THE LARGE SCALE TOPOLOGY AND ALGEBRAIC K-THEORY OF ARITHMETIC GROUPS

BORIS GOLDFARB

To Jed Keesling on his 60th birthday

ABSTRACT. New compactifications of symmetric spaces of noncompact type X are constructed using the asymptotic geometry of the Borel–Serre enlargement. The controlled K-theory associated to these compactifications is used to prove the integral Novikov conjecture for arithmetic groups.

1. STATEMENT OF THE RESULTS

There is a long history of technique called *compactification* or *attaching a boundary* in the study of noncompact symmetric spaces and domains. We will use this term to describe an embedding of the symmetric space as an open subset in a compact Hausdorff space. The boundary points usually carry asymptotic information about the symmetric space which is useful in harmonic analysis and the study of random walks on symmetric spaces. Sometimes these procedures are directly related to compactifications of arithmetic quotients of symmetric spaces. These quotients are moduli spaces of interesting objects, and the boundary points represent the degenerate versions of these objects.

A class of constructions which serve both ends is called Satake compactifications. Each Satake compactification X^{S} is a union of certain strata attached to the symmetric space X, each stratum corresponding to a parabolic subgroup of the connected isometry group of X. Now assume that G is a semisimple linear algebraic group defined over $\mathbb{Q}$ and that X is the symmetric space of maximal compact subgroups of the real points $G(\mathbb{R})$. Attaching only the strata corresponding to $\mathbb{Q}$-parabolic subgroups gives an enlargement $X^{\mathrm{S}}_{\mathbb{Q}}$ of X (no longer compact) which is invariant under any arithmetic subgroup Γ of G. After a suitable change of topology in $X^{\mathrm{S}}_{\mathbb{Q}}$, the quotient $X^{\mathrm{S}}_{\mathbb{Q}}/\Gamma$ becomes a compactification of X/Γ.

Another less singular compactification of X/Γ was constructed by Borel and Serre following the same blueprint. There is an enlargement $X^{\mathrm{BS}}_{\mathbb{Q}}$ of X by certain strata corresponding to $\mathbb{Q}$-parabolic subgroups of G. The quotient $X^{\mathrm{BS}}_{\mathbb{Q}}/\Gamma$ is again compact but the strata are chosen so that the quotient becomes the classifying space $B\Gamma$ for a torsion-free arithmetic group which makes this construction useful for group cohomology computations. There is also an analogue of the Satake compactification of X in this context. Attaching Borel–Serre strata corresponding to all $\mathbb{R}$-parabolic subgroups one gets an enlargement $X^{\mathrm{BS}}_{\mathbb{R}}$ of X which is no longer compact. When the $\mathbb{R}$- and $\mathbb{Q}$-ranks of G coincide, the spaces $X^{\mathrm{BS}}_{\mathbb{R}}$ and $X^{\mathrm{BS}}_{\mathbb{Q}}$ fit together particularly well. We will be interested in this split rank situation which includes all classical groups G.

Following S. Zucker [31] we see that there is a continuous map f from $X^{\mathrm{BS}}_{\mathbb{R}}$ onto the compact Satake space X^{S} which restricts to a continuous map from $X^{\mathrm{BS}}_{\mathbb{Q}}$ onto $X^{\mathrm{S}}_{\mathbb{Q}}$ with either topology. We will construct a compactification X^* of X by attaching a boundary to each fiber of the map f and introducing a compact Hausdorff topology on the resulting set so that f extends to a continuous map $q\colon X^* \to X^{\mathrm{S}}$. Even though the topological space X^* is not metrizable, the

1991 *Mathematics Subject Classification.* Primary: 20F32, 22E40, 53C35; Secondary: 11F06, 18F25, 54D35.
Key words and phrases. Arithmetic group, K-theory, symmetric space, assembly map.
Supported in part by the National Science Foundation through grant DMS-9971319.

construction and interpretation of the boundary points are very geometric. We summarize the most important properties in the first theorem.

Theorem 1. *Given the symmetric space X associated with a split rank algebraic group G, there is an embedding of X in a space X^* such that*

1. *X^* is compact and Hausdorff,*
2. *X is an open dense subset of X^*; in fact, the Borel–Serre enlargement $X^{\mathrm{BS}}_{\mathbb{Q}}$ of X is an open dense subset of X^*,*
3. *X^* is acyclic in the appropriate Čech sense,*
4. *the isometries of X extend to continuous maps of X^*,*
5. *there are continuous equivariant maps from X^* to other compactifications of X such as those of Satake, Bailey–Borel, and Martin at the bottom of the spectrum.*

The compactification X^* with its properties is a major geometric component in splitting the integral assembly map in algebraic K-theory for arithmetic lattices, which is our other main result. We refer to the proceedings [13] for the background, motivation, and careful discussion of Novikov and related conjectures.

Theorem 2. *If Γ is a torsion-free arithmetic group in an algebraic group of split rank, and R is an arbitrary ring, the assembly map $\alpha\colon h(\Gamma, K(R)) \to K(R[\Gamma])$ from the homology of the group Γ with coefficients in the K-theory spectrum $K(R)$ to the K-theory of the group ring $R[\Gamma]$ is a split injection. Here $K(A)$ stands for the nonconnective K-theory spectrum of the ring A.*

The argument uses a refinement of the methods previously successful where geometry of the group possessed some manifestation of nonpositive curvature [7, 8, 15, 16].

We should point out that the topological Novikov conjecture on homotopy invariance of higher signatures has been known for torsion-free lattices in algebraic groups for some time, due to various authors. It is also known, in its integral K-theoretic form as here, for cocompact lattices, cf. [7]. On the other hand, the nonuniform lattices in higher ranks are not bicombable [11, 12] which excludes the possibility of applying techniques from CAT(0) geometry and its analogues to these groups.

A concrete class of arithmetic groups are *congruence subgroups* defined as the kernels of surjective maps $G(\mathbb{Z}) \to G(\mathbb{Z}_\ell)$ induced by reduction mod ℓ for various levels ℓ. The congruence subgroups of SL_n of all levels $\ell \neq 2$ are torsion-free, and every arithmetic subgroup contains a suitable congruence subgroup according to the solution of the congruence subgroup problem. This identifies a particular system of groups to which our theorem applies.

The paper is organized as follows. In section 2, we review necessary details from algebraic groups mainly to establish notation. In section 3, we describe the constructions of Satake and Borel–Serre and maps between them that we use in section 4 to construct the space X^*. Section 5 establishes topological properties of X^* and studies certain geometric properties of the boundary in X^*. Section 6 shows how to apply these properties to split the integral assembly maps.

2. Symmetric Spaces. Algebraic Groups. Arithmetic Subgroups

2.1. **Symmetric Spaces.** A *symmetric space of noncompact type* is a complete simply-connected Riemannian manifold of nonpositive sectional curvature such that for each point $x \in X$ the geodesic symmetry $s_x\colon X \to X$ given by $\exp_x(v) \mapsto \exp_x(-v)$ for all $v \in T_xX$ is an isometry of X, and X is not compact but contains no Euclidean space as a Riemannian factor.

The connected isometry group $G = I_0(X)$ is a semisimple Lie group with no compact factors and with trivial center. It is transitive on X, and $X \cong G/K$ where K is the maximal compact subgroup of G stabilizing a point $x \in X$. If G is a semisimple Lie group with finite center and no compact factors, and if K is a maximal compact subgroup, then the homogeneous space G/K is a symmetric space of noncompact type. A *k-flat* in X is a complete totally geodesic k-dimensional submanifold with zero sectional curvature. The rank of X is the maximal dimension of a k-flat in X.

Every nonpositively curved manifold may be compactified by attaching the *ideal boundary* ∂X and introducing the cone topology on $\varepsilon X = X \cup \partial X$. The points of ∂X are asymptotic classes of geodesic rays, so the isometric action of $G = I_0(X)$ on X extends to ∂X.

2.2. **Linear Algebraic Groups.** Given a linear algebraic group H defined over a subfield k of the complex numbers, we use the notation $H(k)$ for the k-points of H. The connected component of the identity is denoted by H^0. The Zariski topology is always understood in $H(k)$ when $k \neq \mathbb{R}$, and the classical Lie group topology when $k = \mathbb{R}$. If H is connected, put

$$^0H \stackrel{\text{def}}{=} \bigcap_{\chi \in X(H)} \ker(\chi^2),$$

where $X(H)$ is the group of rational characters. The group ^0H is normal in H and is defined over k. Let S be a maximal k-split torus of the radical RH. Then $H(\mathbb{R}) = A \ltimes {}^0H(\mathbb{R})$, a semi-direct product, where $A = S(\mathbb{R})^0$, and $^0H(\mathbb{R})$ contains every compact subgroup of $H(\mathbb{R})$, and also, if $k = \mathbb{Q}$, every arithmetic subgroup of H. If R_uH denotes the unipotent radical of H, then $\hat{L}_H = H/R_uH$ is the canonical reductive Levi quotient. It is also defined over k. Let $\hat{M}_H = {}^0\hat{L}_H$.

2.2.1. *Notation.* An object associated to the reductive Levi quotient $\hat{L}_H$ rather than the group H itself will usually indicate this by wearing a 'hat'.

The totality of all parabolic subgroups of H will be denoted by $\mathcal{P} = \mathcal{P}(H)$. If $k' \subseteq k$ is a subfield then $\mathcal{P}_{k'} = \mathcal{P}_{k'}(H)$ will denote all parabolic subgroups defined over k'. Similar notation $\mathcal{B} = \mathcal{B}(H)$ and $\mathcal{B}_{k'} = \mathcal{B}_{k'}(H)$ will be used for Borel subgroups. The projection $\pi_H \colon H \to \hat{L}_H$ induces a bijection $\mathcal{P}_k(H) \leftrightarrow \mathcal{P}_k(\hat{L}_H)$ preserving conjugacy classes over k, and likewise $\mathcal{P}_k(\hat{L}_H) \leftrightarrow \mathcal{P}_k(\hat{L}_H/\hat{C}_H)$, where $\hat{C}_H$ is the center of $\hat{L}_H$.

2.2.2. *Notation.* Let $\hat{T}_H$ be a maximal k-split torus of $\hat{L}_H/\hat{C}_H$. If $\hat{\Delta}_H$ is a system of simple roots with respect to $\hat{T}_H$, let $\hat{P}_\Theta$ (resp. P_Θ) denote the standard k-parabolic subgroup of $\hat{L}_H/\hat{C}_H$ (resp. of H) relative to $\hat{T}_H$ and $\hat{\Delta}_H$ corresponding to the choice of $\Theta \subseteq \Delta_H$.

This correspondence $\Theta \mapsto P_\Theta$ defines a lattice isomorphism between the power set of $\Delta_H \equiv \hat{\Delta}_H$ and the set of standard parabolic k-subgroups of H. Moreover, each $P \in \mathcal{P}_k(H)$ can be written as $^hP_\Theta := hP_\Theta h^{-1}$ for some $h \in H(k)$ and a uniquely determined $\Theta(P) \subseteq \hat{\Delta}_H$.

Now suppose H is a semisimple group with a set of simple $\mathbb{Q}$-roots Δ. They are the vertices in the connected Dynkin diagram. Let T be a nonempty subset of Δ. For $\Theta \subseteq \Delta$ let $\kappa_T(\Theta)$ be the union of all connected components of Θ that meet T.

2.2.3. **Definition.** Given any $P \in \mathcal{P}_\mathbb{Q}(H)$, it determines a subset $\Theta = \Theta(P) \subseteq \Delta$ such that $P = {}^gP_\Theta$. Let $Q = {}^gP_{\kappa_T(\Theta)}$ and call Q a *T-connected* parabolic subgroup associated to P.

Let P_0 be the standard minimal parabolic $\mathbb{Q}$-subgroup of G, let A be the maximal $\mathbb{Q}$-split torus of G contained in P_0, and K be the maximal compact subgroup in $G(\mathbb{R})$ whose Lie algebra is orthogonal (relative to the Killing form) to the Lie algebra of $A(\mathbb{R})$. Let

$$A_t = \{a \in A(\mathbb{R})^0 : \alpha(a) \le t, \forall \alpha \in \Delta\}.$$

Recall that $P_0 = Z_G(A) \cdot R_u(P_0)$. Furthermore, $Z_G(A) \approx A \cdot F$ where F is the largest connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $Z_G(A)$. From the Iwasawa decomposition, $G(\mathbb{R}) = K \cdot P(\mathbb{R})$. This yields the following decomposition:

$$G(\mathbb{R}) = K \cdot A(\mathbb{R})^0 \cdot F(\mathbb{R}) \cdot R_uP_0(\mathbb{R}).$$

Recall that a *Siegel set* in $G(\mathbb{R})$ is a set of the form

$$\Sigma_{t,\eta,\omega} = K \cdot A_t \cdot \eta \cdot \omega,$$

where η and ω are compact subsets of $F(\mathbb{R})$ and $R_uP_0(\mathbb{R})$ respectively.

2.2.4. **Theorem** (Borel). *There are a Siegel set $\Sigma = \Sigma_{t,\eta,\omega}$ and a finite set $C \subseteq G(\mathbb{Q})$ such that $\Omega = C \cdot \Sigma$ is a fundamental set for Γ.*

2.3. **Arithmetic Groups.** Let G be a linear algebraic group defined over $\mathbb{Q}$ and write $G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.

2.3.1. **Definition.** A subgroup Γ of $G(\mathbb{Q})$ is *arithmetic* if Γ and $G(\mathbb{Z})$ are commensurable, that is, if the subgroup $\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$. A discrete group Γ is arithmetic if it is isomorphic to an arithmetic subgroup of some group G.

Consider the real points $G(\mathbb{R})$ of G. It is a real Lie group, and $\Gamma \subseteq G(\mathbb{R})$ is a discrete subgroup. When G is semisimple, Γ acts freely and properly discontinuously on the symmetric space X associated to $G(\mathbb{R})$. The quotient manifold $M = X/\Gamma$ is not necessarily compact unless $\operatorname{rank} G = 0$ but always has finite invariant volume, that is, Γ is a nonuniform lattice in $G(\mathbb{R})$. According to Margulis, such lattices are always arithmetic if G is simple with finite center, and $\operatorname{rank}(G) \geq 2$. This is true for nonuniform lattices in SL_n for $n \geq 3$.

2.3.2. **Example.** The most prominent class of arithmetic groups are *congruence subgroups* defined as the kernels of surjective maps $G(\mathbb{Z}) \to G(\mathbb{Z}_\ell)$ induced by reduction mod ℓ for various levels ℓ. Every arithmetic group contains a normal torsion-free subgroup of finite index, but, according to Minkowski, the congruence subgroups of special linear groups SL_n of all levels $\ell \neq 2$ are themselves torsion-free. Same is true for other classical groups such as symplectic groups Sp_{2n}. In these cases, when $\operatorname{rank}(G) \geq 2$, every arithmetic subgroup contains a suitable congruence subgroup by the solution of the congruence subgroup problem.

3. Enlargements of the Symmetric Space

3.1. **Borel–Serre Enlargements.** Let G be a semisimple algebraic group defined over $\mathbb{Q}$ and Γ be an arithmetic subgroup. It is a lattice in the real Lie group of real points $G(\mathbb{R})$ and acts on the symmetric space of maximal compact subgroups $X = G(\mathbb{R})/K$ so that X is a model for $E\Gamma$ if Γ is torsion-free.

3.1.1. **Definition.** An *enlargement* of a topological space is an embedding in a superspace as an open subset. A *compactification* is an enlargement to a compact Hausdorff space.

Borel and Serre [3] form a contractible enlargement $\overline{X}_{\mathbb{Q}}$ of X which depends only on the $\mathbb{Q}$-structure of G so that the action of Γ on X extends to a proper action on $\overline{X}_{\mathbb{Q}}$. (This is the space called $X^{\mathrm{BS}}_{\mathbb{Q}}$ in Theorem 2.) It is a new model for $E\Gamma$ but with the free action that is cocompact. Our goal is to construct a compactification X^* of $\overline{X}_{\mathbb{Q}}$.

Using notation from section 2.2, let $P \in \mathcal{P}_k(G)$, and let $\hat{S}_P$ denote the maximal k-split torus of $\hat{C}_P$, and $\hat{A}_P = \hat{S}_P(\mathbb{R})^0$. The dimension of $\hat{A}_P$ is the parabolic k-rank of P. To each $x \in X$ is associated the Cartan involution θ_x of G that acts trivially on the corresponding maximal compact subgroup. There is a unique θ_x-stable lift $\tau_x \colon \hat{L}_P(\mathbb{R}) \to P(\mathbb{R})$ which gives θ_x-stable liftings $A_{P,x} = \tau_x(\hat{A}_P)$, $S_{P,x} = \tau_x(\hat{S}_P(\mathbb{R}))$, and $M_{P,x} = \tau_x(\hat{M}_P(\mathbb{R}))$ of the subgroups $\hat{A}_P$, $\hat{S}_P(\mathbb{R})$, and $\hat{M}_P(\mathbb{R})$.

3.1.2. **Definition.** The *geodesic action* of $\hat{A}_P$ on X is given by $a \circ x = a_x \cdot x$, where $a_x = \tau_x(a) \in A_{P,x}$ is the lifting of $a \in \hat{A}_P$.

Now X can be viewed as the total space of a principal $\hat{A}_P$-bundle under the geodesic action. The group $\hat{A}_P$ can be openly embedded in $\mathbb{R}^{\operatorname{card}(\hat{\Delta}-\Theta(P))}$ via

$$\hat{A}_P \longmapsto (\mathbb{R}^*_+)^{\operatorname{card}(\hat{\Delta}-\Theta(P))}.$$

Let $\bar{A}_P$ be the 'corner' consisting of $\hat{A}_P$ together with positive $\operatorname{card}(\hat{\Delta} - \Theta(P))$-tuples where the entry ∞ is allowed with the obvious topology making it diffeomorphic to $(0,\infty]^{\operatorname{card}(\hat{\Delta}-\Theta(P))}$. The group $\hat{A}_P$ acts on $\bar{A}_P$, and the *corner* $X(P)$ associated to P is the total space of the associated bundle $X \times_{\hat{A}_P} \bar{A}_P$ with fiber $\bar{A}_P$. Denote the common base $X/\hat{A}_P$. of these two bundles by $e(P)$. In particular, $e(G^0) = X$.

3.1.3**. Definition.** The *Borel–Serre enlargement*

$$\overline{X}_k = \bigsqcup_{P \in \mathcal{P}_k(G)} e(P)$$

has a natural structure of a manifold with corners in which each corner $X(P) = \bigsqcup_{Q \supseteq P} e(Q)$ is an open submanifold with corners. The action of $Q(k)$ on X extends to the enlargement $\overline{X}_k$. The faces $e(P)$, $P \in \mathcal{P}_k(G)$, are permuted under this action.

We will borrow a term from [32]. Let $q_P \colon X \to e(P)$ denote the bundle map. For any open subset $V \subseteq e(P)$ a cross-section σ of q_P over V determines a translation of V from the boundary of $\overline{X}_k$, $k = \mathbb{Q}$ or $\mathbb{R}$, into the interior X. For any $t \in \hat{A}_P$ put

$$\hat{A}_P(t) = \{a \in \hat{A}_P :\ a^\alpha > t^\alpha \text{ for all } \alpha \in \Delta_P\},$$

where Δ_P is the set of those simple roots with respect to a lifting of $\hat{T}_P$ that occur in $R_u P$ (transported back to $\hat{A}_P$). It is complementary to $\Theta(P)$ in $\hat{\Delta}$.

3.1.4**. Definition.** For any cross-section $\sigma(V)$, a set of the form $\hat{W}(V, \sigma, t) = \hat{A}_P(t) \circ \sigma(V)$ will be called an *open set defined by geodesic influx from V into X*. There is a natural isomorphism

$$\mu_\sigma \colon \hat{A}_P(t) \times V \xrightarrow{\cong} \hat{W}(V, \sigma, t)$$

which extends to a diffeomorphism

$$\bar{\mu}_\sigma \colon \bar{A}_P(t) \times V \xrightarrow{\cong} W(V, \sigma, t).$$

Now $W(V, \sigma, t)$ is a neighborhood of V in $\overline{X}_k$ (for $k = \mathbb{Q}$ or $\mathbb{R}$) with

$$\bar{\mu}_\sigma(\{(\infty, \dots, \infty)\} \times V) = V.$$

We will call it an *open neighborhood defined by geodesic influx from V into X*.

All of that done so far works for more general homogeneous H-spaces than symmetric spaces for semisimple H. Borel and Serre call them spaces of type $S{-}k$. For each $Q \in \mathcal{P}_k(G)$, $e(Q)$ is such a space. So

$$\overline{e(Q)}_k = \bigsqcup_{P \in \mathcal{P}_k(Q)} e(P) = \bigsqcup_{Q \supseteq P \in \mathcal{P}_k(G)} e(P)$$

can be formed; it is diffeomorphic to the closure $\overline{e(Q)}$ of $e(Q)$ in $\overline{X}_k$. In fact, whenever $P \subseteq Q$, $\hat{A}_Q$ is canonically a subgroup of $\hat{A}_P$ so that the geodesic actions are compatible. The group $\hat{A}_P$ acts geodesically on $e(Q)$ through $\hat{A}_P/\hat{A}_Q$ with quotient $e(P)$. The stratum $e(P) \subseteq \overline{e(Q)}$ is the set of limit points of this geodesic action.

Recall that the parabolic k-subgroups index the simplices $W(P)$ of the Tits k-building of G. The dimensions of the strata $e(P)$ and the incidence relations among their closures reflect the structure of the building as follows:

$$\dim e(P) + \dim W(P) = \dim X - 1,$$

$$e(P) \cap \overline{e(Q)} \neq \emptyset \iff e(P) \subseteq \overline{e(Q)} \iff W(Q) \subseteq W(P) \iff P \subseteq Q.$$

The minimal parabolic (Borel) k-subgroups correspond to the strata $e(P)$ of dimension $\dim X - \operatorname{rank}_k G$, and to the top simplices of the building.

3.1.5**. Remark.** When B is a Borel $\mathbb{R}$-subgroup of G, we have the Iwasawa decomposition $G(\mathbb{R}) = KA_BN_B(\mathbb{R})$. Then $X \approx A_BN_B(\mathbb{R})$, and the geodesic action of A_B on X coincides with multiplication. The quotient $e(B)$ can be viewed as the underlying space of the nilpotent group $N_B(\mathbb{R})$.

3.2. **Actions on Strata.** For $k = \mathbb{Q}$ or $\mathbb{R}$, let P be a parabolic k-subgroup of G. Recall the projection $\pi_P \colon P \to \hat{L}_P$ from §2.2. The real points of the reductive Levi quotient split as a direct product

$$\hat{L}_P(\mathbb{R}) = \hat{M}_P(\mathbb{R}) \times \hat{A}_P,$$

where $\hat{M}_P = {}^0\hat{L}_P$, and there is the Langlands decomposition

$$P(\mathbb{R}) = M_{P,x} A_{P,x} L_{P,x}.$$

Recall that K_x is the stabilizer of x in $G(\mathbb{R})$ acting on X. Then $K_{P,x} = K_x \cap P(\mathbb{R})$ is the stabilizer of x in $P(\mathbb{R})$. The Borel–Serre stratum $e(P) = P(\mathbb{R})/K_{P,x}A_{P,x}$ is a space of type S for P. Notice that it is acted upon from the left by $R_uP(\mathbb{R})$.

3.2.1**. Definition.** The quotient $\hat{e}(P)$ is called the *reductive Borel–Serre stratum*.

Denote the quotient map by $\mu_P \colon e(P) \to \hat{e}(P)$. Let $\hat{K}_P = \pi_P(K_{P,x})$, then $\hat{K}_P$ is a maximal compact subgroup of $\hat{M}_P(\mathbb{R})$ and is lifted to $K_{P,x}$ by τ_x. From the Langlands decomposition,

$$\hat{e}(P) = R_uP(\mathbb{R})\backslash P(\mathbb{R})/K_{P,x}A_{P,x} = \hat{L}_P(\mathbb{R})/\hat{K}_P\hat{A}_P \cong \hat{M}_P(\mathbb{R})/\hat{K}_P$$

is the space of type S associated to the reductive group $\hat{L}_P$: in general, it may not be connected, and it may have trivial $\mathbb{R}_+^*$ factors.

3.2.2**. Proposition.** *For each $P \in \mathcal{P}_{\mathbb{R}}(G)$, the principal $R_uP(\mathbb{R})$-fibration μ_P extends to a principal fibration*

$$\bar{\mu}_P \colon \overline{e(P)} \longrightarrow \overline{\hat{e}(P)}.$$

Proof. Let $Q \subseteq P$ be proper parabolic subgroups with the unipotent radicals $R_uQ \supseteq R_uP$, then Q determines a parabolic subgroup

$$Q^P = \pi_P(Q) = Q/R_uP \subseteq \hat{L}_P = P/R_uP$$

with the unipotent radical $R_uQ^P = R_uQ/R_uP$. Now A_{Q^P} is canonically identified with $A_{P,B}$, in the notation of Borel and Serre [3]. The geodesic actions of A_Q on $e(P)$ and $\hat{e}(P)$ commute with μ_P, so $X_P(Q)$ is a principal $R_uP(\mathbb{R})$-bundle over $X_{\hat{L}_P}(Q^P)$, and the projection $\tau_Q \colon X_P(Q) \to X_{\hat{L}_P}(Q^P)$ extends μ_P. These fibrations τ_* are compatible with the order in the lattice $\mathcal{P}(P)$ in the sense that for each pair $Q_1 \subseteq Q_2 \subseteq P$ the restriction of τ_{Q_1} to $e(Q_2)$ is the projection of a principal $R_uP(\mathbb{R})$-fibration with base $e(Q_2^P)$. So the principal fibrations τ_* are also compatible with the inclusions $X(Q_2) \hookrightarrow X(Q_1)$ and match up to give a principal fibration structure for $\overline{e(P)}$ over $\overline{\hat{e}(P)}$. $\triangledown$

3.2.3**. Proposition** (Zucker [30])**.** *There is a diffeomorphism*

$$F \colon R_uP(\mathbb{R}) \times \hat{e}(P) \longrightarrow e(P)$$

given by

$$F(u, z\hat{K}_P\hat{A}_P) = u \cdot \tau_x(z)K_{P,x}A_{P,x} \in e(P) = P(\mathbb{R})/K_{P,x}A_{P,x}.$$

Here, $z\hat{K}_P\hat{A}_P \in \hat{e}(P) = \hat{L}_P(\mathbb{R})/\hat{K}_P\hat{A}_P$. The map F depends on the choice of the basepoint x which determines the lift τ_x.

Lemma (7.8) of [17] gives a formula for the action of $P(\mathbb{R})$ on $e(P)$ in terms of the coordinates that F provides. Notice that $g \cdot \tau_x\mu_P(g^{-1}) \in \ker(\mu_P) = R_uP(\mathbb{R})$ for any $g \in P(\mathbb{R})$, so

$$g \cdot u \cdot \tau_x\mu_P(g^{-1}) = gug^{-1} \cdot g\tau_x\mu_P(g^{-1}) \in R_uP(\mathbb{R})$$

for all $g \in P(\mathbb{R})$, $u \in R_uP(\mathbb{R})$.

3.2.4**. Lemma.** *The action of $P(\mathbb{R})$ on $R_uP(\mathbb{R}) \times \hat{e}(P)$ is given by*

$$g \cdot (u, z\hat{K}_P\hat{A}_P) = (g \cdot u \cdot \tau_x\mu_P(g^{-1}), \mu_P(g) \cdot z\hat{K}_P\hat{A}_P).$$

This formula shows that $R_uP(\mathbb{R})$ acts only on the first factor by translation. It also follows that there are other equivariant enlargements where the strata are reductive Borel–Serre strata.

3.2.5**. Definition.** The *reductive Borel–Serre enlargement* $\overline{X}^{\rho}_k$ ($k = \mathbb{Q}$ or $\mathbb{R}$) of X is the topological space obtained from the corresponding Borel–Serre enlargement $\overline{X}_k$ by collapsing each nilpotent fiber of the projection $\mu_P\colon e(P) \to \hat{e}(P)$ to a point. These projections combine to give a quotient map $\mu\colon \overline{X}_k \to \overline{X}^{\rho}_k$. The quotient $\overline{X}^{\rho}/\Gamma$ is called the reductive Borel–Serre compactification.

3.3. **Comparison with Satake Compactifications.** Workers in different fields mean different objects when they speak of Satake compactifications. The earlier constructions [27] are compactifications of a globally symmetric space which were later compared to Martin and Furstenberg compactifications and have applications in analysis; the later ones [28] are compactifications of (locally symmetric) arithmetic quotients of symmetric spaces which are the quotients of certain rational portions of the first construction with a properly redefined topology. We are interested in the first construction and the techniques used to study the second. The references for this material are [17, 27, 28, 31].

Let G be as in §3.1 and $\tau\colon G(\mathbb{R}) \to \mathrm{SL}(V)$ be a finite-dimensional representation with finite kernel. For an admissible inner product on V, let v^* denote the adjoint of v. The admissibility of the inner product means that $\tau(g)\tau\big(\theta_K(g)\big)^* = I$. So the mapping $\tau_0(g) := \tau(g)\tau(g)^*$ descends to X. Each $\tau_0(g)$ is a self-adjoint endomorphism of V. Factoring out the action of the scalars, we get $\tau_0\colon X \to \mathrm{PS}(V)$ which is an equivariant embedding. Taking the closure of the image, one gets the *Satake compactification* X^{S}_{τ}. The G-action on X extends to X^{S}_{τ} and the boundary $X^{\mathrm{S}}_{\tau} - X$ decomposes into orbits of certain subgroups of G called *boundary components*. The subgroups are the parabolic subgroups which are τ-connected in the appropriate sense. They also correspond to T-connected subsets of Δ for some $T \subseteq \Delta$ as in Definition 2.2.3. We will use interchangeable notation X^{S}_{τ} and X^{S}_{T}. The spaces X^{S}_{T} are certainly compact and Hausdorff as the closures of bounded subspaces in $\mathrm{PS}(V)$.

3.3.1**. Example** (Minimal Satake Compactifications)**.** These correspond to subsets T consisting of a single root. The boundary components of minimal Satake enlargements ${}_kX^{\mathrm{S}}_i$ are in bijective correspondence with the maximal parabolic k-subgroups.

3.3.2**. Example** (Maximal Satake Compactification)**.** This is the compactification corresponding to $T = \Delta$. There is always a continuous projection from X^{S}_{Δ} onto any other Satake compactification X^{S}_{Θ} for $\Theta \subseteq \Delta$.

Interpreting Zucker [31], Satake compactifications can be viewed as targets of surjective maps from the Borel–Serre enlargements. He describes them as quotients of the rational reductive Borel–Serre enlargement as follows. For $\Theta \subseteq \Delta$ and $P = {}^{g}P_{\Theta} \in \mathcal{P}_k(G)$, let $Q = {}^{g}P_{\kappa_T(\Theta)}$. Then there is a projection

$$p_{T,P}\colon \hat{e}(P) = \hat{e}({}^{g}P_{\Theta}) \longrightarrow \hat{e}(Q).$$

The stratum $\hat{e}(Q)$ as a space of type S for $\hat{L}_Q$. It is the product of the symmetric space for $\hat{L}^{\mathrm{red}}_Q(\mathbb{R})^0$ and the real points of a factor of its center, the orbit of an anisotropic torus. The corresponding Satake boundary component $s(Q)$ is the non-Euclidean factor of $\hat{e}(Q)$. Let

$$q_Q\colon \hat{e}(Q) \longrightarrow s(Q)$$

be the coordinate projection. Now

$${}_kX^{\mathrm{S}}_T = \bigcup_{Q \in \mathcal{P}_T} s(Q),$$

where $\mathcal{P}_T$ is the set of all T-connected $\mathbb{Q}$-parabolic subgroups, and in general the union is not disjoint. Under the

Assumption: $\mathrm{rank}_{\mathbb{Q}}(G) = \mathrm{rank}_{\mathbb{R}}(G)$

which we make from now on unless state otherwise, the set Δ plays the role of simple $\mathbb{Q}$- and $\mathbb{R}$-roots, and the torus action above is trivial, so q_Q is an equivalence. If $T = \Delta$, the map $p_{T,P}$ is an equivalence, and we may identify the reductive Borel–Serre enlargement $\overline{X}^{\rho}_k$ with the maximal Satake enlargement ${}_k X^{\mathrm{S}}_{\Delta}$, cf. [30, §4.2].

3.3.3. **Theorem** (Zucker [31]). *The composition* $\Phi_P = q_Q \circ p_{T,P}$ *is the restriction of a map* $\Phi_T \colon \overline{X}^{\rho}_k \to {}_k X^{\mathrm{S}}_T$. *This map factors through other* $\Phi_\Theta \colon \overline{X}^{\rho}_k \to {}_k X^{\mathrm{S}}_\Theta$ *for* $\Theta \supseteq T$. *Composing* Φ_T *with* μ *from Definition* 3.2.5 *gives* $\Phi_T \colon \overline{X}_k \to {}_k X^{\mathrm{S}}_T$. *In particular, there is a continuous map* $\Phi = \Phi_\Delta \colon \overline{X}_{\mathbb{R}} \to {}_{\mathbb{R}} X^{\mathrm{S}}_\Delta$.

Proof. This theorem is essentially contained in [31, §§2–3]. Zucker is more interested in the restriction of Φ_T to $\overline{X}$ but his §2 is very general and §3 works over $\mathbb{R}$ under our assumption. ▽

4. Compactification of $\overline{X}_{\mathbb{Q}}$

4.1. **Malcev Spaces and their Compactification.** We start by constructing special compactifications of connected simply-connected nilpotent groups.

Let Γ be a torsion-free finitely generated nilpotent group. According to Malcev [26], it can be embedded as a uniform lattice in a connected simply-connected nilpotent Lie group N. By [26, Lemma 4] the subgroup Γ has generators $\{\gamma_1, \ldots, \gamma_r\}$, where $r = \dim N$, with the three properties:

1. (1) each $\gamma \in \Gamma$ can be written as $\gamma = \gamma_1^{n_1} \cdots \gamma_r^{n_r}$,
2. (2) each subset $\Gamma_i = \{\gamma_i^{n_i} \cdots \gamma_r^{n_r}\}$ is a normal subgroup of Γ, and
3. (3) the quotients Γ_i/Γ_{i+1} are infinite cyclic for all $1 \le i < r$.

Let $C_i = c_i(t)$ be the one-parameter subgroup of N with $c_i(1) = \gamma_i$, $1 \le i \le r$. It is easily seen that N satisfies analogues of the three properties of Γ:

1. (1) $N = C_1 \cdots C_r$, and the representation of $g \in N$ as $g = g_1 \cdots g_r$, $g_i \in C_i$, is unique,
2. (2) if $N_{r+1} = \{e\}$, $N_i = C_i \cdots C_r$, $1 \le i \le r$, then N_i are Lie subgroups of N, $\dim N_i = r - i + 1$, and $N_i \lhd N$ for $1 \le i < r$,
3. (3) $C_i \cong \mathbb{R}$ for all $1 \le i \le r$.

If $\mathfrak{n}$ is the Lie algebra of N then $e_1 = \log \gamma_1$, …, $e_r = \log \gamma_r$ becomes a basis in $\mathfrak{n}$ so that each set

$$\mathfrak{n}_i = \{\alpha_i e_i + \alpha_{i+1} e_{i+1} + \cdots + \alpha_r e_r\} \subseteq \mathfrak{n}$$

is an ideal. So $\{\gamma_i\}$ produce special *canonical Malcev coordinates of the first kind*. The correspondences

$$\log \colon g \mapsto \log g,$$

$$\sigma \colon \sum_{k=1}^{r} \alpha_k e_k \mapsto \sum_{k=1}^{r} \alpha_k (0, \ldots, \overset{k}{\widehat{1}}, \ldots, 0)$$

define diffeomorphisms between N, $\mathfrak{n}$ and $\mathbb{R}^r$ and induce flat metrics in N and $\mathfrak{n}$ from the standard Euclidean metric in $\mathbb{R}^r$.

Let $M_i = N/N_i = C_1 \cdots C_{i-1}$. Since $\mathfrak{n}_i$ is an ideal in $\mathfrak{n}$, for any $a \in N$ the Poisson bracket $[a, e_i]$ is in N_{i+1}. Denote the coordinates of $p, g \in N$ by ξ_i, η_i respectively, then the coordinates $\zeta_i(t)$ of $p \cdot g$ satisfy

$$(*) \qquad \zeta_i = \xi_i + \eta_i + q_i(\xi_1, \ldots, \xi_{i-1}, \eta_1, \ldots, \eta_{i-1}),$$

where q_i are polynomials determined by the Campbell–Hausdorff formula. This shows that if $p \in N_j$ then $\xi_1 = \cdots = \xi_{j-1} = 0$ and ζ_k, $k < j$, are independent of $\xi_j, \ldots, \xi_r$. We can conclude that $p \cdot g$ lies in the hyperplane $(\zeta_1, \ldots, \zeta_{j-1}, *, \ldots, *)$ parallel to N_j. So N acts from the right on the set of hyperplanes parallel to N_j. Similar arguments apply to the left multiplication action.

Consider the enlargement of N, as a set, by the ends of rays in M_{j+1} parallel to C_j for each $j = 1, \ldots, r-1$. In order to visualize and parametrize the resulting enlargement, it is helpful to embed N as $(-1,1)^r \subseteq \mathbb{R}^r$ in the most obvious fashion so that the order of the coordinates in $\mathbb{R}^r$ coincides with the order of the index of $C_i \subseteq N$, and the parallelism relation is preserved. We want to define a sequence of certain topological collapses. The collapses are performed in the boundary of the cube I^r and its successive quotients. The first collapse contracts

$$\{(x_1, \ldots, x_{r-1}, *) \in I^r : \exists 1 \leq i \leq r-1 \text{ with } x_i = \pm 1\} \longrightarrow \text{point}.$$

We give this point the projective coordinates $(x_1, \ldots, x_{r-1}, \flat)$. The set

$$\{(x_1, \ldots, x_{r-1}, \flat) : \exists 1 \leq i \leq r-1 \text{ with } x_i = \pm 1\}$$

is the boundary of I^{r-1}. Now we induct on the dimension of the cube. For example, the collapse at the m-th stage can be described a

$$\{(x_1, \ldots, x_{r-m}, *, \flat, \ldots, \flat) \in I^{r-m+1} : \exists 1 \leq i \leq r-m \text{ with } x_i = \pm 1\} \longrightarrow (x_1, \ldots, x_{r-m-1}, \flat, \ldots, \flat).$$

The process effectively stops after $r-1$ steps when the points $(\pm 1, \flat, \ldots, \flat)$ do not get identified. The result is a topological ball B^r with the CW-structure consisting of two cells of each dimension $0, 1, \ldots, r-1$ and one r-dimensional cell and a continuous composition of collapses $\rho \colon I^r \to B^r$. Each lower dimensional cell is the quotient of the appropriate face in ∂I^r: if the face F was defined by $x_i = \pm 1$ then $\dim \rho(F) = i$.

4.1.1. **Definition.** Let N^* be the enlargement of N by endpoints of rays in M_{j+1} parallel to C_j for all $1 \leq j \leq r-1$. The topology in N^* is induced via the identification with the quotient of the Euclidean cube I^r. The identification also defines a cellular structure on N^*.

4.1.2. **Proposition.** *The enlargement N^* is a compactification which is both left and right equivariant with respect to the left and right multiplication actions of N on itself. The orbits of the two actions in $\partial N = N^* - N$ coincide with the cells in the cellular decomposition of the boundary sphere.*

Proof. The fact that N^* is a compactification of N follows from the evident properties of the quotient of the cube I^r. Since the formulas $(*)$ are polynomial, the right multiplication action has a continuous extension to ∂N. Similar formulas for the left action are also polynomial. The cells are invariant because the actions preserve the parallelism relation among the relevant rays. ∇

4.2. **Construction 1.** Retopologizing the target of the continuous map $\Phi \colon \overline{X}_{\mathbb{R}} \to {}_{\mathbb{R}}X^S_\Delta$ of Zucker, we get a continuous map $\Phi \colon \overline{X}_{\mathbb{R}} \to X^S_\Delta$ onto the maximal Satake compactification of X. The fibers of this map are still the nilpotent radicals of the corresponding $\mathbb{R}$-parabolic subgroups. We know from Lemma 3.2.4 that the unipotent radicals act in the fibers by translation. The first step is to compactify each fiber equivariantly.

If $P_1 \subseteq P_2$ are arbitrary standard $\mathbb{R}$-parabolic subgroups, then $R_u P_1 \supseteq R_u P_2$. Let B_0 be the standard Borel subgroup. There is a choice of Malcev coordinates in B_0 which restricts to Malcev coordinates in all $R_u P_\Theta$, $\Theta \subseteq \Delta$. In particular, we have a fixed ordering of all chosen coordinates in $R_u P_\Theta$. For arbitrary $y \in X^S_\Delta$, if $y \in s({}^g P_\Theta)$ then $\Phi^{-1}(y) \subseteq e({}^g P_\Theta)$, so $\Phi^{-1}(y) \cong R_u P_\Theta$. We apply the construction from section 4.1 to each $\Phi^{-1}(y)$, $y \in X^S - X$, denote the result by $\Phi^{-1}(y)^*$, and put

$$\delta X = \bigcup_{y \in X^S \setminus X} \partial \Phi^{-1}(y).$$

4.2.1. **Definition.** Put $X^* = \overline{X}_{\mathbb{R}} \sqcup \delta X$.

There is the obvious set projection $q \colon X^* \to X^S_\Delta$ extending Φ from Theorem 3.3.3 collapsing $q^{-1}(y)^* \mapsto y$. The topology in X^* will be introduced using the following fact.

4.2.2. **Proposition** (Bourbaki [4]). *Let X be a set. If to each $x \in X$ there corresponds a set $\mathcal{N}(x)$ of subsets of X such that*

(1) *every subset of* X *containing one from* $\mathcal{N}(x)$ *itself belongs to* $\mathcal{N}(x)$,
(2) *a finite intersection of sets from* $\mathcal{N}(x)$ *belongs to* $\mathcal{N}(x)$,
(3) *the element* x *belongs to every set in* $\mathcal{N}(x)$,
(4) *for any* $N \in \mathcal{N}(x)$ *there is* $W \in \mathcal{N}(x)$ *such that* $N \in \mathcal{N}(y)$ *for every* $y \in W$,

then there is a unique topology on X *such that, for each* $x \in X$, $\mathcal{N}(x)$ *is the set of neighborhoods of* x*, that is, subsets which contain an open superset of* x*.*

The space $\overline{X}_{\mathbb{R}}$ has the topology in which each corner $X(P)$ is open. For $y \in \overline{X}_{\mathbb{R}}$ let $\mathcal{N}(y) = \{\mathcal{O} \subseteq X^* : \mathcal{O}$ contains an open neighborhood of y in $\overline{X}_{\mathbb{R}}\}$.

4.2.3. *Notation.* Set-theoretically, each $e(P)$, $P \in \mathcal{P}_{\mathbb{R}}$, is enlarged to $s(P) \times R_u P_{\Theta(P)}(\mathbb{R})^*$. We denote this set with the product topology by $\varepsilon(P)$. Given an open subset $U \subseteq \varepsilon(P)$, let $\mathcal{O}(U) = q_P^{-1}(V)$, the total space of the restriction to $V = U \cap e(P)$ of the trivial bundle q_P over $e(P)$ with fiber A_B. Then define $C(U) = \{z \in \overline{X}_{\mathbb{R}} :$ there is $\mathcal{O} \in \mathcal{N}(z)$ such that $\mathcal{O} \cap X \subseteq \mathcal{O}(U)\} \cup \{z \in \delta X :$ there is an open $U' \subseteq \varepsilon(P)$ with $\Phi(z) \in s(P)$ such that $z \in U'$ and $\mathcal{O}(U') \subseteq \mathcal{O}(U)\}$.

Now for $y \in \delta X$, let $\mathcal{N}(y) = \{\mathcal{O} \subseteq X^* :$ there is an open set $U \subseteq \varepsilon(P)$ with $\Phi(y) = s(P)$ containing y with $C(U) \subseteq \mathcal{O}\}$. This defines a system of neighborhoods $\mathcal{N}(y)$ for any $y \in X^*$.

4.2.4. **Definition.** For a subset $S \subseteq \hat{X}$ let $\mathcal{N}(S) = \{\mathcal{O} \subseteq X^* : \mathcal{O} \in \mathcal{N}(y)$ for every $y \in S\}$ and call S *primary open* if $S \in \mathcal{N}(S)$.

4.2.5. **Proposition.** *If* $P \in \mathcal{P}_{\mathbb{R}}$*, and* $U, U_1, U_2 \subseteq \varepsilon(P)$ *are open subsets, then*
(1) $C(U)$ *is open in* X^*,
(2) $C(U_1) \cap C(U_2) = C(U_1 \cap U_2)$.

Proof. These properties follow formally from the definition. ▽

4.2.6. **Theorem.** *The primary open subsets of* X^* *form a well-defined topology.*

Proof. We need to check that the four characteristic properties from Proposition 4.2.2 are satisfied by the system of neighborhoods $\mathcal{N}(x)$, $x \in X^*$. Parts (1) and (3) are clear from definitions. Part (2) follows from Proposition 4.2.5(2). Given any $N \in \mathcal{N}(x)$, $x \in s(P) \times \partial R_u P(\mathbb{R})$, there is $U \subseteq \varepsilon(P)$ such that $C(U) \subseteq N$. Take $W = C(U)$. By Proposition 4.2.5(1), $N \in \mathcal{N}(y)$ for any $y \in W$. Thus (4) is also satisfied. ▽

4.2.7. **Definition.** The set X^* with the *primary* topology will be denoted by X_1^*. It is easy to see that the primary topology on X_1^* is not Hausdorff.

The *secondary* topology on X^* is the q-pull-back of the topology on X_Δ^S. Let X_2^* be the resulting topological space. Again, X_2^* is non-Hausdorff.

Let X^* be the space topologized by the product of the primary and secondary topologies.

4.2.8. **Example.** Consider the algebraic group $G = SL_2$ and an arbitrary proper parabolic $\mathbb{R}$-subgroup P of G. It acts on X fixing a point $p(P)$ in $\partial\mathbb{E}$, that is, P permutes geodesics abutting to $p(P)$. The corners $X(P)$ are constructed by attaching a line at $p(P)$ parametrizing these geodesics. If P fixes a rational point then $X(P) \subseteq \overline{X}$. Complete each stratum to $e(P)^* = e(P) \cup \{-\infty, +\infty\}$. The resulting set is X^*. Every $X(P)$ is declared to be open, so typical open neighborhoods of $z \in e(P)$ in X^* are the open neighborhoods of z in $X(P)$. Given a line $e(P)$ and one of its endpoints y, a typical *primary* neighborhood of y consists of

- y itself and a ray in $e(P)$ converging to y,
- the set U in $\mathbb{E}$ swept out by the hyperbolic geodesics abutting to $p(P)$ representing the points of the ray in $e(P)$,
- points in all strata $e(B)^*$, $B \in \mathcal{P}_{\mathbb{R}}$, such that the hyperbolic geodesic connecting $p(B)$ to $p(P)$ is properly inside U,
- the ray in $e(R)$, where $p(R)$ is the opposite end of the geodesic representing the vertex of the ray in $e(P)$, represented by geodesics contained in U and its limit point in the boundary $\partial e(R)$.

In this example, as in any case with rank $G = 1$, taking product with the secondary topology does not affect the primary topology on X^*, cf. [15]. With this topology, the subspace $X \subseteq X^*$ has the hyperbolic metric topology, and $X^* - X$ is simply $S^1 \times I$ with an analogue of the lexicographic order topology. In terms of the usual description of the lexicographic ordering on the unit square $I \times I$, the analogue we refer to is the quotient topology on $S^1 \times I$ associated to the obvious identification $(0, y) \sim (1, y)$ for all $y \in I$. In particular, the boundary $X^* - X$ is compact but not separable and, therefore, not metrizable.

4.3. **Construction 2.** The construction in section 4.2 compactifies all strata $e(P)$ simultaneously. Often it is more convenient to use an inductive description of the same primary topology X_1^* as above. The induction is over the rank of the spaces of type S associated to $\mathbb{R}$-parabolic subgroups of G, or, in other terms, over the cardinality in the lattice of subsets Θ of simple roots Δ starting with $\Theta = \Delta$ and finishing with $\Theta = \emptyset$.

So we start by Γ_B-equivariantly compactifying each $e(B)$, $B \in \mathcal{B}_{\mathbb{R}}$. Again, by Remark 3.1.5 and Lemma 3.2.4, $e(B) \approx N_B = R_u B(\mathbb{R})$, and the action is precisely the left multiplication action of Γ_B as a subgroup of N_B. Each $e(B)$ can be compactified as in section 4.1. In order to make these compactifications compatible, we make our choice of Malcev coordinates in $e(B_0)$ for the standard Borel subgroup B_0 as in section 4.2 and take the resulting compactification $e(B_0)^*$. For any standard parabolic P_Θ, the conjugation action of $P_\Theta(\mathbb{R})$ permutes the Borel strata $e({}^g B)$ adjacent to $e(P_\Theta)$. Define the space

$$\Upsilon(\Delta, \Theta) \stackrel{\text{def}}{=} P_\Theta(\mathbb{R}) \underset{B_0(\mathbb{R})}{\times} e(B_0)^*.$$

Inductively, given a standard parabolic subgroup P_Θ, $\Theta \subseteq \Delta$, and compactifications $e(P_T)^*$ for $T \supseteq \Theta$, define

$$\Upsilon(T, \Theta) \stackrel{\text{def}}{=} P_\Theta(\mathbb{R}) \underset{P_T(\mathbb{R})}{\times} e(P_T)^* \quad \text{and} \quad \Upsilon(\Theta) \stackrel{\text{def}}{=} \bigcup_{T \subseteq \Theta} \Upsilon(T, \Theta).$$

Warning. The space $\Upsilon(\Theta)$ comes with the identification topology which we are going to use in the ensuing construction, but it will not be the subspace topology induced from the resulting topology on X^*.

4.3.1**. Definition.** Define the set $e(P_\Theta)^* = \varepsilon(P_\Theta) \sqcup \Upsilon(\Theta)$.

The space $\overline{e(P_\Theta)}_{\mathbb{R}}$ has the topology in which each corner $X(P_T)$ is open for all $T \supseteq \Theta$. The enlargement $\varepsilon(P_\Theta)$ has the product topology as in §4.2. For $y \in \overline{e(P_\Theta)}_{\mathbb{R}} \cup \varepsilon(P_\Theta)$ let $\mathcal{N}(y) = \{\mathcal{O} \subseteq e(P_\Theta)^* : \mathcal{O}$ contains an open neighborhood of y in $\overline{e(P_\Theta)}_{\mathbb{R}} \cup \varepsilon(P_\Theta)\}$. Given an open subset $U \subseteq \Upsilon(T)$ for $T \supseteq \Theta$, let $\mathcal{O}(U) = q_{\Theta,T}^{-1}(V)$, the total space of the restriction to $V = U \cap e(P_T)$ of the trivial bundle $q_{\Theta,T}$ over $e(P_T)$ with fiber $A_{\Theta,T}$. If U is any open subset of $\Upsilon(\Theta)$, let

$$\mathcal{O}(U) = \bigcup_{P' \in \mathcal{P}_{\mathbb{R}}} \mathcal{O}(U \cap \Upsilon(P'))$$

where $\Upsilon(P') = {}^g\Upsilon(T)$ for $P' = {}^g P_T \subseteq P_\Theta$. Then define $C(U) = \{z \in \overline{e(P_\Theta)}_{\mathbb{R}} \cup \varepsilon(P_\Theta) :$ there is $\mathcal{O} \in \mathcal{N}(z)$ such that $\mathcal{O} \cap e(P_\Theta) \subseteq \mathcal{O}(U)\} \cup \{z \in \Upsilon(P_\Theta) \backslash \overline{e(P_\Theta)}_{\mathbb{R}}$: there is an open $U' \subseteq \Upsilon(P_\Theta)$ such that $z \in U'$ and $\mathcal{O}(U') \subseteq \mathcal{O}(U)\}$.

Now for $y \in \Upsilon(P_\Theta) \backslash \overline{e(P_\Theta)}_{\mathbb{R}}$, let $\mathcal{N}(y) = \{\mathcal{O} \subseteq e(P_\Theta)^*$: there is an open set $U \subseteq \Upsilon(\Theta)$ containing y with $C(U) \subseteq \mathcal{O}\}$. This defines a system of neighborhoods $\mathcal{N}(y)$ for any $y \in e(P_\Theta)^*$. Again, the primary open subsets form a well-defined topology on $e(P_\Theta)^*$; for $\Theta = \emptyset$ one gets the primary topology on X^*. It is easy to see that this is the same topology as in X_1^* using the compatibility of geodesic actions in the Borel–Serre strata as in Proposition 3.2.2. So using this description in conjunction with the secondary topology X_2^* as in §4.2 gives the same space X^*.

4.3.2**. Remark.** One can also use any of the minimal Satake compactifications and the map $\Phi_T : \overline{X}_{\mathbb{R}} \to {}_{\mathbb{R}}X_T^S$ to induce the secondary topology. This eventually gives the same topology on

X^* if used in the inductive construction of this section, where during the inductive step all of the lower rank strata are assumed to already have the expected topology.

4.4. **Construction 3.** Recall from §2.1 that every irreducible symmetric space of noncompact type X associated to some semisimple group G can be isometrically embedded as a totally geodesic submanifold of $X(\mathrm{SL}_n)$ for $n = \dim(G)$. The algebraic group SL_n has split rank. The closure of this embedding in $X(\mathrm{SL}_n)^*$ is a compactification of X. This description is harder to handle than the explicit constructions above. Note, however, that there are no additional assumptions about G such as split rank.

5. Topological and Other Properties

5.1. **Hausdorff Property.** For $x_1, x_2 \in X^*$, if $q(x_1) = q(x_2) \in X^{\mathrm{S}}$ then either $x_1, x_2 \in q^{-1}(y)$ for some $y \in X^{\mathrm{S}} - X$ or $x_1 = x_2 \in X$. Now each $q^{-1}(y)$ is Hausdorff, so the points are separated in the primary topology. If $q(x_1) \neq q(x_2) \in X^{\mathrm{S}}$ then the points are separated in the secondary topology since X^{S} is Hausdorff.

5.2. **Compactness.** It can be shown that X_1^* is compact. Unfortunately compactness of X_1^* and X_2^* alone does not imply compactness of X^*. This follows from

5.2.1**. Lemma.** *For each $y \in X_\Delta^{\mathrm{S}}$ and any open neighborhood U of $q^{-1}(y)$ in X^* there exists an open neighborhood V of y such that $q^{-1}(y) \subseteq U$.*

Proof. The topology in X_Δ^{S} can be described by making a sequence convergent if and only if it converges to a maximal flat and its projection onto the flat converges in Taylor's polyhedral compactification [20, 29].

The claim is a tautology for $y \in X$. The question is easily reduced by induction on the rank or dimension to the case of $y = s(B)$ for some $B \in \mathcal{B}_{\mathbb{R}}(G)$. Here $q^{-1}(y) = R_u B(\mathbb{R}) = e(B)$. Given a neighborhood $U \supseteq e(B)^*$, choose an open neighborhood N of $\partial e(B) = e(B)^* \setminus e(B)$ in $e(B)^*$ and a section σ of q_B so that $U \supseteq C(N) \cup W(\sigma) \supseteq e(B)^*$. The geodesic influx set $W(\sigma)$ is an open neighborhood of $e(B)$ in $\overline{X}_{\mathbb{R}}$ and $W^{\mathrm{S}}(\sigma) = qW(\sigma)$ is an open neighborhood of y in ${}_{\mathbb{R}}X_\Delta^{\mathrm{S}}$. Consider the subset $R = q_B^{-1}(\complement N) \cap \complement W^{\mathrm{S}}(\sigma)$ of X_Δ^{S} and its closure $\mathrm{cl}(R)$ in X_Δ^{S}. Notice that $y \notin \mathrm{cl}(R)$ because y is a vertex in the polyhedral compactifications of the flats asymptotic to y which are precisely the fibers of q_B, and the corresponding corners are contained in $W(\sigma)$. Clearly, $q^{-1}(\mathrm{cl}\, R) \supseteq \mathrm{cl}(q_B^{-1}(\complement N) \cap \complement W(\sigma)) = \complement(C(N) \cup W(\sigma))$. Now $q^{-1}(\complement\, \mathrm{cl}\, R) = \complement q^{-1}(\mathrm{cl}\, R) \subseteq C(N) \cup W(\sigma) \subseteq U$. So we can take $V = \complement\, \mathrm{cl}(R)$. ▽

5.2.2**. Corollary.** *The space X^* is a compactification of both the symmetric space X and the rational Borel–Serre enlargement $\overline{X}$.*

Proof. Let $\mathcal{U}$ be an arbitrary open covering of $\hat{X}$. Since $q^{-1}(y)$ is compact for each $y \in X^S$, let $U_{y,1}, \dots, U_{y,n_y}$ be a finite collection of elements of $\mathcal{U}$ with

$$q^{-1}(y) \subseteq \bigcup_{i=1}^{n_y} U_{y,i}.$$

By Lemma 5.2.1 there is V_y such that

$$q^{-1}(V_y) \subseteq \bigcup_{i=1}^{n_y} U_{y,i}.$$

By compactness of X^S there is a finite collection of points $y_1, \dots, y_k$ with $X^S = V_{y_1} \cup \cdots \cup V_{y_k}$. Then

$$\hat{X} = \bigcup_{i=1}^{n_{y_1}} U_{y_1,i} \cup \cdots \cup \bigcup_{i=1}^{n_{y_k}} U_{y_k,i}.$$

▽

5.3. **Čech-acyclicity.** We will need to use homological triviality of our compactifications in §6. The homology theory involved here is a version of Čech homology.

5.3.1**. Definition** (Carlsson–Pedersen [8])**.** A finite rigid covering of a topological space Z is a set function β from Z to open subsets of Z which takes only finitely many values and satisfies (1) $x \in \beta x$ for all $x \in Z$ and (2) $\mathrm{cl}(\beta^{-1}U) \subseteq U$ for all $U \in \mathrm{im}(\beta)$. Set the nerve $N(\beta)$ to be the simplicial nerve of the infinite covering $\{\beta(x) : x \in Z\}$. The *modified Čech homology* of Z with coefficients in a spectrum S is the simplicial spectrum

$$\check{h}(Z; S) = \underset{\overleftarrow{\mathrm{Cov}\, Z}}{\mathrm{holim}}\, (N\_ \wedge S),$$

where Cov Z is the partially ordered category of finite rigid open coverings. This is a generalized Steenrod homology theory.

We will see that all Satake compactifications of X are acyclic. Since the continuous map $q\colon X^* \to X^{\mathrm{S}}_{\Delta}$ has contractible point inverses, it would be desirable to have an analogue of the Vietoris–Begle theorem for the modified Čech theory. We proved a weak Vietoris–Begle theorem in [15, Theorem 7.4.1]. Recall that the Chogoshvili homology theory is the unique extension of the Steenrod–Sitnikov homology to compact Hausdorff spaces from the category of metric compacta satisfying certain axioms of Berikashvili. The fibers need only be Chogoshvili-acyclic for the result of Inassaridze used in that proof, so we have

5.3.2**. Theorem.** *If $f\colon X \to Y$ is a surjective continuous map, where Y and $f^{-1}(y)$ are Chogoshvili-acyclic for each $y \in Y$, then $\check{f}\colon \check{h}(X; KR) \to \check{h}(Y; KR)$ is a weak homotopy equivalence. So both X and Y are Čech-acyclic.*

5.3.3**. Theorem.** *Each space X^{S}_{Θ} is Chogoshvili-acyclic.*

Proof. The metric space X^{S}_{Θ} needs to be Steenrod-acyclic. We use $H_*(\_)$ to denote the Steenrod–Sitnikov homology and apply the following version of the Vietoris–Begle theorem.

5.3.4**. Theorem** (Nguen Le Ahn [25])**.** *Let $f\colon X \to Y$ be a continuous surjective map of metrizable compacta so that $\widetilde{H}_i(f^{-1}(y); G) = 0$ for all $y \in Y$, $i \leq n$. Then the induced homomorphism $H_q(f)\colon H_q(X; G) \to H_q(Y; G)$ is an isomorphism for $0 \leq q \leq n$ and an epimorphism for $q = n + 1$.*

According to [20], the maximal Satake compactification X^{S}_{Δ} is homeomorphic to the Martin compactification $X^{\mathrm{M}}(\lambda_0)$ of X at the bottom of the positive spectrum λ_0. There is also the Karpelevič compactification X^{K} which is defined inductively in [23] and maps equivariantly onto $X^{\mathrm{M}}(\lambda_0)$. Theorem 5.3.4 applies to this map $f\colon X^{\mathrm{K}} \to X^{\mathrm{M}}(\lambda_0)$ because the fibers of f are easily seen to be genuinely contractible using the result of Kushner [24] that X^{K} is homeomorphic to a ball. The same result applied to X^{K} itself shows that all of the spaces in

$$D^n \cong X^{\mathrm{K}} \xrightarrow{f} X^{\mathrm{M}}(\lambda_0) \cong X^{\mathrm{S}}_{\Delta} \xrightarrow{\Phi} X^{\mathrm{S}}_{T}$$

are Steenrod- or Chogoshvili-acyclic. ▽

5.3.5**. Corollary.** *Compactifications X^* are Čech-acyclic.*

Note however that X^* is unlikely to be contractible.

5.4. **Equivariance.** If Γ is an arithmetic subgroup of $G(\mathbb{Q})$, it is immediate from the construction that this compactification is Γ-equivariant. In fact, the action of $G(\mathbb{R})$ on X extends to X^* which is in contrast to the fact that this action does not extend to $\overline{X}$.

5.4.1**. Remark.** The space X^* is certainly not a topological ball. This disproves a version of the conjecture of Lizhen Ji [22], p. 82, that an equivariant compactification of X such that the closure of each flat is a topological ball should be homeomorphic to the closed unit ball in the tangent space $T_x X$. The closures of all maximal flats in X^* are in fact contained in $\overline{X}_{\mathbb{R}}$ and are topological balls. The construction of X^* demonstrates that continuity of the extended action must be a necessary condition in the statement.

5.5. **Boundaries of Arithmetic Groups.** The notion of a boundary for a discrete group has roots in the theory of Fuchsian groups. Classically, the boundary circle is used to classify and study isometries of the hyperbolic disk. One attempt to incorporate existing generalizations in a formal definition for a discrete group Γ is due to M. Bestvina.

5.5.1**. Definition** (Bestvina [2])**.** A *boundary* of Γ is a topological space Y such that there is a space Z with the following properties:

(1) Z is compact, metrizable, finite-dimensional, contractible and locally contractible containing Y as a Z-set,
(2) $Z - Y$ has a free properly discontinuous action of Γ with compact quotient,
(3) for every open cover $\mathcal{U}$ of Z and every compact subset $K \subseteq Z - Y$ all but finitely many translates of K are $\mathcal{U}$-small.

In the literature on Novikov conjecture, such a compactification Z of $E\Gamma = Z - Y$ is called *good*; property (3) is usually expressed by saying that the action of Γ on $Z - Y$ is *small at infinity*.

This definition is motivated by useful geometric boundaries for torsion-free Gromov hyperbolic groups and CAT(0) groups. The latter class includes all uniform lattices in a semisimple Lie group in which case Y is the ideal boundary of the associated symmetric space X.

The construction of X^* provides a useful generalization of the notion of boundary, namely $Y = X^* - \overline{X}$, in this case of an arithmetic group Γ. The space X^* contains $\overline{X}$ as an open dense Γ-subset, in particular Γ acts continuously on $\overline{X}$ as before.

5.5.2**. Definition.** The metric that we use in $\overline{X}$ is a transported Γ-invariant metric. It can be obtained by first introducing a bounded metric in the compact space $\overline{X}/\Gamma$, then taking the metric in $\overline{X}$ to be the induced path metric where the measured path-lengths are the lengths of the images in $\overline{X}/\Gamma$ under the covering projection.

With this metric, the diameter of a fundamental domain Ω is bounded by some number D as is also the diameter of any Γ-translate of the domain. Beware that this metric is very different from the one Borel and Serre used in section 8.3 in [3]. The general metrization theorems of Palais they used produce metrics which are bounded at infinity.

The important property of this metric is that by choosing a base point x_0 in Ω and taking its orbit under the Γ-action we can embed the group Γ with a word metric quasi-isometrically in $\overline{X}$. Now $\overline{X}$ and Γ have the same large scale geometry, therefore the boundary Y must contain the same asymptotic information about both spaces, and we can think of Y as a boundary of Γ. The accumulation points of $\Gamma \subseteq X^*$ is the analogue of the limit set of a Fuchsian group; it is a closed subset of Y. In Example 4.2.8, this boundary is a subset of the the top and the bottom circles of the cylinder $X^* - X$ with the lexicographic topology.

The space Y does not satisfy many of the properties from Definition 5.5.1. From sections 5.1—5.3, we know it does satisfy a weakening of property (1) and the crucial property (2). We also find it natural to insist that the boundary be Γ-equivariant, and our Y satifies this additional property together with all of Bestvina's examples. On the other hand, this assignment of the boundary Y to Γ is not canonical and depends on the chosen continuous model $\overline{X}$ of Γ. One should not expect even the weak naturality properties established in [2] for the boundaries of Definition 5.5.1. It seems that naturality is restricted to hereditary properties such as the fact that for an algebraic subgroup H of G, $Y(H)$ embeds in $Y(G)$.

Property (3) is the next most desirable feature. The way it comes up in Bestvina's context is always via geodesic combings on the groups and the spaces $Z - Y$ which are essential for the constructions of the corresponding boundaries Y. According to [11], nonuniform arithmetic lattices are not combable. This suggests that failure of property (3) should be generally unavoidable for arithmetic groups.

In order to make the boundary Y useful in proving the Novikov conjecture for Γ, one needs to look for certain equivalence classes of boundary points and relativize the notion of size for the translates of compact subsets K.

5.5.3. **Definition.** For any subset K of a metric space (X,d) let $K[D]$ denote the set $\{x \in X : d(x,K) \le D\}$. If (X,d) is embedded in a topological space X^* as an open dense subset, a set $A \subseteq Y = X^* - X$ is *boundedly saturated* if for every closed subset C of $\hat{X}$ with $C \cap Y \subseteq A$, the closure of each D-neighborhood of $C \backslash Y$ for $D \ge 0$ satisfies $\overline{(C\backslash Y)[D]} \cap Y \subseteq A$.

It is easy to see that in sufficiently nice spaces, including all spaces in this paper, the collection of boundedly saturated subsets of Y is closed with respect to taking complements, intersections and unions. In other words, it is a Boolean algebra of sets BA. It is clearly independent of the choice of bounded metric in $\overline{X}/\Gamma$. Since all arithmetic subgroups of the given G are commensurable, this gives an invariant of arithmetic subgroups of G.

Our next goal is to generate a convenient subalgebra of BA.

5.5.4. **Definition** (Cubical Cellular Decompositions). Let $I^r = [-1,1]^r$ be the r-dimensional cube embedded in $\mathbb{R}^r$. It has 2^n vertices indexed by various r-tuples with entries either 1 or -1. Let us denote this set by $V_{(-1)}$. We also say that $V_{(-1)}$ is derived from $I_{(-1)} = \{\pm 1\}$ and write this as $V_{(-1)} = I^r_{(-1)}$. Now define the following subsets of I:

$$I_{(0)} = \{-1,0,1\}, \quad I_{(1)} = \left\{-1,-\frac{1}{2},0,\frac{1}{2},1\right\}, \quad \dots$$

where

$$I_{(i)} = \left\{-1,\dots,\frac{k}{2^i},\frac{k+1}{2^i},\dots,1\right\}, \quad k \in \mathbb{Z}, \quad -2^i \le k \le 2^i,$$

for $i \in \mathbb{N}$. We also get the corresponding derived subsets of I^r:

$$V_{(0)}, V_{(1)}, \dots, V_{(i)} = \{v_i(s_1,\dots,s_r)\} = I^r_{(i)}, \dots$$

where

$$v_i(s_1,\dots,s_r) \stackrel{\text{def}}{=} \left(\frac{s_1}{2^i},\dots,\frac{s_r}{2^i}\right), \quad s_j \in \mathbb{Z}, \quad -2^i \le s_j \le 2^i.$$

At each stage $V_{(i)}$ is the set of vertices of the obvious cellular decomposition of I^r, where the top dimensional cells are r-dimensional cubes with the j-th coordinate projection being an interval

$$\left[\frac{k_j}{2^i},\frac{k_j+1}{2^i}\right] \subseteq I, \quad 1 \le j \le i.$$

These cells can be indexed by the n-tuples $\{(k_1,\dots,k_j,\dots,k_r) : -2^i \le k_j < 2^i\}$, the coordinates of the lexicographically smallest vertex, $2^{(i+1)r}$ of the r-tuples at all.

These decompositions behave well with respect to the sequence of collapses from §4.1 and induce cellular decompositions of the result from the (-1)-st derived decomposition of I^r and the corresponding CW-structure in B^r. We will refer to this isomorphism of CW-structures as $\Upsilon \colon \partial B^r \to \tau N$.

There are cubical analogues of links and stars of the usual simplicial notions. Thus the *star* of a vertex is the union of all cells which contain the vertex in the boundary. The *open star* is the interior of the star. For the i-th derived cubical decomposition, the open star of the vertex $v_i(s_1,\dots,s_r)$ will be denoted by $\mathrm{Star}^o(v_i(s_1,\dots,s_r))$. These sets form the *open star covering* of I^r.

By *vertices* in δN we mean the image $\Upsilon\rho(V_{(n)} \cap \partial I^r)$. Let $v \in \Upsilon\rho(V_{(n)} \cap \partial I^r)$ then

$$\mathrm{Star}^o((\Upsilon\rho)^{-1}(v) \cap V_{(n)}) = \bigcup_{v_n \in V_{(n)}, \Upsilon\rho(v_n) = v} \mathrm{Star}^o(v_n)$$

is an open neighborhood (the open star) of $(\Upsilon\rho)^{-1}(v)$, and, in fact,

$$\mathrm{Star}^o_n(v) \stackrel{\text{def}}{=} \Upsilon\rho(\mathrm{Star}^o(\rho^{-1}\Upsilon^{-1}(v) \cap V_{(n)}))$$

is an open neighborhood of v which we call the *open star* of v. The map $\Upsilon\rho$ is bijective in the interior of I^r, so $\mathrm{Star}^{\circ}_n(v)$ can be defined by the same formula for $v \in \Upsilon\rho(V_{(n)} \cap \mathrm{int} I^r)$.

In order to determine the geometry of open sets in $\hat{X}$ and ultimately saturated sets in Y, we need to study the geometric question: describe the family of flats asymptotic to the given two chambers or walls at infinity of a symmetric space X. One answer is well-known in terms of horocycles.

5.5.5. **Theorem** (Im Hof [21]). *If $y, z \in \partial X$ are contained in Weyl chambers $W(y)$, $W(z) \subseteq \partial X$, let N_y, N_z be the nilpotent components in the corresponding Iwasawa decompositions. For an arbitrary point $x \in X$ the intersection of the horocycles $N_y \cdot x \cap N_z \cdot x$ parametrizes the set of all flats asymptotic to both $W(y)$ and $W(z)$.*

The minimal strata $e(B)$ for $B \in \mathcal{B}_{\mathbb{R}}$ parametrize the flats which are asymptotic to $W(B)$.

5.5.6. **Definition.** Define the subsets $\mathcal{A}(B, B') \subseteq e(B)$ to be the geodesic projections $q_B(N_B \cdot x \cap N_{B'} \cdot x)$ in the sense that they consist of $a \in e(B)$ such that the flat $q_B^{-1}(a)$ is asymptotic to $W(B')$.

This parametrization is more convenient for us because each $\xi \in e(B)$ is precisely the point of intersection $e(B) \cap q_B^{-1}(\xi) = \{\xi\}$. Now given an open subset $U \subseteq e(B)$, the corresponding open set $C(U) \subseteq \hat{X}$ can be described as $\complement(\mathrm{cl}\, q_B^{-1}(\complement U))$, and $q_B^{-1}(\complement U)$ can be identified easily by examining the closure of each flat $q_B^{-1}(\xi)$, $\xi \notin U$.

5.5.7. **Proposition.** *Given B, $B' \in \mathcal{B}_{\mathbb{R}}$, the flats which are asymptotic to both $W(B)$ and $W(B')$ are parametrized by*

$$\mathcal{A}(B, B') \overset{\sigma}{\rightsquigarrow} \mathcal{A}(B', B).$$

If $S \subseteq \mathcal{A}(B, B')$ then $\sigma(S) \subseteq \mathcal{A}(B', B)$ is contained in the closure $\mathrm{cl}(q_B^{-1}(S))$.

5.5.8. **Corollary.** *If B, $B' \in \mathcal{B}_{\mathbb{R}}$ and $U \subseteq \varepsilon(B)$ is an open subset then $y \in \varepsilon(B')$ is contained in $C(U)$ if and only if either*

1. *$y \in e(B')$ and its orthogonal projection π_B onto $\mathcal{A}(B', B)$ is not contained in the subset $A_B(U)$ corresponding bijectively to $U \cap \mathcal{A}(B, B')$, or*
2. *$y \in \partial e(B')$ and $y \notin \mathrm{cl}(\pi_B^{-1} A_B(U))$.*

The intersections of $C(U)$ with $\varepsilon(P)$, $P \in \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}$, are open product subsets.

From this description easily follows

5.5.9. **Proposition** (Weak Summability). *Given arbitrary open subsets U_1 and $U_2 \subseteq \varepsilon(B)$, it may not be true that*

$$C(U_1 \cup U_2) = C(U_1) \cup C(U_2).$$

However, the open stars in any derived decomposition of $\varepsilon(B)$ from Definition 5.5.4 *do have this property.*

5.5.10. **Corollary.** *Given a finite collection of open subsets $\Omega_1, \ldots, \Omega_n \subseteq \hat{X}$ with $\varepsilon(B) \subseteq \bigcup \Omega_i$ there is another finite collection of open subsets $U_1, \ldots, U_m \subseteq \varepsilon(B)$ so that*

- $\varepsilon(B) \subseteq \bigcup U_j$,
- $\forall\ 1 \le j \le m\ \exists\ 1 \le i \le n$ with $C(U_j) \subseteq \Omega_i$,
- $C(\bigcup U_j) = \bigcup C(U_j)$.

5.5.11. **Theorem.** *The following subsets of Y are boundedly saturated:*

- *each $\varepsilon(P) = s(P) \times R_u P(\mathbb{R})^*$ for $P \in \mathcal{P}_{\mathbb{R}} \backslash \mathcal{P}_{\mathbb{Q}}$,*
- *each product cell in $\varepsilon(P) \cap Y$ for all $P \in \mathcal{P}_{\mathbb{Q}}$.*

This defines a partition $\mathcal{E}$ of Y into disjoint boundedly saturated subsets.

Proof. The proof is entirely similar to that in section 8 of [15]. One uses Proposition 5.5.7 and Corollary 5.5.8 to create 'barriers' consisting of translates of fundamental domains that isolate the boundedly saturated subsets of the boundary. As explained in those proofs, one may use general Siegel fundamental sets in place of the geometrically explicit fundamental domains of Garland–Raghunathan for rank one lattices or Grenier for lattices in SL_3. $\triangledown$

5.5.12**. Definition.** The boundedly saturated sets identified in Proposition 5.5.11 generate a Boolean subalgebra of sets BA.

6. Proof of Theorem 1

The general plan of the proof is common with [7, 8, 15, 16]

6.1. **Outline.** Given a discrete group Γ whose classifying space $B\Gamma$ is a finite complex, we assume there is a compactification Z of the universal cover $E\Gamma$ such that the free action of Γ on $E\Gamma$ extends to Z, and Z is acyclic with respect to the modified Čech homology as in Definition 5.3.1, with coefficients in $K(R)$. The idea is to interpret $\alpha(\Gamma)$ as a Γ-fixed point map of two Γ-spectra in the following commutative diagram.

$$\begin{array}{ccc} B\Gamma_+ \wedge K(R) & \xrightarrow{\alpha(\Gamma)} & K(R[\Gamma]) \\ \simeq\downarrow & & \simeq\downarrow \\ \mathcal{R}^\Gamma & \xrightarrow{\pi_*^\Gamma} & \mathcal{T}^\Gamma \end{array}$$

Here $\mathcal{R}$ and $\mathcal{T}$ are the K-theory spectra of certain categories of free R-modules parametrized over $E\Gamma \times I$. The important feature is that the resulting spectra depend only on the global behavior of the supports of the modules. This is manifested in the equivalence $\mathcal{R} \simeq \check{h}(Y;KR)$ when Γ satisfies the assumptions above.

There are canonical maps from fixed points to homotopy fixed points and the commutative square

$$\begin{array}{ccc} \mathcal{R}^\Gamma & \xrightarrow{\pi_*^\Gamma} & \mathcal{T}^\Gamma \\ \rho^*\downarrow & & \downarrow \\ \mathcal{R}^{h\Gamma} & \xrightarrow{\pi_*^{h\Gamma}} & \mathcal{T}^{h\Gamma} \end{array}$$

where ρ^* happens to be an equivalence in this situation.

6.1.1**. Definition.** Let C_1 and C_2 be two closed subsets of Y. The pair (C_1, C_2) is called *excisive* if there is an open subset $V \subseteq Z$ such that $C_2 - C_1 \subseteq V$ and $\overline{V} \cap C_1 \subseteq C_2$. For two arbitrary subsets U_1 and U_2, the pair (U_1, U_2) is excisive if every compact subset C of $U_1 \cup U_2$ is contained in $C_1 \cup C_2$ where (C_1, C_2) is an excisive pair of closed subsets with $C_i \subseteq U_i$. A collection of subsets $U_i \subseteq Y$ is called excisive if every pair in the Boolean algebra of sets generated by U_i is excisive.

We make an additional assumption that the boundary $Y = Z - E\Gamma$ contains a Γ-invariant family $\mathcal{F}$ of excisive boundedly saturated subsets that cover Y. This guarantees that there is a map

$$\sigma\colon \mathcal{T} \longrightarrow \operatorname*{\underleftarrow{holim}}_{A \in \mathcal{A}} NA \wedge K(R),$$

where $\mathcal{A}$ is a contractible Γ-category of finite rigid coverings A of Y by the sets from $\mathcal{F}$. The composition $\sigma^{h\Gamma} \circ \pi_*^{h\Gamma}$ is induced from a Γ-equivariant map

$$\theta\colon \check{h}(Y;K(R)) = \operatorname*{\underleftarrow{holim}}_{U \in \operatorname{Cov} Y} NU \wedge K(R) \longrightarrow \operatorname*{\underleftarrow{holim}}_{A \in \mathcal{A}} NA \wedge K(R)$$

which we describe next. It is a general fact that if θ is a (nonequivariant) equivalence then $\theta^{h\Gamma}$ is also an equivalence, and one has $\alpha(\Gamma)$ as the first map in a composition which is an equivalence.

Note that very little is known about the other maps in the composition but this still makes $\alpha(\Gamma)$ a split injection.

In the simplest case when $\mathcal{F}$ are open sets, θ coincides with the restriction map induced by the inclusion $\mathcal{A} \subseteq \mathrm{Cov}\, Y$. To identify θ in our more general situation, we need to make a sensible choice of boundedly saturated sets $\mathcal{F}$.

The following is the summary of the required conditions on $\mathcal{A}$.

(1) There is a subcategory $\mathcal{O}rd\, Y$ of $\mathcal{C}ov\, Y$ such that the inclusion $\jmath\colon \mathcal{O}rd\, Y \hookrightarrow \mathcal{C}ov\, Y$ induces a weak homotopy equivalence;
(2) For each set $U = \phi(y)$ for $\phi \in \mathcal{O}rd\, Y$ there is an open set $V(U) \subseteq \hat{X}$ with the following properties: (1) $V \cap Y = U$ and (2) $\{V(U) : U \in \mathrm{im}\, \phi\}_{\mathcal{O}rd\, Y}$ form a cofinal system of finite coverings of Y by open subsets of $\hat{X}$;
(3) Given a covering $\phi \in \mathcal{O}rd\, Y$, there is an assignment (which we call *saturation* and denote by **sat**) of a based boundedly saturated subset $A_y \subseteq Y$ to each set $\phi(y)$ so that **sat** induces a natural transformation
$$\mathbf{sat}_\ast \colon N\_ \wedge K(R) \longrightarrow N\mathbf{sat}(\_) \wedge K(R),$$
and the collection $\mathcal{A}$ above is precisely the result of applying saturation to $\mathcal{O}rd\, Y$. We require the resulting collection to be *excisive* in the sense defined in [8]. We require that each morphism $\mathbf{sat}_\ast$ is a weak equivalence of spectra by Quillen's Theorem A applied to $\mathbf{sat}_\ast \colon N\_ \to N\mathbf{sat}(\_)$.

6.2. **Orderly coverings.** We will construct a cofinal family of finite open coverings of Y that satisfies conditions (1) and (2). Recall that a rigid covering $\beta \in \mathcal{C}ov\, Y$ of Y consists of pairs $x \in U(x)$ where $x \in Y$ and the values $U(x)$ lie in a finite open covering of Y. Let $\mathcal{U}$ be the underlying finite open covering $\mathrm{im}\, \beta$.

Fix a Borel subgroup $B \in \mathcal{B}_{\mathbb{R}}$. There is a number $\ell_B \in \mathbb{N}$ with $\ell_B \geq n_B$ and an open neighborhood $U_B \ni f(B)$ in X^S with
$$\mathrm{PreInf}_{\ell_B, U_B}(v) \overset{\mathrm{def}}{=} Y \cap C(\mathrm{Star}^{\mathrm{o}}_{\ell_B}(v)) \cap p^{-1}U_B \subseteq \beta(x)$$
for each $v \in V_{(\ell_B)}$ and some $x \in Y$. Let $\mathfrak{I}$ be the set consisting of all $P \in \mathcal{P}_{\mathbb{R}} \setminus \mathcal{B}_{\mathbb{R}}$ such that $f(P) \cap \complement U_B \neq \varnothing$. Let $\mathfrak{f}$ consist of all $B' \in \mathcal{B}_{\mathbb{R}}$ such that
$$\mathcal{A}(B,P) \cap \mathrm{Star}^{\mathrm{o}}_{\ell_B}(v) = \varnothing \quad \text{and} \quad \mathcal{A}(B,P) \cap \overline{\mathrm{Star}^{\mathrm{o}}_{\ell_B}(v)} \neq \varnothing.$$
Now we can define $V_B(U) \subseteq U_B$ such that
$$U_B \backslash V_B = U_B \cap \bigcup_{B \not< P \in \mathfrak{I}} \overline{f(P)}$$
and
$$\mathrm{Inf}_{\ell_B, U_B}(v) \overset{\mathrm{def}}{=} \mathrm{PreInf}_{\ell_B, U_B}(v) \cap p^{-1}(V_B) \backslash \bigcup_{B' \in \mathfrak{f}} \varepsilon(B').$$
The union of these sets over all $v \in V_{(\ell_B)}$ is an open neighborhood of $\varepsilon(B)$ in Y by the weak summability property.

Using compactness of $\hat{X}$, compactness of each $e(P)\hat{}$, $P \in \mathcal{P}_{\mathbb{R}}$, and relative compactness of $\varepsilon(P)$, one can choose finite subsets $\mathfrak{B} \subseteq \mathcal{B}_{\mathbb{R}}$ and $\mathfrak{P} \subseteq \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}$ and numbers $0 < m_P$, $k_P \in \mathbb{N}$ for $P \in \mathfrak{P}$ satisfying

(1) $\forall B \in \mathfrak{B}\ \exists P \in \mathfrak{P}$ such that $B < P$,
(2) $Y = \bigcup_{B \in \mathfrak{B}} \mathrm{Inf}_{\ell_B, U_B}(v) \cup \bigcup_{P \in \mathfrak{P}} \varepsilon(P)$,

and the following properties: fix $P \in \mathfrak{P}$ and use the notation $\mathfrak{B}(P) := \{B \in \mathfrak{B} : B < P\}$, then

(3) for some $0 < k_P \in \mathbb{N}$ and $w(P) \in \partial \hat{e}(P) \cap V_{(k)}$
$$Y \cap \delta(\hat{e}(P)) = \bigcup_{B \in \mathfrak{B}} \mathrm{Inf}_{\ell_B, U_B}(v) \cap p^{-1}\mathrm{Star}^{\mathrm{o}}_{k_B}(w) \cap \delta(\hat{e}(P)),$$

(4) $\mathcal{O}_{m,k,P}$ refines the restriction of $\mathcal{U}$ to $\varepsilon(P)$,
(5) $m_P \geq \max_{B\in\mathfrak{B}(P)}(\ell_B)$, $k_P \geq \max_{B\in\mathfrak{B}(P)}(k_B)$,
(6) each open star in the associated k_P-th cubical derived decomposition of $\varepsilon(\hat{e}(P))$ contains at most one point from $\{W(B) : B \in \mathfrak{B}(P)\}$,
(7) for each $w \in \partial\hat{e}(P) \cap V_{(k_P)}$ there exists $B \in \mathfrak{B}$ such that

$$\text{either } W(B) \in \mathrm{Star}^{\mathrm{o}}_{k_P}(w) \text{ or } p^{-1}(\mathrm{Star}^{\mathrm{o}}_{k_P}(w)) \subseteq \bigcup_{v\in V_{(\ell_B)}} \mathrm{Inf}_{\ell_B,U_B}(v).$$

For a Borel subgroup $B(w)$ define

$$\mathrm{Ord}_{\ell_B,U_B,k_P}(v;w) \overset{\text{def}}{=} \left(\mathrm{Inf}_{\ell_B,U_B}(v)\backslash e(P)^{\wedge}\right) \cup p^{-1}\mathrm{Star}^{\mathrm{o}}_{k_P}(w)$$

and

$$\mathrm{ExcOrd}_{\ell_B,U_B,k_P}(v;w) \overset{\text{def}}{=} \mathrm{Ord}_{\ell_B,U_B,k_P}(v;w)\backslash \bigcup_{B<P'} e(P')^{\wedge}.$$

Consider the category *ExcOrdY* of finite open coverings by the sets (Exc)$\mathrm{Ord}_{\ell_B,U_B,k_P}(v;w)$ and $\mathcal{O}_{m,k,P}$ for all choices of β, $\mathfrak{B}$, $\mathfrak{P}$, etc., and generate all finite rigid coverings $\omega \in \mathcal{C}ov\,Y$ which satisfy

- $\mathrm{im}\,\omega \in \mathit{ExcOrdY}$,
- $\omega(y) = \mathrm{Ord}_{\ell_P,k}(v;w)$ for some $P \in \mathfrak{P}$ if and only if $y \in \varepsilon(P)$,
- if $y \in \varepsilon(B)$ for some $B \in \mathfrak{P}$ then

$$\omega(y) = \mathrm{ExcOrd}_{\ell_{P(w)},k}(v;w)$$

for some v where $\chi(W(B)) = \mathrm{Star}^{\mathrm{o}}_k(w)$ for a fixed finite rigid covering χ of $\varepsilon X(M^0_1)$ by open stars $\mathrm{Star}^{\mathrm{o}}_k(z)$, $z \in V_{(k)}$,
- $\omega(y) \in \mathcal{O}_{m,k}$ if $y \in \varepsilon(P)$.

The resulting coverings form a full subcategory $\textsc{PreOrd}Y \subseteq \mathcal{C}ov\,Y$. This procedure may look asymmetric as to the roles of maximal strata played in corners

$$\overline{X(B)} = e(P')^{\wedge} \cup e(P'')^{\wedge}$$

when P', $P'' \in \mathfrak{P}$ and $y \in \varepsilon(B)$: there is a choice of w and, hence, of particular $P^{(j)}$ involved here. The asymmetry disappears after the next step when one generates the smallest full subcategory $\mathcal{O}rd\,Y$ of $\mathcal{C}ov\,Y$ containing $\textsc{PreOrd}Y$ and closed under intersections.

The category $\mathcal{O}rd\,Y$ is not cofinal; however the map

$$\jmath^*\colon \check{h}(Y;KR) \longrightarrow \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\mathrm{holim}}\,(N\_ \wedge KR)$$

induced by the inclusion $\jmath\colon \mathcal{O}rd\,Y \hookrightarrow \mathcal{C}ov\,Y$ is a weak homotopy equivalence by Quillen's Theorem A, cf. [15].

6.3. **Definition of $\mathcal{A}$.** The boundedly saturated coverings we produce are outcomes of actual saturation with respect to a Boolean algebra of boundedly saturated sets. The construction is by induction on the rank. Saturation enlarges the sets in $\mathcal{O}rd\,Y$ using the chosen coverings α_B, $B \in \mathfrak{B}$, and π_P, $P \in \mathfrak{P}\backslash\mathfrak{B}$. It suffices to present the construction of boundedly saturated coverings $\alpha(\omega, \alpha_B, \pi_P)$ based on generators $\omega \in \textsc{PreOrd}Y$.

6.3.1. **Definition.** For $B \in \mathcal{B}_{\mathbb{R}}(G)$ use the notation $\alpha_{i,B}$ for the finite rigid covering of the cell $\sigma_{i,B}$ given by $\alpha_{i,B}(y) = \alpha_B(y) \cap \sigma_{i,B}$ for each $y \in \sigma_{i,B}$. The same formula associates $\alpha_{i,B}(y) \subseteq \sigma_{i,B}$ to each $y \in \partial e(B)$. For $P > B$ of type i, define $\Pi_{B,P}\colon \delta e(B) \to \mathrm{im}\,\pi_P$ by $\Pi_{B,P}(y) = \alpha_{i,B}(y) \times (\hat{e}(P))^S_{\mathbb{Q}}$ where $B' \in \mathcal{B}_{\mathbb{R}}$ and the vertices v, w are from

$$\omega(y) = \mathrm{ExcOrd}_{\ell_{B'(w)},U_{B'},k_P}(v;w).$$

Now set

$$\alpha^{\text{int}}(y) = \begin{cases} \pi_P(y) & \text{if } y \in \varepsilon(P),\ P \in \mathfrak{P} \cap \mathcal{P}_{\mathbb{Q}} \\ \omega(y)\backslash\varepsilon(B) \cup \Pi_{B,P}(y) & \text{if } y \in \varepsilon(B),\ B \in \mathfrak{B}, \\ \omega(y) \cup \Pi_{B,P(j)}(y) & \text{if } y \in \sigma_{j,B},\ B \notin \mathfrak{B}, \\ \omega(y) & \text{otherwise.} \end{cases}$$

The *saturation* of a subset S with respect to a Boolean algebra of sets is the union of elements of BA which intersect S nontrivially. Define $\alpha(\beta)$ as the finite rigid covering of Y by the saturations of sets S in $\alpha^{\text{int}}(\beta)$ with respect to the Boolean algebra BA from Definition 5.5.12. The equivariant category $\mathcal{A}$ is the collection of all such α.

Each of the two steps in this construction preserves the homotopy type of the nerve of ω and α^{int}. Now the natural transformation $N\_ \to N\alpha^{\text{int}}(\_) \to N\alpha(\_)$ is composed of homotopy equivalences. So

$$\underset{\overleftarrow{\mathcal{O}rd\,Y}}{\text{holim}}\,(N\_ \wedge KR) \xrightarrow{\simeq} \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\text{holim}}\,(N\alpha^{\text{int}}(\_) \wedge KR) \xrightarrow{\simeq} \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\text{holim}}\,(N\alpha(\_) \wedge KR).$$

This procedure also defines a left cofinal saturation functor $\mathbf{sat}\colon \mathcal{O}rd\,Y \to \mathcal{A}$ so that the induced map

$$\mathbf{sat}_*\colon \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\text{holim}}\,(N\alpha(\_) \wedge KR) \xrightarrow{\simeq} \underset{\overleftarrow{\mathcal{A}}}{\text{holim}}\,(N\_ \wedge KR)$$

is a weak equivalence. The composition of all equivalences above is the required equivalence

$$\theta\colon \check{h}(Y;KR) \simeq \underset{\overleftarrow{\mathcal{A}}}{\text{holim}}\,(N\_ \wedge KR).$$

This completes the proof of Theorem 1.

References

[1] W. Ballmann, E. Ghys, A. Haefliger, P. de la Harpe, E. Salem, R. Strebel, and M. Troyanov, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Birkhäuser (1989).

[2] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J. **43** (1996), 123–139.

[3] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comm. Math. Helv. **48** (1973), 436–491.

[4] N. Bourbaki, *General topology. Chapters 1–4*, Springer (1989).

[5] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions, and localizations*, Lecture Notes in Mathematics **304**, Springer (1972).

[6] G. Carlsson and B. Goldfarb, *Algebraic K-theory of geometric groups*, in preparation.

[7] G. Carlsson and E.K. Pedersen, *Controlled algebra and the Novikov conjecture for K- and L-theory*, Topology **34** (1993), 731–758.

[8] ______, *Čech homology and the Novikov conjectures*, Math. Scand. **82** (1998), 5–47.

[9] G. Carlsson, E.K. Pedersen, and W. Vogell, *Continuously controlled algebraic K-theory of spaces and the Novikov conjecture*, Math. Ann. **310** (1998), 169–182.

[10] G. Carlsson, E.K. Pedersen, and J. Roe, *Controlled C^*-algebra theory and the injectivity of the Baum–Connes map*, preprint (1996).

[11] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S.V.F. Levy, M.S. Paterson, and W.P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers (1992).

[12] B. Farb, *Combing lattices in semisimple Lie groups*, in *Groups–Korea '94* (Pusan), de Gruyter (1995), 57–67.

[13] S.C. Ferry, A. Ranicki, and J. Rosenberg, editors, *Novikov conjectures, index theorems and rigidity* (Oberwolfach, 1993), 2 vols., Cambridge U. Press (1995).

[14] S.C. Ferry and S. Weinberger, *A coarse approach to the Novikov conjecture*, in *Novikov conjectures, index theorems and rigidity, Vol. 1* (Oberwolfach, 1993), Cambridge U. Press (1995), 147–163.

[15] B. Goldfarb, *Novikov conjectures for arithmetic groups with large actions at infinity*, *K*-theory **11** (1997), 319–372.

[16] ______, *Novikov conjectures and relative hyperbolicity*, Math. Scand. **85** (1999), 169–183.

[17] M. Goresky, G. Harder, and R. MacPherson, *Weighted cohomology*, Invent. Math. **116** (1994), 139–214.

[18] M. Gromov, *Hyperbolic groups*, in *Essays in group theory* (S.M. Gersten, ed.), M.S.R.I. Publication **8** (1987), 75–264.

[19] ______, *Asymptotic invariants of infinite groups*, in *Geometric group theory*, Vol. 2 (G.A. Niblo and M.A. Roller, eds.), Cambridge U. Press (1993), 1–295.

[20] Y. Guivarc'h, L. Ji, and J. Taylor, *Compactifications of symmetric spaces*, C. R. Acad. Sci. Paris **317** (1993), 1103–1108.

[21] H.-C. Im Hof, *Visibility, horocycles, and the Bruhat decompositions*, in *Global differential geometry and global analysis* (D. Ferus et al., eds.), Lecture Notes in Mathematics **838** (1981), 149–153.

[22] L. Ji, *Satake and Martin compactifications of symmetric spaces are topological balls*, Math. Res. Letters **4** (1997), 79–89.

[23] F.I. Karpelevič, *The geometry of geodesics and the eigenfunctions of the Beltrami-Laplace operator on symmetric spaces*, Trans. Moscow Math. Soc. **14** (1965), 51–199.

[24] G.F. Kushner, *The Karpelevič compactification is homeomorphic to a ball*, Proc. Seminar in Vector and Tensor Analysis **19**, Moscow U. Press (1979), 95–111. (Russian)

[25] N. Le Ahn, *On the Vietoris–Begle theorem*, Mat. Zametki **36** (1984), 847–854. (Russian)

[26] A.I. Malcev, *On one class of homogeneous spaces*, Izv. AN SSSR, Ser. Mat. **13** (1949), 9–32. (English translation: Amer. Math. Soc. Transl. **39** (1951), 276–307.)

[27] I. Satake, *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math. **77** (1960), 77–110.

[28] ———, *On compactifications of the quotient spaces for arithmetically defined discontinuous groups*, Ann. of Math. **72** (1960), 555–580.

[29] J.C. Taylor, *Compactifications defined by a polyhedral cone decomposition of* $\mathbb{R}^n$, in *Harmonic analysis and discrete potential theory* (M.A. Picardello, ed.), Plenum Press (1992), 1–14.

[30] S. Zucker, L_2*-cohomology of warped products and arithmetic groups*, Invent. Math. **70** (1982), 169–218.

[31] ———, *Satake compactifications*, Comm. Math. Helv. **58** (1983), 312–343.

[32] ———, L_2*-cohomology and intersection homology of locally symmetric varieties, II*, Comp. Math. **59** (1986), 339–398.

Department of Mathematics and Statistics, SUNY, Albany, NY 12222
E-mail address: goldfarb@math.albany.edu