Solution to 4(b).

In this problem, $\{\mathcal{T}_\alpha\}$ is a family of topologies on X. We already know from part (a) that $\bigcap \mathcal{T}_\alpha$ is a topology but $\bigcup \mathcal{T}_\alpha$ is not necessarily.

About the first question: we know there are topologies $\mathcal{T}$ on X that contain all of $\mathcal{T}_\alpha$, for example the discrete topology $\mathcal{D}$. There is a partial order on all topologies according to inclusion in each other. We are looking for "a smallest" topology $\mathcal{T}_{min}$ among those that contain all $\mathcal{T}_\alpha$.

I say take $\bigcap \mathcal{T}$. By part (a) it is a topology. It is still of the same kind: all $\mathcal{T}_\alpha$ are contained in $\bigcap \mathcal{T}$.

Notice that this by itself proves uniqueness: suppose there is another $\mathcal{T}_{min}$. Then $\bigcap \mathcal{T} \subseteq \mathcal{T}_{min}$ because $\mathcal{T}_{min}$ is one of $\mathcal{T}$. And $\mathcal{T}_{min} \subseteq \bigcap \mathcal{T}$ because this is exactly what "minimal" means with respect to inclusions.

Now for the second question. Consider all topologies $\mathcal{T}$ contained in all of $\mathcal{T}_\alpha$. We could try to take their union $\bigcup \mathcal{T}$ to mimic the answer to 1st question — but we know from (a) this wouldn't be a topology necessarily. Instead, we can do something more basic. Take $\bigcap \mathcal{T}_\alpha$. This is a topology. Also, $\bigcap \mathcal{T}_\alpha \subseteq \mathcal{T}_\alpha$ for all α. If $\mathcal{T} \subseteq \mathcal{T}_\alpha$

for all α then $\mathcal{T} \subseteq \bigcap \mathcal{T}_\alpha$, so $\bigcap \mathcal{T}_\alpha$ is a largest such topology. For uniqueness, compare $\mathcal{T}_{max}$ to $\bigcap \mathcal{T}_\alpha$ as before.

By the way, my description of the answer in class was simply spelling out what the elements of $\bigcap \mathcal{T}_\alpha$ look like.