Solution to #3 on page 126.

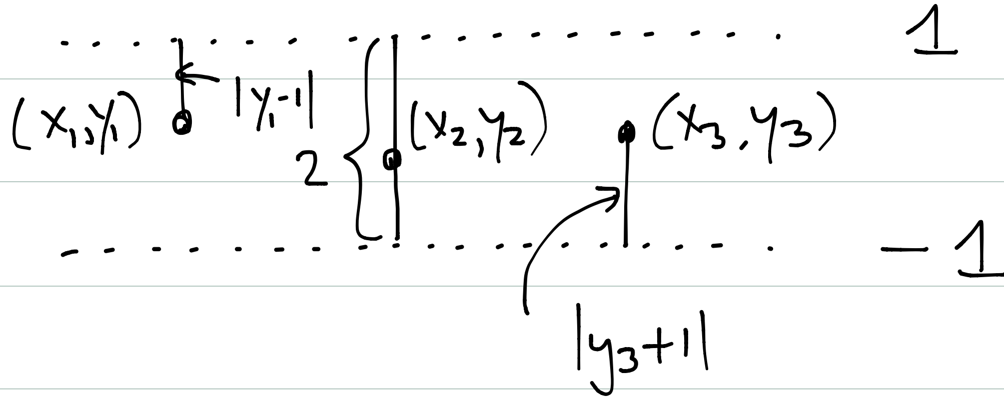
What, didn't I give a metric for this in class already? Yes, but there was a mistake — that number 100 was too large. Replace it by 1, and that's a good metric. This solution is to show that kind of metric is not the only one possible.

First, it suffices to give a metric for the subspace $\mathbb{R} \times (-1, 1)$ since it is homeomorphic to $\mathbb{R} \times \mathbb{R}$ via $h(x, y) = (x, \tan y)$.

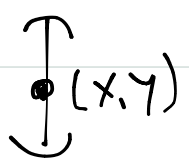
Define $d((x_1, y_1), (x_2, y_2)) =$

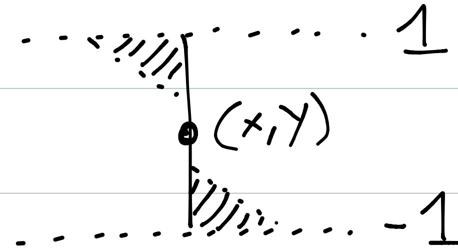
$$\begin{cases} |x_1 - x_2| + |y_1 - 1| + |y_2 + 1| & \text{if } x_1 < x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$

d is clearly reflexive, symmetric, and the Δ-inequality is generally true with the margin of 2 units:



Metric balls for d look like this:

either 

or 

This basis of metric balls is generated (as unions) by balls of the first kind, that is by $B_d((x,y),\varepsilon)$ such that $\varepsilon < \min\{|y-1|,|y+1|\}$. This is the basis for the dictionary order topology. So the metric topology is indeed the dictionary order topology.

Comment 2 | My description of the metric balls for the uniform metric allow to compare the uniform topology to the product topology and the box topology rather easily: $\mathcal{T}_b \supsetneq \mathcal{T}_u \supsetneq \mathcal{T}_p$.

a basic open set with size of factors decreasing to 0 is not in $\mathcal{T}_u$ b/c open balls in $\bar{\rho}$ have fixed size factor projections

a metric ball for $\bar{\rho}$ with a small radius ε ($< \frac{1}{2}$ for example) is not open in $\mathcal{T}_p$