MAT 540A Test No. 5 Fall 2014

Please turn in the solutions on Monday, November 10, in class.

**Question 1** (Qual Exam from January 2014)**.** Let $X$ be a metrizable and separable topological space. Prove that $X$ is normal and 2nd countable.

**Question 2** (Qual Exam from January 2010)**.** Let $f\colon X \to Y$ be a closed, continuous surjection. Assume that $X$ is normal. Prove that $Y$ is normal.

**Question 3** (Qual Exam from January 2009)**.** Let $X$ be a path-connected space. Given a subset $A \subset X$ and two points $a, b \notin A$, we will say that $A$ *separates* $a$ from $b$ if $a$ and $b$ are in different path components of the complement of $A$, that is, if every path from $a$ to $b$ in $X$ meets $A$. Let $A_1 \supset A_2 \supset A_3 \supset \ldots$ be a decreasing sequence of closed subsets of $X$, and let $A$ be the intersection of all subsets $A_1, A_2, A_3, \ldots$.

If each $A_i$ separates $a$ from $b$, show that $A$ separates $a$ from $b$.

**Question 4** (Qual Exam from January 2008)**.** Recall that the *dictionary order topology* on the unit square $I^2 = [0,1] \times [0,1]$ is generated by the subbase consisting of the subsets

$$S_{a,b,c,d} = \{a\} \times (b,1] \cup (a,c) \times [0,1] \cup \{c\} \times [0,d)$$

for all quadruples of numbers $a, b, c, d$ with $0 \leq a < c \leq 1$, $0 < b < 1$, $0 < d < 1$, and the subsets

$$T_{k,l,m} = \{k\} \times (l,m)$$

for all triples $k, l, m$ with $0 \leq k \leq 1$, $0 \leq l < m \leq 1$. We denote the unit square with this topology by $I^2_\ell$.

(a) Show that $I^2_\ell$ is compact.

(b) Show that $I^2_\ell$ is not second countable.

(Note: all compact metric spaces are second countable, so this shows $I^2_\ell$ is not metrizable.)

**Question 5.** This is an example that shows that products, even finite, of normal spaces are not necessarily normal.

(a) Show that $\mathbb{R}_\ell$ is normal.

(b) Show that $\mathbb{R}^2_\ell$ is not normal.