilon \end{pmatrix} \in \mathbb{P}_{\nvDash}(\mathbb{R}) :\ \mathbb{F} \in \mathbb{SL}^{\pm}_{\nvDash}(\mathbb{R}),\ \epsilon = \pm\nVdash \right\},$$

$$SL_2^{\pm}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} :\ a,b,c,d \in \mathbb{R},\ \det = \pm 1 \right\},$$

$$A_1 = \left\{ \begin{pmatrix} a^{-1} & & \\ & a^{-1} & \\ & & a^2 \end{pmatrix} \in P_1(\mathbb{R}) :\ \text{Ↄ} \in (\mathbb{R}^*)^+ \right\}.$$

The group

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & n_3 \\ & 1 & n_2 \\ & & 1 \end{pmatrix} \in P_1(\mathbb{R}) \right\}$$

is the unipotent radical $R_u P_1(\mathbb{R})$. Denote by I_3 the 3×3 identity matrix and by d_2 the diagonal matrix $\mathrm{diag}(1, -1, -1)$. Then M_1 has two connected components $M_1 = \{d_2, I_3\} \times M_1^0$ with $M_1^0 \cong SL_2$. So the generalized symmetric space associated to P_1 has the following product structure:

$$X(P_1) = X(M_1^0) \times N_1.$$

The proper parabolic subgroups of P_1 are all Borel, in the obvious bijective correspondence with the Borel subgroups of M_1^0. In fact, the product structure extends to

$$\bar{X}_{\mathbb{Q}}(P_1) = \bar{X}_{\mathbb{Q}}(M_1^0) \times N_1$$

(see [11]). This space is a cocompact $E\Gamma$ for any arithmetic subgroup $\Gamma \subseteq P_1(\mathbb{Q})$. There is also the real version of this product

$$\bar{X}_{\mathbb{R}}(P_1) = \bar{X}_{\mathbb{R}}(M_1^0) \times N_1.$$

Each of the three first coordinate projections—μ, $\bar{\mu}_{\mathbb{Q}}$, $\bar{\mu}_{\mathbb{R}}$—is an N_1-principal fibration.

Recall from Remark 8.5.1 that each stratum $e(B)$, $B \in \mathcal{P}_{\mathbb{R}}(P_1)$, is the underlying space of the 3-dimensional Heisenberg group. The standard Borel subgroup

$$P_0(\mathbb{Q}) = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} \subseteq \mathbb{P}_{\not{k}}(\mathbb{Q})$$

has the unipotent radical whose real points

$$N_0 = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} \approx e(P_0)$$

have the natural Euclidean metric. It makes sense, therefore, to speak of the ideal compactification $\varepsilon(P_0)$ of $e(P_0)$.

To construct an enlargement of $X(P_1)$ other than $\bar{X}_{\mathbb{R}}(P_1)$, compactify each fiber of μ by its ideal boundary. In other words, embed $e(P_1)$ in

$$\varepsilon(P_1) \overset{\text{def}}{=} X(M_1^0) \times \varepsilon N_1.$$

The formula from Lemma 9.2.4 shows that $P_1(\mathbb{R})$ acts on $X(P_1)$ by bundle automorphisms.

Proposition C.1.1. *The inclusion $X(P_1) \subseteq \varepsilon(P_1)$ is $P_1(\mathbb{R})$-equivariant.*

Proof. The added points are the parallelism classes of rays in fibers of μ. We are going to check that the bundle automorphisms from $P_1(\mathbb{R})$ map lines to lines in a way which preserves the parallelism relation and that the formulae for the action are polynomial.

A line in the μ-fiber over $\hat{z} = z\hat{K}_1\hat{A}_1$ can be parametrized as

$$(\hat{z}, \{u_t\}) = \big(\hat{z}, \{0, y_1 + y_2 t, z_1 + z_2 t)\}\big) \subseteq \{\hat{z}\} \times N_1.$$

An element $g \in P_1(\mathbb{R})$ maps the fiber to $\{\mu(g) \cdot \hat{z}\} \times N_1$. If $g = (g_{ij})$, $1 \le i, j \le 3$, $g_{31} = g_{32} = 0$, then

$$g \cdot (0, y_1 + y_2 t, z_1 + z_2 t) \cdot g^{-1} =$$
$$\big(0, g_{11} z_1 + g_{12} y_1 + (g_{11} z_2 + g_{12} y_2) t, g_{21} z_1 + g_{22} y_1 + (g_{21} z_2 + g_{22} y_2) t\big).$$

Recall that the N_1-coordinate of $g \cdot (\hat{z}, u_t)$ is given by $gu_tg^{-1} \cdot g\tau_x\mu(g^{-1})$, where $g\tau_x\mu(g^{-1}) \in N_1$. Since the right multiplication in N_1 is simply the coordinatewise addition, $g \cdot (\hat{z}, u_t)$ is a line with the slope coefficients depending polynomially only on y_2 and z_2. $\square$

The proper Borel–Serre strata for SL_2 are disjoint in the boundary of $\bar{X}_{\mathbb{R}}(M_1^0)$, and so are the closures of the strata in $\bar{X}_{\mathbb{R}}(P_1) - X$ where $P_1(\mathbb{R})$ operates by automorphisms. If the strata $e(B)$, $B \in \mathcal{P}(P_1)$, have Euclidean metrics induced from $N_0 = e(P_0)$ via the conjugation by fixed $g_B \in \tau M_1^0(\mathbb{R})$ then $P_1(\mathbb{R})$ also acts on the disjoint union of the ideal compactifications $\varepsilon(B)$, i.e., on

$$\delta(P_1) \stackrel{\text{def}}{=} \varepsilon(P_0) \underset{P_0(\mathbb{R})}{\times} P_1(\mathbb{R}).$$

Notice that all fibers of $\bar{\mu}_{\mathbb{R}}|(\bar{X}_{\mathbb{R}} - X)$ are totally geodesic in the corresponding strata.

Definition C.1.2. Define the set $\hat{X}(P_1) = \varepsilon(P_1) \sqcup \delta(P_1)$.

Rationale . Since $M_1^0 \cong SL_2$, each level in $\bar{X}_{\mathbb{R}}(M_1^0) \times N_1$ can be compactified appropriately as in Example 9.4.7. The result could be easily compactified by embedding

$$\hat{X}(M_1^0) \times N_1 \hookrightarrow \hat{X}(M_1^0) \times \varepsilon N_1.$$

This compactification would contain both of the Γ-enlargements considered before. However, the $P_1(\mathbb{R})$-action on $X(P_1)$ would not extend continuously to this product. This is precisely the defect we correct by using the ball $\varepsilon(P_0)$ instead of $\varepsilon L_0 \times \varepsilon N_1 := \varepsilon(P_0^{P_1}) \times \varepsilon N_1$ to compactify $e(P_0)$.

The product topology on $\bar{X}_{\mathbb{R}}(M_1^0) \times \varepsilon N_1$ could be used to introduce a topology on a major part of $\hat{X}(P_1)$. However, only a quotient of the product becomes a subset of $\hat{X}(P_1)$ when one contracts the fibers of $\psi\colon L_0 \times \partial N_1 \to \partial N_1$ and the analogues in other strata. Let σ be the meridian circle $\partial N_1 \subseteq e(P_0)$. The translates of σ under the action of the stable lift of $M_1^0(\mathbb{R})$ in various strata are well-defined. Now the complement of this subset consists of hemispheres in $\delta(P_1)$. For these points, open neighborhoods can be specified similarly to the open influx neighborhoods of the endpoints $y \in \hat{X}(SL_2)$ in Example 9.4.7. Something similar has to be done for points on the meridian circles $g \cdot \sigma$ in $\delta(P_1)$ in order for the resulting space to be compact. It is easy to see, however, that if the complete geodesic bundle over an open neighborhood of a point on the meridian circle σ is to be contained inside its neighborhood in $\hat{X}(P_1)$ then the space comes out non-Hausdorff. The correction we use is the requirement that the neighborhoods of points on circles project to neighborhoods of their images under the most obvious set projection

$$p\colon \hat{X}(P_1) \longrightarrow \varepsilon X(M_1^0).$$

In other words, the projection has to be continuous.

The space $\varepsilon(P_1)$ comes with the product topology. For $y \in \varepsilon(P_1) \subseteq \hat{X}(P_1)$ let

$$\mathcal{N}(y) = \{\mathcal{O} \subseteq \hat{X}(P_1) : \mathcal{O} \text{ contains an open neighborhood of } y \text{ in } \varepsilon(P_1)\}.$$

Also, $\bar{X}_{\mathbb{R}}(P_1) \subseteq \hat{X}(P_1)$ is the $\mathbb{R}$-Borel–Serre construct and has topology in which each corner $X(B)$, $B \in \mathcal{P}_{\mathbb{R}}(P_1)$, is open. For $y \in \bar{X}_{\mathbb{R}}(P_1)$ let

$$\mathcal{N}(y) = \{\mathcal{O} \subseteq \hat{X}(P_1) : \mathcal{O} \text{ contains an open neighborhood of } y \text{ in } \bar{X}_{\mathbb{R}}(P_1)\}.$$

Notation . Given an open subset $U \subseteq \varepsilon(B)$, let $\mathcal{O}(U) = q_B^{-1}(V)$, the total space of the restriction to $V = U \cap e(B)$ of the trivial bundle q_B over $e(B)$ with fiber A_B. If U is any open subset of $\delta(P_1)$, let

$$\mathcal{O}(U) = \bigcup_{B \in \mathcal{P}_{\mathbb{R}}} \mathcal{O}\big(U \cap e(B)\big)$$

and

$$\begin{aligned} \mathcal{C}(U) = \{z \in \varepsilon(P_1) \cup \bar{X}_{\mathbb{R}}(P_1) : \exists\, \mathcal{O} \in \mathcal{N}(z) \ni \mathcal{O} \cap e(P_1) \subseteq \mathcal{O}(U)\} \cup \\ \{z \in \delta(P_1) \backslash \bar{X}_{\mathbb{R}}(P_1) : \exists \text{ open } U' \subseteq \delta(P_1) \ni z \in U' \text{ and } \mathcal{O}(U') \subseteq \mathcal{O}(U)\}. \end{aligned}$$

Now let $y \in \partial e(B)$ and define

$$\mathcal{N}(y) = \{\mathcal{O} \subseteq \hat{X}(P_1) : \exists \text{ open set } U \subseteq \delta(P_1) \text{ with } y \in U \text{ and } \mathcal{C}(U) \subseteq \mathcal{O}\}.$$

This gives a system of neighborhoods $\mathcal{N}(y)$ for any $y \in \hat{X}(P_1)$.

For a subset $\mathcal{S} \subseteq \hat{X}(P_1)$, let $\mathcal{N}(\mathcal{S}) = \{\mathcal{O} \subseteq \hat{X}(P_1) : \mathcal{O} \in \mathcal{N}(y) \text{ for every } y \in \mathcal{S}\}$.

Theorem C.1.3. *We call $\mathcal{S}$* open *if $\mathcal{S} \in \mathcal{N}(\mathcal{S})$. The open subsets of $\hat{X}(P_1)$ form a well-defined topology.*

Proof. Routine. It is completely analogous to Theorem 9.4.5. □

As we have mentioned before, this topology on $\hat{X}(P_1)$ is non-Hausdorff. Another non-Hausdorff topology is obtained as a pull-back via the projection $p\colon \hat{X}(P_1) \to \varepsilon X(M_1^0)$.

Definition C.1.4. We define the topology on $\hat{X}(P_1)$ to be the product of the topology given by Theorem C.1.3 and the topology pulled back from $\varepsilon X(M_1^0)$.

Theorem C.1.5 (Topological Properties)**.** *The space $\hat{X}(P_1)$ is* (1) *compact,* (2) *Hausdorff, and* (3) *Čech-acyclic.*

Proof. (1) Let $\mathcal{U}$ be an arbitrary open cover of $\hat{X}(P_1)$. Since each space $\varepsilon(B)$, $B \in \mathcal{P}_{\mathbb{R}}(P_1)$, is compact, there are finitely many elements $U_{1,B}, \ldots, U_{s_B,B}$ from $\mathcal{U}$ which cover $\varepsilon(B)$. It is clear that there is an open neighborhood $V(B) \subseteq \varepsilon X(M_1^0)$ of $p\big(\varepsilon(B)\big)$ such that $p^{-1}\big(V(B)\big) \subseteq \bigcup_{i=1}^{s_B} U_{i,B}$. Let $\mathfrak{B}$ be a finite subset of $\mathcal{B}(P_1)$ with $\varepsilon(M_1^0) = \bigcup_{B \in \mathfrak{B}} V(B)$. The open subspace $\varepsilon(P_1)$ is relatively compact in $\varepsilon X(M_1^0) \times \varepsilon N_1$. It follows easily that there is a finite collection $\{U_a \in \mathcal{U} : a \in \mathfrak{A}\}$ with $\varepsilon(P_1) \subseteq \bigcup_{a \in \mathfrak{A}} U_a$. Now U_a, $a \in \mathfrak{A}$, together with all $U_{1,B}, \ldots, U_{s_B,B}$, $B \in \mathfrak{B}$, form the desired finite subcovering of $\mathcal{U}$.

(2) Recall that the projection $\hat{X}(P_1) \to \varepsilon X(M_1^0)$ is made continuous with our topology. Since $\varepsilon(P_1)$ is Hausdorff and projects onto the open subset $X(M_1^0)$ of the normal space $\varepsilon X(M_1^0)$, it suffices to show that $\delta(P_1)$ with the subspace topology is Hausdorff. This is essentially done by noticing that $\delta(P_1)$ projects onto Hausdorff $\partial X(M_1^0)$, and each fiber $\varepsilon(B)$ of this projection is Hausdorff.

(3) Apply Theorem 11.3.1. □

C.2. **Proof of the Splitting.** As in §13, the proof consists of constructing a composite weak homotopy equivalence that completes the ambient diagram of spectra. The first factor in the weak equivalence is again induced by the inclusion of an appropriate category $\mathcal{O}rd\, Y \hookrightarrow \mathcal{C}ov Y$ for $Y = \hat{X}(P_1) - \bar{X}_{\mathbb{Q}}(P_1)$ which satisfies the hypotheses of §4.4.

Cellular Decompositions. The open sets used in $\mathcal{O}rd\,Y$ are based on open stars in cubical decompositions which this time have to be carefully coordinated. This is done by specifying the composition of collapses and a homeomorphism

$$I^3 \xrightarrow{\rho} B^3 \xrightarrow{\overline{\Upsilon}} \varepsilon N_0$$

similar to those in §7.5. The sequence of collapses ρ in ∂I^3 is precisely the one from §7.5. The homeomorphism $\overline{\Upsilon}$ can always be arranged to be "linear" so that each plane $\{(x_1, x_2, x_3) : -1 < x_2, x_3 < 1\}$ is mapped to a coset of $N_1 \subseteq N_0$ and $(\pm 1, \flat, \flat)$ are the limit points of the parallel lines $\{(x_1, x_2, x_3) : -1 < x_3 < 1\}$. Now induce the cubical cellular decomposition of εN_0 via $\overline{\Upsilon}\rho$. Similarly, other $\varepsilon(B)$ have decompositions induced from $\varepsilon N_0 = \varepsilon(P_0)$ via the conjugation by fixed $g_B \in \tau M_1^0(\mathbb{R})$. This last restriction is important—it guarantees that $v \in \partial N_0$ is mapped to $g_B \cdot v \in \partial B$ in the closure of the same lift of $X(M_1^0)$ in $\varepsilon(P_1)$.

One can also identify $\varepsilon X(M_1^0)$ with B^2 and induce a cubical decomposition from I^2 via the collapse ρ. This is just one particular way of forming derivable decompositions of $\varepsilon X(M_1^0)$ with decreasing mesh and the nerves of the corresponding open star coverings always contractible. Notice that ∂N_1 is a cellular subcomplex in each derived decomposition of ∂N_0. A choice of natural numbers m, k gives a decomposition of ∂N_1 and $\varepsilon X(M_1^0)$ of corresponding orders. This leads to the product cellular structures on $\partial N_1 \times \varepsilon X(M_1^0)$ and the corresponding open star coverings. The induced open star coverings of $\varepsilon(P_1) = \partial N_1 \times X(M_1^0)$ will be denoted $\mathcal{O}_{m,k}$.

Bounded Saturation. The standard fundamental domain $\mathcal{D}$ in N_0 is the product of the domain $\mathcal{D}_0$ in $e(P_0^{P_1})$ and the domain $\mathcal{D}_1$ in N_1 (cf. Example 6.2.4). We choose our domain in $\bar{X}_{\mathbb{Q}}(P_1)$ to be the product of the completion of the usual hyperbolic triangle T (cf. §B.2) by $\mathcal{D}_0$—a domain in $\bar{X}_{\mathbb{Q}}(M_1^0)$—and $\mathcal{D}_1$. Now all of the arguments using "barriers" from §12.4 can be repeated.

Corollary C.2.1. *The following subsets of Y are boundedly saturated:*

- *each subset of the form*

  $$(\xi_k \cup \vartheta_l) \times X^{\mathrm{S}}_{\mathbb{Q}}(M_1^0) \subseteq \partial N_1 \times X^{\mathrm{S}}_{\mathbb{Q}}(M_1^0) \subseteq Y, \quad k, l \in \{1, 2\},$$

  where ξ_1, ξ_2 and ϑ_1, ϑ_2 are the two 0-cells and two 1-cells in ∂N_1, and $X^{\mathrm{S}}_{\mathbb{Q}}(M_1^0)$ is the union of X and the rational ideal bounary points (i.e., the rational Satake enlargement),
- *each 2-cell in ∂N_0 and their conjugates in the closures of other rational strata,*
- *each $\varepsilon(B)$, $B \in \mathcal{B}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{Q}}$, by the analogue of Corollary 12.5.2.*

Definition C.2.2 (cf. Definition 13.2.1)**.** The covering $\mathcal{A}_0$ of ∂N_0 by the sets

$$\overline{\Upsilon}\rho\big(\mathrm{Star}^{\mathrm{o}}(v_{-1}(s_1, s_2, s_3))\big)$$

is finite but not open. This choice generates the category $\{\alpha_0\}$ of finite rigid coverings α_0 of ∂N_0 with $\operatorname{im}\alpha_0 = \mathcal{A}_0$. Notice that all $\partial e(B)$, $B \in \mathcal{B}_{\mathbb{R}}$, have well-defined cellular decompositions and, therefore, coverings $\mathcal{A}_B$ analogous to $\mathcal{A}_0$. There are also associated rigid coverings $\{\alpha_B\}$ with $\operatorname{im}\alpha_B = \mathcal{A}_B$.

Remark C.2.3. It is easy to see that requirement (2) in Definition 4.2.3 forces any finite rigid covering of ∂N_0 by sets of the form $\overline{\Upsilon}\rho\big(\mathrm{Star}^{\mathrm{o}}(v_{-1}(s_1, s_2, s_3))\big)$ to actually employ all such sets, i.e., $\operatorname{im}\alpha_B = \mathcal{A}_B \Leftrightarrow \operatorname{im}\alpha_B \subseteq \mathcal{A}_B$ in the above definition.

Definition C.2.4. Consider the finite covering $\mathcal{A}_1$ of ∂N_1 by the sets

$$\overline{\Upsilon}\rho\big(\mathrm{Star}^{\mathrm{o}}(v_{-1}(s_1, s_2))\big) \cap \partial N_1$$

This choice generates the category $\{\alpha_1\}$ of finite rigid coverings α_1 of ∂N_1 with $\operatorname{im} \alpha_1 = \mathcal{A}_1$. The same remark applies here.

There are six cells in each $\partial e(B)$, $B \in \mathcal{B}_{\mathbb{R}}$, which can be viewed as images of appropriate sides of the cube I^3:

$$\mathrm{Cell}_i^{\pm} = \overline{\Upsilon}\rho\{x_i = \pm 1,\ -1 < x_j < 1,\ j \neq i\}, \quad i = 1, 2, 3.$$

From compactness of $\varepsilon(B)$ and property (2) of finite rigid coverings from Definition 4.2.3, it follows that for each choice of α_B there is $0 < n_B \in \mathbb{N}$ such that if $v \in \overline{\Upsilon}\rho(V_{(n_B)}) \cap \mathrm{Cell}_i^{\pm}$ then

$$y \in \overline{\Upsilon}\rho\big(\mathrm{Star}^{\mathrm{o}}_{n_B}(v)\big) \implies \mathrm{Cell}_i^{\pm} \subseteq \alpha_B(y).$$

We will simplify the notation $\overline{\Upsilon}\rho(V_{(k)})$ to $V_{(k)}$.

Similar numbers $0 < m_1 \in \mathbb{N}$ exist for the coverings $\alpha_1 \in \{\alpha_1\}$.

Definition C.2.5. Let $\{\pi\}$ be the category of finite rigid coverings of $\partial N_1 \times X^{\mathrm{S}}_{\mathbb{Q}}(M_1^0)$ by the four boundedly saturated sets

$$(\xi_i \cup \vartheta_j) \times X^{\mathrm{S}}_{\mathbb{Q}}(M_1^0), \quad i, j \in \{1, 2\}.$$

For each $\pi \in \{\pi\}$ and $k \in \mathbb{N}$, there exists $0 < m(\pi) \in \mathbb{N}$ such that if $v \in \zeta_i \times X^{\mathrm{S}}_{\mathbb{Q}}(M_1^0)$ then

$$y \in \mathrm{Star}^{\mathrm{o}}_{m,k}(v) \cap \varepsilon(P_1) \implies \zeta_i \times X^{\mathrm{S}}_{\mathbb{Q}}(M_1^0) \subseteq \pi(y),$$

where ζ denotes either ξ or ϑ and $i \in \{1, 2\}$.

Orderly Coverings. Given a covering $\beta \in \mathcal{C}ovY$, let $\mathcal{U} = \operatorname{im} \beta \in \mathcal{C}\mathrm{ov}^{\mathrm{s}}Y$. Using compactness of $\hat{X}(P_1)$ and relative compactness of $\varepsilon(P_1)$, one can choose a finite collection $\mathfrak{P}$ of proper parabolic subgroups $P \in \mathcal{B}_{\mathbb{R}}$ and numbers $0 < m, k, \ell_P \in \mathbb{N}$ with the following properties:

(1) for some $0 < k_P \in \mathbb{N}$ and $w(P) \in \partial X(M_1^0) \cap V_{(k)}$

$$\delta X(P_1) \cap Y = \bigcup_{P \in \mathfrak{P}} \mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_{\ell_P}(v)\big) \cap p^{-1}\mathrm{Star}^{\mathrm{o}}_{k_P}(w) \cap Y,$$

where $\mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_{\ell_P}(v)\big)$ means $\mathrm{Star}^{\mathrm{o}}_{\ell_P}(v)$ when $v \in e(P)$,

(2) $\{\mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_{\ell_P}(v)\big) \cap p^{-1}\mathrm{Star}^{\mathrm{o}}_{k_P}(w) \cap Y\}$ refines the restriction of $\mathcal{U}$ to $\delta X(P_1)$,

(3) $\mathcal{O}_{m,k}$ refines the restriction of $\mathcal{U}$ to $\varepsilon(P_1)$,

(4) $m \geq \max_{P \in \mathfrak{P}}(\ell_P)$, $k \geq \max_{P \in \mathfrak{P}}(k_P)$,

(5) each open star in the associated k-th cubical derived decomposition of $\varepsilon X(M_1^0)$ contains at most one point from $\{W(P) : P \in \mathfrak{P}\}$,

(6) for each $w \in \partial X(M_1^0) \cap V_{(k)}$ there exists $P \in \mathfrak{P}$ such that either $W(P) \in \mathrm{Star}^{\mathrm{o}}_k(w)$ or

$$p^{-1}\big(\mathrm{Star}^{\mathrm{o}}_k(w)\big) \subseteq \bigcup_{v \in V_{(\ell_P)}} \mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_{\ell_P}(v)\big). \tag{$\dagger$}$$

For a given w as in property (6) out of all groups $P_1, \ldots, P_s \in \mathfrak{P}$ satisfying condition (†) pick $P(w)$ with $\ell_P = \max\{\ell_{P_1}, \ldots, \ell_{P_s}\}$. Define

$$\mathrm{Ord}_{\ell_P,k}(v; w) \stackrel{\mathrm{def}}{=} Y \cap \mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_{\ell_P}(v)\big) \cap p^{-1}\mathrm{Star}^{\mathrm{o}}_k(w)$$

and

$$\mathrm{ExcOrd}_{\ell_P,k}(v;w) \overset{\mathrm{def}}{=} \mathrm{Ord}_{\ell_P,k}(v;w)\backslash\varepsilon(P).$$

Consider the category $\mathcal{E}\mathrm{xc}\mathcal{O}\mathrm{rd}^{\mathrm{s}}Y$ of finite open coverings $\Omega \in \mathcal{C}\mathrm{ov}^{\mathrm{s}}Y$ by the sets $(\mathrm{Exc})\mathrm{Ord}_{\ell_P,k}(v;w)$ and $\mathcal{O}_{m,k}$ for all possible choices of β, $\mathfrak{P}$, etc. Now generate all finite rigid coverings $\omega \in \mathcal{C}ov Y$ which satisfy

- $\mathrm{im}\,\omega \in \mathcal{E}\mathrm{xc}\mathcal{O}\mathrm{rd}^{\mathrm{s}}Y$,
- $\omega(y) = \mathrm{Ord}_{\ell_P,k}(v;w)$ for some $P \in \mathfrak{P}$ if and only if $y \in \varepsilon(P)$,
- if $y \in \varepsilon(B)$ for some $B \in \mathfrak{P}$ then

  $$\omega(y) = \mathrm{ExcOrd}_{\ell_{P(w)},k}(v;w)$$

  for some v where $\chi\big(W(B)\big) = \mathrm{Star}^{\mathrm{o}}_k(w)$ for a fixed finite rigid covering χ of $\varepsilon X(M^0_1)$ by open stars $\mathrm{Star}^{\mathrm{o}}_k(z)$, $z \in V_{(k)}$,
- $\omega(y) \in \mathcal{O}_{m,k}$ if $y \in \varepsilon(P_1)$.

The resulting coverings form a full subcategory $\mathrm{P{\scriptsize RE}O{\scriptsize RD}}Y$ of $\mathcal{C}ov Y$. Now mimic the rest of §13.1 to get a category $\mathcal{O}rd\,Y$ with

$$\jmath^* \colon \check{h}(Y;KR) \xrightarrow{\simeq} \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\mathrm{holim}}\,(N\_ \wedge KR).$$

Completion of the Proof. The rest of the proof also follows §13.2 and §13.3 closely. Fix $\pi \in \{\pi\}$ and $\alpha_P \in \{\alpha_P\}$ for each $P \in \mathcal{P}_{\mathbb{R}}(P_1)$. We present the construction of boundedly saturated coverings $\alpha(\omega,\alpha_P,\pi)$ on generators $\omega \in \mathrm{P{\scriptsize RE}O{\scriptsize RD}}Y$. It is in two steps.

Definition C.2.6 ($\boldsymbol{\alpha}^{\mathrm{int}}$). For $B \in \mathcal{P}_{\mathbb{R}}(P_1)$ use the notation $\alpha_{1,B}$ for the finite rigid covering of σ_B given by $\alpha_{1,B}(y) = \alpha_B(y) \cap \sigma_B$ for each $y \in \sigma_B$. The same formula associates $\alpha_{1,B}(y) \subseteq \sigma_B$ to each $y \in \partial e(B)$. Define $\Pi_B \colon \delta e(B) \to \mathrm{im}\,\pi$ by

$$\Pi_B(y) = \begin{cases} \alpha_{1,B}(y) \times X^{\mathrm{S}}_{\mathbb{Q}}(M^0_1) & \text{if } B \in \mathfrak{P}, \\ \alpha_{1,P}(v) \times X^{\mathrm{S}}_{\mathbb{Q}}(M^0_1) & \text{otherwise,} \end{cases}$$

where v and w are the vertices in $\omega(y) = \mathrm{ExcOrd}_{\ell_{P(w)},k}(v;w)$. Now define

$$\alpha^{\mathrm{int}}(y) = \begin{cases} \pi(y) & \text{if } y \in \varepsilon(P_1), \\ \omega(y)\backslash\varepsilon(P) \cup \Pi_P(y) & \text{if } y \in \varepsilon(P),\ P \in \mathfrak{P}, \\ \omega(y) \cup \Pi_P(y) & \text{if } y \in \sigma_P,\ P \notin \mathfrak{P}, \\ \omega(y) & \text{otherwise.} \end{cases}$$

Each $A \in \mathrm{im}\,\alpha^{\mathrm{int}}$ is a union of a (possibly empty) boundedly saturated set and a (possibly empty) subset of $Y\backslash\big(\partial N_1 \times X^{\mathrm{S}}_{\mathbb{Q}}(M^0_1)\big)$ which is a disjoint union of boundedly saturated 2-cells in $\varepsilon(B)$, $B \in \mathcal{P}_{\mathbb{Q}}$, and balls $\varepsilon(B)$, $B \in \mathcal{P}_{\mathbb{R}}\backslash\mathcal{P}_{\mathbb{Q}}$. It makes sense to talk about envelopes of A with respect to the Boolean algebra of subsets of Y generated by those identified in Corollary C.2.1. The next step $\alpha^{\mathrm{int}}(\omega) \rightsquigarrow \alpha(\omega)$ is the saturation. Using the argument as in §13.3 and Quillen's theorem A one can see that each step preserves the homotopy type of the nerve:

$$\imath(\omega) \colon N\omega \xrightarrow{\simeq} N\alpha^{\mathrm{int}}(\omega) \xrightarrow{\simeq} N\alpha(\omega).$$

The argument is completed as in §13.3 by a comparison with the obvious "hypothetical" non-equivariant compactification.

Appendix D. Novikov Conjectures for Lattices in SL_3

The material of Appendix C can be used at the last stage of a three-step inductive argument to prove

Theorem D . *If Γ is a torsion-free arithmetic subgroup of $SL_3(\mathbb{Q})$ then α is a split injection.*

The base case here turns out to be that of torsion-free arithmetic subgroups of SL_2 which were described in detail in Examples 8.3.3, 9.4.7, and Appendix B. We have noticed that all of these groups are free. The Novikov conjecture for free groups has been known for some time. On the contrary, by a theorem of Furstenberg [38, Theorem 5], no arithmetic subgroup of SL_3 can be free. More generally, no such group can act freely on a simplicial tree ([77, §6.6, Theorem 16]). Again, the congruence subgroups of SL_3 are important examples of groups for which this theorem is a new result. Incidentally, a study of the homology of congruence subgroups of SL_3 was the basis for the computation of the order $\circ\big(K_3(\mathbb{Z})\big) = \ntrianglerighteq\nleftarrow$ in [57]. Every arithmetic group contains a normal torsion-free subgroup of finite index, but, according to Minkowski, the congruence subgroups of SL_n of all levels $\ell \neq 2$ are themselves torsion-free (see [15, p. 40]). For $n \geq 3$ every arithmetic subgroup contains a suitable congruence subgroup according to the solution of the congruence subgroup problem ([6]). This identifies a particular cofinal system of torsion-free arithmetic subgroups in SL_3 to which our Theorem 1 applies.

Since the center of $SL_3(\mathbb{R})$ is finite and the projective group $PSL_n(\mathbb{R})$ is simple when $n \geq 3$ (cf. [55, XIII, Theorem 9.3]), the arithmeticity result of Margulis ([61, IX]) shows that all torsion-free non-uniform lattices in SL_n are arithmetic. (This makes Theorem 5 from Introduction a corollary of Theorem D.) By [28, §10.4] or a recent result of Farb ([29]), such lattices are not bicombable which excludes the possibility of applying techniques from CAT(0) geometry and its analogues to these groups. At the moment there are no examples known which would distinguish the classes of automatic groups and bicombable groups ([3]) although the classes seem to be fundamentally different (cf. [14]).

D.1. **Construction of $\hat{X}(SL_3)$.** This will be another compactification of the symmetric space $X(G)$ for the semi-simple algebraic group $G = SL_3$. What makes it different from the Martin, Satake, Karpelevič, and the ideal compactifications is that it also contains the Borel–Serre enlargements $E\Gamma = \bar{X}_{\mathbb{Q}} \subseteq \bar{X}_{\mathbb{R}}$ as open dense subspaces. In general, there may appear complications in the way $\bar{X}_{\mathbb{Q}}$ and $\bar{X}_{\mathbb{R}}$ fit together arizing from the fact that the $\mathbb{Q}$-rank of G may be different from the $\mathbb{R}$-rank. Restricting our attention to the case of the special linear group SL_3 avoids such phenomena.

D.1.1. *Basic Case: SL_2.* The truly basic case is that of the Lie group $L = \mathbb{R}$ acting on itself by addition. The associated symmetric space is L, and the equivariant compactification we want is the obvious completion $\hat{L} = L \cup \{-\infty, +\infty\}$.

Now consider the standard parabolic $\mathbb{R}$-subgroup P_0 of $G = SL_2$. The corresponding reductive stratum in $\bar{X}^{\rho}_{\mathbb{R}}$ is a point, and

$$e(P_0) \cong R_u P_0(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} : \curvearrowleft \in \mathbb{R} \right\} \cong \mathbb{L}.$$

Recall from Example 8.3.3 that a proper parabolic subgroup $P \subseteq SL_2$ stabilizes a point $p(P) \in \partial\mathbb{E}$ and permutes the geodesics abutting to $p(P)$. The stratum $e(P) \cong L$ parametrizes those geodesics. Complete each stratum: $\varepsilon(P) \cong \hat{L}$.

The group SL_2 is of $\mathbb{R}$-rank one. We discussed such groups in Part IV. Autonomous simpler description of the compactification $\hat{X}(SL_2)$ and proofs of such properties of it as being Hausdorff, compact, and Čech-acyclic are given in §B.

D.1.2. *Minimal Borel–Serre Strata.* For a torsion-free arithmetic subgroup Γ of $G(\mathbb{Q})$ and any rational Borel subgroup B, $\Gamma \cap B(\mathbb{Q})$ is the largest subgroup which acts in $e(B)$. It is a cocompact nilpotent discrete subgroup $\Gamma \cap N_B$ of the nilpotent component N_B in the Langlands decomposition of $B(\mathbb{R})$. Indeed, for the standard Borel subgroup P_0 of $G = SL_3$, $N_{P_0} = N_0$ as in Example 8.3.4, $\Gamma \cap N_0$ is the discrete Heisenberg group, and N_0/Γ_0 is the compact 3-dimensional Heisenberg nilmanifold.

By analogy with the construction of $\hat{X}(SL_2)$, we first compactify each $e(B)$, $B \in \mathcal{B}_{\mathbb{R}}$, $\Gamma_B = \Gamma \cap B(\mathbb{R})$-equivariantly, then provide the new points with certain neighborhoods which would form a part of the basis for the topology on $\hat{X}$. Recall that $e(B)$ can be identified with N_B (Remark 8.5.1) and that Γ_B acts on N_B via left multiplication (Lemma 9.2.4). In Part III we performed Γ_N-equivariant compactifications of general simply-connected nilpotent groups N. The 3-dimensional Heisenberg group $U_3(\mathbb{R})$ was the principal example. Refer to Example 6.2.4 for the discussion of the ideal compactification of $e(P_0) \cong N_0 = U_3(\mathbb{R})$ which is both left- and right-equivariant.

Definition D.1.1 (Certain Projections)**.** Let us introduce two auxiliary constructions which are themselves enlargements of $e(P_0)$.

The stratum $e(P_0)$ can be fibered in two different ways over $\mathbb{R}^{\nVdash}$ as a plane bundle corresponding to the two semi-direct product structures on N_0. These structures are analogous to the discrete Heisenberg group case; they are in terms of the following four subgroups:

$$L_1 = \{(x,0,0)\} \cong \mathbb{R}^{\nVdash}, \quad Z_1 = \{(0,y,z)\} = N_1(\mathbb{R}) \cong \mathbb{R}^{\nvDash},$$
$$L_2 = \{(0,y,0)\} \cong \mathbb{R}^{\nVdash}, \quad Z_2 = \{(x,0,z)\} = N_2(\mathbb{R}) \cong \mathbb{R}^{\nvDash}.$$

The corresponding $\mathbb{R}^{\nvDash}$-bundles are

$$\Phi_i \colon Z_i \times L_i \longrightarrow L_i, \quad i = 1 \text{ or } 2.$$

The unipotent radicals Z_i can be ideally compactified as flat Euclidean spaces. There are associated fiber bundles

$$\phi_i \colon \varepsilon Z_i \times L_i \longrightarrow L_i, \quad i = 1 \text{ or } 2,$$

and line bundles over circles:

$$\psi_i \colon \partial Z_i \times L_i \longrightarrow \delta Z_i, \quad i = 1 \text{ or } 2.$$

If σ_1 and σ_2 denote the meridian circles in $\varepsilon(P_0)$ given by $x_2 = 0$ and $y_2 = 0$ respectively then the projections we wish to record are

$$\pi_i \colon \varepsilon Z_i \times L_i \longrightarrow \varepsilon Z_i \times L_i/\psi_i = e(P_0) \cup \sigma_i \hookrightarrow \hat{e}(P_0).$$

Definition D.1.2 (Cubical Cellular Decompositions)**.** Refer to §7.5 for the construction of cubical cellular decompositions of $I^3 \subseteq \mathbb{R}^{\nVdash}$. In each i-th derived CW-structure there must be $2^{3(i+1)}$ cells involved. These decompositions behave well with respect to the collapse performed in the boundary of the cube I^3 which contracts faces

$$\{(x_1,*,*) \in I^3 : x_1 = \pm 1\} \longrightarrow (\pm 1, \flat, \flat).$$

This *symmetric* collapse is different from the ones constructed in §7.5. The result is a topological ball B^3 with the CW-structure consisting of two cells of dimension 0, four cells of each dimension 1 and 2, one 3-dimensional cell and a continuous collapse $\rho\colon I^3 \to B^3$. Notice that every old derived cubical CW-structure in I^3 induces a CW-decomposition of the image in the obvious way.

Recall the study of the right action of N_0 or Γ_0 on ∂N_0. We have noticed that the poles with $x_2 = y_2 = 0$ are fixed, and the circles of points with $x_2 = 0$ or $y_2 = 0$ are themselves invariant sets. In particular, the four open arcs complementary to the fixed poles are invariant, and the four connected components of the complement to the circles are also invariant. Consider the (-1)-st derived decomposition of I^3 and the corresponding CW-structure in B^3. Notice that the cells in ∂B^3 are in bijective correspondence with the invariant cells just described. We will refer to this isomorphism of CW-structures as $\Upsilon\colon \partial B^3 \to \partial N_0$.

Again, there are cubical analogues of links and stars defined by the same formulae as before.

D.1.3. *Maximal Borel–Serre Strata.* We proceed to Γ_{P_i}-equivariantly enlarge each of $e(P_i)$, $i = 1, 2$. Recall the projection map

$$\mu_{P_i}\colon e(P_i) = R_u P_i(\mathbb{R}) \times \hat{}(\mathbb{P}_{\beth}) \longrightarrow \hat{}(\mathbb{P}_{\beth}).$$

The real version

$$\bar{\mu}_{P,\mathbb{R}}\colon \overline{e(P)}_{\mathbb{R}} \longrightarrow \overline{\hat{e}(P)}_{\mathbb{R}}$$

of the principal fibration $\bar{\mu}_P$ from Proposition 9.2.2 has the related product structure which extends $\bar{\mu}_P$. The total space $\overline{e(P_i)}_{\mathbb{R}}$ is the first partial enlargement of $e(P_i)$ we have in mind.

Another enlargement is obtained by compactifying each flat fiber of the principal fibration μ_{P_i} by its ideal boundary. In other words, we embed $e(P_i)$ in

$$\varepsilon(P_i) \overset{\text{def}}{=} \varepsilon\big(R_u P_i(\mathbb{R})\big) \times \hat{}(\mathbb{P}_{\beth}).$$

The formula from Lemma 9.2.4 shows that $P_i(\mathbb{R})$ acts on $e(P_i)$ by bundle automorphisms. It can be used to see that in general the action extends to $\varepsilon(P_i)$. In our low-dimensional situation, the concrete calculations in Propositions C.1.1 and D.1.3 may be more satisfying.

Proposition D.1.3. *The inclusion*

$$e(P_2) \subseteq \varepsilon\big(R_u P_2(\mathbb{R})\big) \times \hat{}(\mathbb{P}_{\ngeq})$$

is a $P_2(\mathbb{R})$-equivariant enlargement.

Proof. This is identical to the proof of Proposition C.1.1 but the formulae are different. This reflects the non-symmetric equivariance in the corner associated to P_0.

If $g = (g_{ij})$, $1 \le i,j \le 3$, $g_{21} = g_{31} = 0$, and

$$u_t = (x_1 + x_2 t, 0, z_1 + z_2 t) \in R_u P_2(\mathbb{R}),$$

then

$$g u_t g^{-1} = g_{11}\left(\frac{g_{33}}{D}(x_1 + x_2 t) - \frac{g_{32}}{D}(z_1 + z_2 t), 0, \frac{g_{22}}{D}(z_1 + z_2 t) - \frac{g_{23}}{D}(x_1 + x_2 t)\right),$$

where $D = g_{22}g_{33} - g_{23}g_{32}$. $\square$

Now it is clear that we can define $e(P_2)\widehat{\ } = \hat{X}(P_2)$ in exactly the same way Definitions C.1.2 and C.1.4 do that for $\hat{X}(P_1)$. Then Theorems C.1.3 and C.1.5 and their proofs apply verbatim. This is true for all $e(P)\widehat{\ }$, $P \in \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}(SL_3)$.

D.1.4. *Definition of* $\hat{X}(SL_3)$. The conjugation action of $SL_3(\mathbb{R})$ on $\hat{X}(SL_3)$ permutes the Borel–Serre strata associated to the three standard proper parabolic subgroups P_i, $i = 0, 1, 2$. Each stratum may be compactified as in §D.1.2 and D.1.3. Denoting $\varepsilon(P_0)$ by $e(P_0)\widehat{\ }$, define

$$Y_i = \widehat{e(P_i)} \underset{P_i(\mathbb{R})}{\times} SL_3(\mathbb{R}).$$

Since $\widehat{e(P_0)} \subseteq \widehat{e(P_1)}$, $\widehat{e(P_0)} \subseteq \widehat{e(P_2)}$, we have $Y_0 \subseteq Y_1$, $Y_0 \subseteq Y_2$ and can form

$$\delta(\mathcal{P}_3) \overset{\text{def}}{=} Y_1 \underset{Y_0}{\cup} Y_2.$$

A warning similar to the one in §9.4 has to be issued here. The identification space $\delta(\mathcal{P}_3)$ will soon be retopologized. This is similar to the way the non-compact space $\delta(P_1) = \coprod_B \varepsilon(B)$ was used in §C.1 to introduce a compact topology.

The space $\delta X = \delta(\mathcal{P}_3)$ contains a subspace $Y_{\mathbb{R}}$ which can be constructed similarly by replacing the compactifications of strata in the formulae above by $\overline{e(P_i)}_{\mathbb{R}}$. It is easy to see that $Y_{\mathbb{R}}$ is homeomorphic to $\bar{X}_{\mathbb{R}} - X$. An argument showing that the rational version of this space $Y_{\mathbb{Q}}$ is diffeomorphic to $\bar{X}_{\mathbb{Q}} - X$ is given in [57, §6].

Definition D.1.4. Define the set $\hat{X} = \bar{X}_{\mathbb{R}} \cup \delta X = X \sqcup \delta X$.

The space $\bar{X}_{\mathbb{R}}$ is the Borel–Serre enlargement and has the topology in which each corner $X(P)$ is open. For $y \in \bar{X}_{\mathbb{R}}$ let

$$\mathcal{N}(y) = \{\mathcal{O} \subseteq \hat{X} : \mathcal{O} \text{ contains an open neighborhood of } y \text{ in } \bar{X}_{\mathbb{R}}\}.$$

Given a maximal parabolic $\mathbb{R}$-subgroup P and an open subset $U \subseteq \widehat{e(P)}$, let $\mathcal{O}(U) = q_P^{-1}(V)$, the total space of the restriction to $V = U \cap e(P)$ of the trivial bundle q_P over $e(P)$ with fiber $\hat{A}_P$. If U is any open subset of δX, let

$$\mathcal{O}(U) = \bigcup_{B \in \mathcal{P}_{\mathbb{R}}} \mathcal{O}\big(U \cap e(P)\big).$$

Notation . Let $y \in \partial e(B)$ for some $B \in \mathcal{B}_{\mathbb{R}}$ and let P, $Q \in \mathcal{P}_{\mathbb{R}}$ such that $B \subseteq P$, Q. Then for any open neighborhood Ω of y in δX, $\Omega \cap \varepsilon(B)$ contains an open neighborhood U of y in $\varepsilon(B)$ such that $q_P^{-1}(V) \cup q_Q^{-1}(V) \subseteq \Omega$, where $V = U \cap e(B)$, and $q_P \colon e(P) \to e(B)$ and $q_Q \colon e(Q) \to e(B)$ are the associated bundles with fibers $\hat{A}(P, B)$ and $\hat{A}(Q, B)$ respectively. The point is that

$$\mathcal{O}\big(V \cup q_P^{-1}(V) \cup q_Q^{-1}(V)\big) = q_B^{-1}(V).$$

It is convenient to denote this set also by $\mathcal{O}(U)$ even though $U \subseteq \varepsilon(B)$ is not open in δX.

Let U be again an open subset of δX. Define

$$\begin{aligned} \mathcal{C}(U) = &\{z \in \bar{X}_{\mathbb{R}} : \exists\ \mathcal{O} \in \mathcal{N}(z) \text{ such that } \mathcal{O} \cap X \subseteq \mathcal{O}(U)\} \cup \\ &\qquad \{z \in \delta X \backslash \bar{X}_{\mathbb{R}} : \exists \text{ open } U' \subseteq \delta X \text{ such that } z \in U' \text{ and } \mathcal{O}(U') \subseteq \mathcal{O}(U)\}. \end{aligned}$$

Now for $y \in \delta X \backslash \bar{X}_{\mathbb{R}}$ let

$$\mathcal{N}(y) = \{\mathcal{O} \subseteq \hat{X} : \exists \text{ open set } U \subseteq \delta X \text{ containing } y \text{ with } \mathcal{C}(U) \subseteq \mathcal{O}\}.$$

This defines a system of neighborhoods $\mathcal{N}(y)$ for any $y \in \hat{X}$. For a subset $\mathcal{S} \subseteq \hat{X}$ let $\mathcal{N}(\mathcal{S}) = \{\mathcal{O} \subseteq \hat{X} : \mathcal{O} \in \mathcal{N}(y) \text{ for every } y \in \mathcal{S}\}$ and call $\mathcal{S}$ *primary open* if $\mathcal{S} \in \mathcal{N}(\mathcal{S})$.

Proposition D.1.5. *The primary open subsets of $\hat{X}$ form a well-defined topology.*

We omit the proof again—cf. Theorem 9.4.5.

Definition D.1.6. The set $\hat{X}$ with the *primary* topology just defined will be denoted by $\hat{X}_1$. It follows from Remark 11.1.1 that the primary topology on $\hat{X}_1$ is not Hausdorff.

Now let X^{S} be either of the two Satake compactifications, say, the maximal one. If one uses a minimal compactification then the construction of topology must become genuinely inductive presupposing the correct topology in each $e(P_i)\widehat{}$ as in Appendix C. For our choice of $X^{\mathrm{S}} = X^{\mathrm{S}}_{\Delta}$ the following definition will be global and quick.

There is the obvious set projection

$$p \colon \hat{X} \longrightarrow X^{\mathrm{S}}$$

extending Φ from Theorem 9.3.1. This map should be identical on X and contract each conjugate of $\varepsilon(P_0)$. The complement consists of the conjugates of $\varepsilon(P_1)$ and $\varepsilon(P_2)$ which are projected onto the corresponding conjugates of the Satake strata $f(P_1) = \hat{e}(P_1)$ and $f(P_2) = \hat{e}(P_2)$ fiberwise.

Definition D.1.7. The *secondary* topology on $\hat{X}$ is the p-pull-back of the topology on X^{S}. Let $\hat{X}_2$ be the resulting topological space. Again, $\hat{X}_2$ is non-Hausdorff.

By the product of two topologies on a set we mean the one generated by the union of bases for each topology.

Definition D.1.8. Let $\hat{X}$ be the space topologized by the product of the primary and secondary topologies on the set $\hat{X}$.

D.2. **Properties of the Compactification.**

D.2.1. *Hausdorff Property.* For x_1, $x_2 \in \hat{X}$, if $p(x_1) = p(x_2) \in X^{\mathrm{S}}$ then either x_1, $x_2 \in p^{-1}(y)$ for some $y \in X^{\mathrm{S}} - X$ or $x_1 = x_2 \in X$. Now each $p^{-1}(y)$ is Hausdorff, so the points are separated in the primary topology. If $p(x_1) \neq p(x_2) \in X^{\mathrm{S}}$ then the points are separated in the secondary topology.

D.2.2. *Calculus of Flats.* In order to determine the geometry of open sets in $\hat{X}$, we need to study the geometric question: *describe the family of flats asymptotic to the given two chambers or walls at infinity of a symmetric space X.* The answer is quite natural in terms of horocycles.

Theorem D.2.1 (Im Hof [49])**.** *If y, $z \in \partial X$ are contained in Weyl chambers $W(y)$, $W(z) \subseteq \partial X$, let N_y, N_z be the nilpotent components in the corresponding Iwasawa decompositions. For an arbitrary point $x \in X$ the intersection of the horocycles $N_y \cdot x \cap N_z \cdot x$ parametrizes the set of all flats asymptotic to both $W(y)$ and $W(z)$.*

When X is a Hermitian symmetric space, one can consider the enlargement $X \subseteq \bar{X}_{\mathbb{R}}$. Now the minimal strata $e(B)$, $B \in \mathcal{B}_{\mathbb{R}}(\operatorname{Isom} X)$, parametrize the flats which are asymptotic to $W(B)$.

Definition D.2.2. Define the subsets $\mathcal{A}(B,B') \subseteq e(B)$ to be the geodesic projections $q_B(N_B \cdot x \cap N_{B'} \cdot x)$—they consist of $a \in e(B)$ such that the flat $q_B^{-1}(a)$ is asymptotic to $W(B')$.

This parametrization is convenient for us because each $\xi \in e(B)$ is precisely the point of intersection $e(B) \cap q_B^{-1}(\xi) = \{\xi\}$. Now given an open subset $U \subseteq e(B)$, the corresponding open set $\mathcal{C}(U) \subseteq \hat{X}$ can be described as $\complement\big(\operatorname{cl} q_B^{-1}(\complement U)\big)$, and $q_B^{-1}(\complement U)$ can be identified once the closure of each flat $q_B^{-1}(\xi)$, $\xi \notin U$, is known. This is easy to do for $X = X(SL_3)$.

Let $W(B)$, $W(B')$ be two adjacent Weyl chambers at infinity in $\partial X(SL_3)$, i.e., there exists $P \in \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}$ with B, $B' \subseteq P$. Using the product structure on $\bar{X}_{\mathbb{R}}(P)$ and the fact that $\gamma_\xi = e(P) \cap \operatorname{cl}\big(q_B^{-1}(\xi)\big)$ are geodesics for each $\xi \in e(B)$, we see that the collection of flats which are asymptotic to both $W(B)$ and $W(B')$ is parametrized by $\mathcal{A}(B,B') = \bar{\mu}_{P,\mathbb{R}}^{-1}(y) \subseteq e(B)$, where $y \in e(B^P)$ is the endpoint of the well-defined hyperbolic geodesic $p(\gamma_\xi) \subseteq f(P)$ connecting $f(B)$ with $f(B')$. In other words, $\mathcal{A}(B,B')$ is precisely the set of such $\xi \in e(B)$.

Now let $W(B)$, $W(B')$ be two Weyl chambers at infinity which are neither adjacent nor in opposition (that is, do not contain ideal points represented by the opposite orientations of the same geodesic). This determines unique chamber $W(B')$ and walls $W(P)$, $W(P')$ with $B' = P \cap P'$, $B < P$, $B'' < P'$. So each flat asymptotic to $W(B)$ and $W(B'')$ will be also asymptotic to $W(B')$, and $\mathcal{A}(B,B'') \subseteq \mathcal{A}(B,B')$. Indeed, $\mathcal{A}(B,B')$ are points in the plane $\bar{\mu}_{P,\mathbb{R}}^{-1}(y)$. Let z be the other end of the geodesic $p(\gamma_\xi)$. There is a bijective correspondence

$$\mathcal{A}(B',B) = \bar{\mu}_{P,\mathbb{R}}^{-1}(z) \leftrightsquigarrow \mathcal{A}(B,B').$$

On the other hand, $\{z\} = f(P) \cap f(P')$, and

$$\mathcal{A}(B',B'') = \bar{\mu}_{P',\mathbb{R}}^{-1}(z) \leftrightsquigarrow \mathcal{A}(B'',B').$$

Now $\bar{\mu}_{P,\mathbb{R}}^{-1}(z)$ and $\bar{\mu}_{P',\mathbb{R}}^{-1}(z)$ are two transverse planes in $e(B')$ intersecting in a line L. It is clear that there are bijections

$$\mathcal{A}(B,B'') \leftrightsquigarrow L \leftrightsquigarrow \mathcal{A}(B'',B).$$

In terms of coordinates (x,y,z) in $e(B)$ induced from $e(P_0)$, $\mathcal{A}(B,B'')$ is a line $(x,y,*)$ with one of the coordinates x or y determined by the choice of $W(B')$ adjacent to $W(B)$, the other—by the choice of $W(B'')$ adjacent to $W(B')$.

Similarly, on one hand, each point in $\mathcal{A}(B,B'')$ uniquely determines a chamber $W(B^{(3)})$ adjacent to $W(B'')$, on the other—it determines a unique flat which is, therefore, the unique flat asymptotic to both $W(B)$ and $W(B^{(3)})$. This verifies the result of [49] in a more descriptive way. We have in mind

Proposition D.2.3. *Given B, $B' \in \mathcal{B}_{\mathbb{R}}(SL_3)$, the flats which are asymptotic to both $W(B)$ and $W(B')$ are parametrized by*

$$\mathcal{A}(B,B') \overset{\sigma}{\leftrightsquigarrow} \mathcal{A}(B',B).$$

If $S \subseteq \mathcal{A}(B,B')$ then $\sigma(S) \subseteq \mathcal{A}(B',B)$ is contained in the closure $\operatorname{cl}\big(q_B^{-1}(S)\big)$.

In view of the discussion above, using the product structures in $\hat{X}(P)$, we can determine the geometry of $\mathcal{C}(U)$ for an open subset $U \subseteq \varepsilon(B)$.

Proposition D.2.4. *If B, $B' \in \mathcal{B}_{\mathbb{R}}(SL_3)$ and $U \subseteq \varepsilon(B)$ is an open subset then $y \in \varepsilon(B')$ is contained in $\mathcal{C}(U)$ if and only if either*

(1) *$y \in e(B')$ and its orthogonal projection π_B onto $\mathcal{A}(B', B)$ is not contained in the subset $A_B(U)$ corresponding bijectively to $U \cap \mathcal{A}(B, B')$, or*
(2) *$y \in \partial e(B')$ and $y \notin \mathrm{cl}\big(\pi_B^{-1} A_B(U)\big)$.*

The intersections of $\mathcal{C}(U)$ with $\varepsilon(P)$, $P \in \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}$, are the obvious open product subsets.

¿From this description easily follows

Proposition D.2.5 (Weak Summability)**.** *Given arbitrary open subsets U_1 and $U_2 \subseteq \varepsilon(B)$, it may not be true that*

$$\mathcal{C}(U_1 \cup U_2) = \mathcal{C}(U_1) \cup \mathcal{C}(U_2).$$

However, the open stars in any derived decomposition of $\varepsilon(B)$ from Definition D.1.2 do have this property.

Corollary D.2.6. *Given a finite collection of open subsets $\Omega_1, \ldots, \Omega_n \subseteq \hat{X}$ with $\varepsilon(B) \subseteq \bigcup \Omega_i$ there is another finite collection of open subsets $U_1, \ldots, U_m \subseteq \varepsilon(B)$ so that*

- $\varepsilon(B) \subseteq \bigcup U_j$,
- $\forall\ 1 \le j \le m\ \exists\ 1 \le i \le n$ with $\mathcal{C}(U_j) \subseteq \Omega_i$,
- $\mathcal{C}(\bigcup U_j) = \bigcup \mathcal{C}(U_j)$.

D.2.3. *Compactness.* It can be shown that $\hat{X}_1$ is compact as in §10 and §11 where summability was used implicitly—it holds obviously in the $\mathbb{R}$-rank one case. Unfortunately compactness of $\hat{X}_1$ and $\hat{X}_2$ alone does not imply compactness of $\hat{X}$. This will follow from

Lemma D.2.7. *For each $y \in X^{\mathrm{S}}_{\Delta} - X$ and any open neighborhood U of $\pi^{-1}(y)$ in $\hat{X}$ there exists an open neighborhood V of y such that $\pi^{-1}(y) \subseteq U$.*

Proof. The topology in X^{S}_{Δ} can be described by making a sequence *convergent* if and only if it converges to a maximal flat and its projection onto the flat converges in Taylor's polyhedral compactification (see [46, 82]).

Suppose $y = f(B)$ for some $B \in \mathcal{B}_{\mathbb{R}}$. Then U is a neighborhood of $\varepsilon(B) \subseteq \hat{X}$. We know from Lemmas 10.1.1, 10.1.2 and Corollary D.2.6 that there is a neighborhood M of $\partial e(B) \subseteq \varepsilon(B)$ and a section $\sigma \colon e(B) \to X$ of q_B so that

$$N(B) := \mathcal{C}(M) \cup W(e(B), \sigma, 0) \subseteq U,$$

and the closure of the complement of $N(B)$ does not intersect $\varepsilon(B)$. In particular, this means that for each flat F asymptotic to $W(B)$,

$$W(B) \notin \overline{F \cap N(B)}.$$

If λ_i is a sequence in $\pi\big(\overline{\complement N(B)}\big) \subseteq X^{\mathrm{S}}_{\Delta}$ converging to $f(B)$ then there is a sequence ϕ_i with the same limit contained in a flat asymptotic to $f(B)$. The preceding discussion shows that ϕ_i would "lift" to a sequence (ϕ_i itself!) converging to $W(B)$ which is impossible. So the closed set

$$\pi\big(\overline{\complement N(B)}\big) \cap f(B) = \emptyset.$$

Now any neighborhood of $f(B)$ in $\mathsf{C}\pi\big(\overline{\mathsf{C}N(B)}\big)$ will do as V.

A simpler argument works for $y \in \operatorname{int} f(P_i)$, $i = 1$ or 2. $\square$

Let $\mathcal{U}$ be an arbitrary open covering of $\hat{X}$. Since $\pi^{-1}(y)$ is compact for each $y \in X^{\mathrm{S}}_{\Delta}$, let $U_{y,1}, \ldots, U_{y,n_y}$ be a finite collection of elements of $\mathcal{U}$ with

$$\pi^{-1}(y) \subseteq \bigcup_{i=1}^{n_y} U_{y,i}.$$

By Lemma D.2.7 there is V_y such that

$$\pi^{-1}(V_y) \subseteq \bigcup_{i=1}^{n_y} U_{y,i}.$$

By compactness of X^{S}_{Δ} there is a finite collection of points $y_1, \ldots, y_k$ with $X^{\mathrm{S}}_{\Delta} = V_{y_1} \cup \cdots \cup V_{y_k}$. Then

$$\hat{X} = \bigcup_{i=1}^{n_{y_1}} U_{y_1,i} \cup \cdots \cup \bigcup_{i=1}^{n_{y_k}} U_{y_k,i}.$$

D.2.4. *Čech-acyclicity.* Recall the weak Vietoris–Begle theorem for the modified Čech theory (Theorem 11.3.1). The fibers need only be Chogoshvili-acyclic for the result of Inassaridze used in that proof, so we have

Theorem D.2.8. *If $f \colon X \to Y$ is a surjective continuous map, where Y and $f^{-1}(y)$ are Chogoshvili-acyclic for each $y \in Y$ and for any abelian coefficient group, then*

$$\check{f} \colon \check{h}(X; KR) \longrightarrow \check{h}(Y; KR)$$

is a weak homotopy equivalence. So both X and Y are Čech-acyclic.

Now the fibers of $p \colon \hat{X} \to X^{\mathrm{S}}$ are either points, disks, or closures of maximal Borel–Serre strata which are all Chogoshvili-acyclic by the theorem of Inassaridze and induction. The Satake compactification X^{S} is Chogoshvili-acyclic by Theorem 11.5.1. So Theorem D.2.8 applies to p, and $\hat{X}$ is Čech-acyclic.

D.3. **Proof of the Splitting.** The proof is best viewed as the induction on the rank of the semi-simple factor of the generalized symmetric space. We will induct from our knowledge of compactifications of the (generalized) symmetric spaces associated to the proper parabolic subgroups P_0 (rank 0) and P_1 (rank 1) of SL_3 (rank 2).

D.3.1. *Partial Cellular Decomposition.* The cells in ∂N_0 described in Definition D.1.2 form a subdivision of the cellular structure of ∂N_0 from Appendix C: there are twice as many 1-cells and 2-cells so that the picture is made symmetric in the standard corner where $N_0 \cong e(P_0)$ is the base of two geodesic bundles $e(P_1)$ and $e(P_2)$. Just as in Definition C.2.2 there is a product structure in each $\varepsilon(P)$, $P \in \mathcal{P}_{\mathbb{Q}} \backslash \mathcal{B}_{\mathbb{Q}}$, so that the closure of each lift of $\hat{e}(P)$ in $e(P)\hat{}$ determines a point in the appropriate meridian $\sigma_{j,B}$ (depending on the type j of P) for each $B < P$. In $e(P_j)\hat{}$ the lift to $\xi_i \times \hat{e}(P_j)$ determines a vertex in σ_{j,P_0}. The complements in each of the two meridians are the 1-cells, and the complements of the circles in ∂N_0 are the 2-cells. Similar decompositions are well-defined in other boundaries of Borel strata: for each $B \in \mathcal{B}_{\mathbb{R}}$ with $B < P'$, P'', $e(B) \cong R_u B(\mathbb{R})$ where $R_u P'(\mathbb{R})$ and $R_u P''(\mathbb{R})$

are subgroups isometrically embedded in the transported flat metric. Their intersection is a geodesic converging to the two 0-cells and $\partial R_u P'(\mathbb{R}) \cup \partial \mathbb{R}_{\approx}\mathbb{P}''(\mathbb{R})$ is the 1-skeleton in $\partial R_u B(\mathbb{R})$. The 2-cells are the four connected components of the complement. Now the derived cells structures in $\varepsilon(B)$ may be introduced so that the induced deriveds in $\partial R_u P(\mathbb{R})$ are compatible with the product structures in $\partial R_u P(\mathbb{R}) \times \varepsilon\big(\hat{(}\mathbb{P})\big)$. Consider derived cubical decompositions of the unit square I^2 and their images under homeomorphisms $\pi_P \colon I^2 \to \varepsilon\big(\hat{e}(P)\big)$. The nerves of the open star coverings of $\varepsilon\big(\hat{e}(P)\big)$ or $\hat{e}(P)$ are obviously contractible. We consider finite open coverings $\mathcal{O}_{m,k,P}$ of $\varepsilon(P)$ by the products of open stars in (unrelated) cubical decompositions of $\partial R_u P(\mathbb{R})$ and $\hat{e}(P)$ (cf. Appendix C).

Let $\xi_{i,B}$ and $\vartheta^{(t)}_{j,B}$ for $i,\, j,\, t \in \{1,2\}$ be the two 0-cells and four 1-cells in $\partial e(B)$ (of course, t is the type of the adjacent maximal parabolic subgroup). We would like to consider
$$\varsigma_{i,j,P} = \left(\xi_{i,B} \cup \vartheta^{(t)}_{i,B}\right) \times \big(\hat{e}(P)\big)^{\mathrm{S}}_{\mathbb{Q}},$$
where P is the maximal parabolic subgroup of type k containing B. It is clear that this definition is independent of the choice of $B < P$.

D.3.2. *Bounded Saturation.* The space $\hat{X}$ contains $\bar{X}$ as an open dense Γ-subset; in particular, Γ acts continuously on $\bar{X}$ as before. The metric that we are going to use in $\bar{X}$ is a transported Γ-invariant metric similar to the ones used before.

Now we can identify a Boolean algebra of boundedly saturated sets fine enough for our purposes. By Proposition 12.3.1 (quasi-isometry invariance), one can substitute the given Γ by $SL_3(\mathbb{Z})$ and use Grenier's fundamental domain. A union $\mathcal{T}$ of this domain and a finite number of its adjacent translates may be taken to intersect $e(P_0)$ in the domain $\mathcal{D}$ for the Heisenberg group action as in Example 6.2.4. If ω is chosen to contain the domain $\mathcal{D}$ then the Siegel set Σ from Theorem 12.2.5 can be chosen (taking $t \geq 2/\sqrt{3}$) to contain $\mathcal{T}$. In fact, the corresponding domain and set for the action of the torsion-free Γ is a union of appropriate translates of $\mathcal{T}$ and Σ respectively as in Remark 12.2.3. Now the domains are arranged so that, using "barriers" as in the proof of Proposition 12.4.1 and the analogue of Lemma 12.5.1, the following is clear.

Proposition D.3.1. *The following subsets of Y are boundedly saturated:*

- *each subset $\varsigma_{i,j,P}$ for $P \in \mathcal{P}_{\mathbb{Q}} \backslash \mathcal{B}_{\mathbb{Q}}$,*
- *each 2-cell in $\partial e(B)$ for $B \in \mathcal{B}_{\mathbb{Q}}$,*
- *each $\varepsilon(P)$ for $P \in \mathcal{P}_{\mathbb{R}} \backslash (\mathcal{P}_{\mathbb{Q}} \cup \mathcal{P}_{\mathbb{R}})$.*

Definition D.3.2. The boundedly saturated sets identified in Proposition D.3.1 generate a Boolean algebra of sets $\mathcal{BA}$.

There are obvious analogues of finite rigid coverings from Definitions C.2.2, C.2.4, C.2.5. Definition C.2.2 applies literally but gives covering sets with a slightly different geometry—they are open hemispheres union a vertex. Definition C.2.4 has analogues for each of the two meridians in $\varepsilon(B)$ producing categories $\{\alpha_{1,B}\}$ and $\{\alpha_{2,B}\}$. The analogues of $\{\pi\}$ make sense for all maximal parabolic subgroups of SL_3 but we are going to use only $\{\pi_P\}$, $P \in \mathcal{P}_{\mathbb{Q}} \setminus \mathcal{B}_{\mathbb{Q}}$.

D.3.3. *Orderly Coverings.* We will construct a family of finite open coverings cofinal in $\mathrm{Cov}^{\mathrm{s}}Y$. Given a finite rigid covering $\beta \in \mathcal{C}ov Y$, the underlying finite open covering is $\mathcal{U} = \operatorname{im}\beta \in \mathrm{Cov}^{\mathrm{s}}Y$.

Fix a Borel subgroup $B \in \mathcal{B}_{\mathbb{R}}(SL_3)$. There is a number $\ell_B \in \mathbb{N}$ with $\ell_B \geq n_B$ and an open neighborhood $U_B \ni f(B)$ in X^{S} with

$$\mathrm{PreInf}_{\ell_B,U_B}(v) \overset{\text{def}}{=} Y \cap \mathcal{C}\big(\mathrm{Star}^{\mathrm{o}}_{\ell_B}(v)\big) \cap p^{-1}U_B \subseteq \beta(x)$$

for each $v \in V_{(\ell_B)}$ and some $x \in Y$. Let $\mathfrak{I}$ be the set consisting of all $P \in \mathcal{P}_{\mathbb{R}} \setminus \mathcal{B}_{\mathbb{R}}$ such that $f(P) \cap \complement U_B \neq \emptyset$. Let $\mathfrak{F}$ consist of all $B' \in \mathcal{B}_{\mathbb{R}}$ such that

$$\mathcal{A}(B,P) \cap \mathrm{Star}^{\mathrm{o}}_{\ell_B}(v) = \emptyset \quad \text{and} \quad \mathcal{A}(B,P) \cap \overline{\mathrm{Star}^{\mathrm{o}}_{\ell_B}(v)} \neq \emptyset.$$

Now we can define $V_B(U) \subseteq U_B$ such that

$$U_B \backslash V_B = U_B \cap \bigcup_{B \not< P \in \mathfrak{I}} \overline{f(P)}$$

and

$$\mathrm{Inf}_{\ell_B,U_B}(v) \overset{\text{def}}{=} \mathrm{PreInf}_{\ell_B,U_B}(v) \cap p^{-1}(V_B) \backslash \bigcup_{B' \in \mathfrak{F}} \varepsilon(B').$$

The union of these sets over all $v \in V_{(\ell_B)}$ is an open neighborhood of $\varepsilon(B)$ in Y by the weak summability property.

Using compactness of $\hat{X}$ and each $e(P)\hat{\ }$, $P \in \mathcal{P}_{\mathbb{R}}(SL_3)$, and relative compactness of $\varepsilon(P)$, one can choose finite subsets $\mathfrak{B} \subseteq \mathcal{B}_{\mathbb{R}}$ and $\mathfrak{P} \subseteq \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}$ and numbers $0 < m_P$, $k_P \in \mathbb{N}$ for $P \in \mathfrak{P}$ with

(1) $\forall B \in \mathfrak{B}\ \exists \mathfrak{P} \in \mathfrak{P}$ such that $B < P$,
(2) $Y = \bigcup_{B \in \mathfrak{B}} \mathrm{Inf}_{\ell_B,U_B}(v) \cup \bigcup_{P \in \mathfrak{P}} \varepsilon(P)$,

and the properties that parallel those from Appendox C: fix $P \in \mathfrak{P}$ and use the notation $\mathfrak{B}(\mathfrak{P}) := \{\mathfrak{B} \in \mathfrak{B} : \mathfrak{B} < \mathfrak{P}\}$, then

(3) for some $0 < k_P \in \mathbb{N}$ and $w(P) \in \partial\hat{e}(P) \cap V_{(k)}$

$$Y \cap \delta\big(\hat{e}(P)\big) = \bigcup_{B \in \mathfrak{B}} \mathrm{Inf}_{\ell_B,U_B}(v) \cap p^{-1}\mathrm{Star}^{\mathrm{o}}_{k_B}(w) \cap \delta\big(\hat{e}(P)\big),$$

(4) $\mathcal{O}_{m,k,P}$ refines the restriction of $\mathcal{U}$ to $\varepsilon(P)$,
(5) $m_P \geq \max_{B \in \mathfrak{B}(\mathfrak{P})}(\ell_B)$, $k_P \geq \max_{B \in \mathfrak{B}(\mathfrak{P})}(k_B)$,
(6) each open star in the associated k_P-th cubical derived decomposition of $\varepsilon\big(\hat{e}(P)\big)$ contains at most one point from $\{W(B) : B \in \mathfrak{B}(\mathfrak{P})\}$,
(7) for each $w \in \partial\hat{e}(P) \cap V_{(k_P)}$ there exists $B \in \mathfrak{B}$ such that either $W(B) \in \mathrm{Star}^{\mathrm{o}}_{k_P}(w)$ or

$$p^{-1}\big(\mathrm{Star}^{\mathrm{o}}_{k_P}(w)\big) \subseteq \bigcup_{v \in V_{(\ell_B)}} \mathrm{Inf}_{\ell_B,U_B}(v).$$

Pick a Borel subgroup $B(w)$ as before in Appendix C. Define

$$\mathrm{Ord}_{\ell_B,U_B,k_P}(v;w) \overset{\text{def}}{=} \Big(\mathrm{Inf}_{\ell_B,U_B}(v) \backslash \widehat{e(P)}\Big) \cup p^{-1}\mathrm{Star}^{\mathrm{o}}_{k_P}(w)$$

and

$$\mathrm{ExcOrd}_{\ell_B,U_B,k_P}(v;w) \overset{\text{def}}{=} \mathrm{Ord}_{\ell_B,U_B,k_P}(v;w) \backslash \bigcup_{B < P'} \widehat{e(P')}.$$

Consider the category $\mathcal{E}\mathrm{xc}\mathcal{O}\mathrm{rd}^{\mathrm{s}}Y$ of finite open coverings by the sets $\mathcal{O}_{m,k,P}$ and $(\mathrm{Exc})\mathrm{Ord}_{\ell_B,U_B,k_P}(v;w)$ for all choices of β, $\mathfrak{B}$, $\mathfrak{P}$, etc. Generate $\mathrm{PreOrd}Y$ as in

Appendix C with obvious modifications. This procedure may look asymmetric as to the roles of maximal strata played in corners

$$\overline{X(B)} = \widehat{e(P')} \cup \widehat{e(P'')}$$

when P', $P'' \in \mathfrak{P}$ and $y \in \varepsilon(B)$: there is a choice of w and, hence, of paticular $P^{(j)}$ involved here. The asymmetry disappears with the next step when one generates the full $\times$-closed subcategory $\mathcal{O}rd\,Y$ of $\mathcal{C}ovY$.

This $\mathcal{O}rd\,Y$ is not cofinal in $\mathcal{C}ovY$ but satisfies the hypotheses on the category $\mathcal{C}$ in §4.4. Recall that the conclusion of that section was that the map

$$\jmath^* \colon \check{h}(Y; KR) \longrightarrow \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\operatorname{holim}} (N\_ \wedge KR)$$

induced by the inclusion $\jmath\colon \mathcal{O}rd\,Y \hookrightarrow \mathcal{C}ovY$ is a weak homotopy equivalence.

D.3.4. *Definition of* $\{\alpha\}$. As explained in §13.3, the boundedly saturated coverings we produce are analogues of outcomes of actual saturation with respect to some Boolean algebra of boundedly saturated sets. In fact, we do saturate after using the chosen coverings α_B and π_P, $B \in \mathfrak{B}$, $P \in \mathfrak{P}$, to alter the sets in $\mathcal{O}\mathrm{rd}^{\mathrm{s}}Y$. We present the construction of boundedly saturated coverings $\alpha(\omega, \alpha_B, \pi_P)$ on generators $\omega \in$ PreOrd$Y$.

Definition D.3.3. For $B \in \mathcal{B}_{\mathbb{R}}(SL_3)$ use the notation $\alpha_{1,B}$ or $\alpha_{2,B}$ for the finite rigid covering of $\sigma_{1,B}$ or $\sigma_{2,B}$ respectively given by $\alpha_{i,B}(y) = \alpha_B(y) \cap \sigma_{i,B}$ for each $y \in \sigma_{i,B}$. The same formula associates $\alpha_{i,B}(y) \subseteq \sigma_{i,B}$ to each $y \in \partial e(B)$. For $P > B$ of type i, define $\Pi_{B,P}\colon \delta e(B) \to \operatorname{im} \pi_P$ by

$$\Pi_{B,P}(y) = \begin{cases} \alpha_{i,B}(y) \times \big(\hat{e}(P)\big)^{\mathrm{S}}_{\mathbb{Q}} & \text{if } B \in \mathfrak{B}, \\ \alpha_{i,B'}(v) \times \big(\hat{e}(P)\big)^{\mathrm{S}}_{\mathbb{Q}} & \text{otherwise}, \end{cases}$$

where $B' \in \mathcal{B}_{\mathbb{R}}$ and the vertices v, w are from $\omega(y) = \mathrm{ExcOrd}_{\ell_{B'(w)}, U_{B'}, k_P}(v; w)$. Now define

$$\alpha^{\mathrm{int}}(y) = \begin{cases} \pi_P(y) & \text{if } y \in \varepsilon(P),\ P \in \mathfrak{P} \cap \mathcal{P}_{\mathbb{Q}} \\ \omega(y) \backslash \varepsilon(B) \cup \Pi_{B,P}(y) & \text{if } y \in \varepsilon(B),\ B \in \mathfrak{B}, \\ \omega(y) \cup \Pi_{B,P(j)}(y) & \text{if } y \in \sigma_{j,B},\ B \notin \mathfrak{B}, \\ \omega(y) & \text{otherwise.} \end{cases}$$

Now let $\alpha(\beta)$ be the finite rigid covering of Y by the envelopes of sets in $\alpha^{\mathrm{int}}(\beta)$ with respect to the Boolean algebra $\mathcal{BA}$ from Definition D.3.2.

The sequence of the two steps in this construction is motivated by the general discussion in §13. Using the ideas from §13.3, Appendix C and the geometry of $\operatorname{im}\omega$ from §D.2.2, we see that each step preserves the homotopy type of the nerve of ω and α^{int}. Now we can apply the general argument: there is a natural transformation $N\_ \to N\alpha^{\mathrm{int}}(\_) \to N\alpha(\_)$ composed of homotopy equivalences. So

$$\underset{\overleftarrow{\mathcal{O}rd\,Y}}{\operatorname{holim}} (N\_ \wedge KR) \xrightarrow{\ \simeq\ } \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\operatorname{holim}} (N\alpha^{\mathrm{int}}(\_) \wedge KR) \xrightarrow{\ \simeq\ } \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\operatorname{holim}} (N\alpha(\_) \wedge KR).$$

This process also defines a left cofinal functor $T\colon \mathcal{O}rd\,Y \to \{\alpha\}$ so that

$$\underset{\overleftarrow{\mathcal{O}rd\,Y}}{\operatorname{holim}} (N\alpha(\_) \wedge KR) \xleftarrow{\ \simeq\ } \underset{\overleftarrow{\{\alpha\}}}{\operatorname{holim}} (N\_ \wedge KR),$$

and we can compose the required weak equivalence $\check{h}(Y; KR) \simeq \operatorname{holim}(N\alpha \wedge KR)$.

Index of Notation

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