Question 1. Suppose f is continuous. For any $A \subset Y$ the subset $\overline{A} \subset Y$ is closed. So $f^{-1}(\overline{A})$ is closed. Now $\overline{f^{-1}(A)}$ is the smallest closed subset containing $f^{-1}(A)$, and $f^{-1}(\overline{A})$ is such, so we get the required inclusion.

Suppose we have the given inclusion for all $A \subset Y$. If A is closed, $A = \overline{A}$. Now $f^{-1}(A) \subset \overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$, so all inclusions are equalities, so $f^{-1}(A) = \overline{f^{-1}(A)}$ and $f^{-1}(A)$ is closed. This shows that f is continuous.

Question 2. Yes and yes. For part (b), in the 3-point space X with the topology $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ the subspace $\{a,b\}$ is discrete and so Hausdorff, but X is not Hausdorff because c cannot be separated from either a or b.

Part 2. First, recall that a metric space is always Hausdorff: given any two distinct points of a metric space M with distance $d(x,y) = d$, the open metric balls of radii $d/3$ separate x and y. If X in question is not Hausdorff, it is also not metrizable. Now (1) is a simple space with just one pair of distinct points that is not Hausdorff, therefore not metrizable. (3) is not Hausdorff because the two origins cannot be separated, so not metrizable either.

Space in (4) is a subspace of the countable product of real lines, so both Hausdorff and metrizable.

The fact that (2) is Hausdorff is not hard: any two points neither of which is the y-axis are separated by small neighborhoods that are in one-to-one correspondence with neighborhoods in $\mathbb{R}^2$. The y-axis and any other points are also separated by open neighborhoods. The one for the y-axis can always be chosen to be a narrow open strip containing the y-axis in $\mathbb{R}^2$. The harder question is whether this space is metrizable. The answer is yes. Here is a way to see this. The function $\tan^{-1}$ gives a homeomorphism from $\mathbb{R}$ to $(-1,1)$, so two components $\tan^{-1}$ give a homeomorphism from $\mathbb{R}^2$ to the open square $(-1,1)^2$. This homeomorphism gives the homeomorphism from $\mathbb{R}^2/\{y-\text{axis}\}$ to $(-1,1)^2/\{y-\text{axis} \cap (-1,1)^2\}$. One can use methods from Test 3 to see that the last space is homeomorphic to the "bow-tie" $\{x \in \mathbb{R}^2 \mid |x_2| < |x_1| < 1\}$. The "bow-tie" is a subspace of $\mathbb{R}^2$, so it is Hausdorff and metrizable.