Question 3. Suppose A does not separate a from b. Then there is a continuous path $p\colon [0,1] \to X\backslash A$ with $p(0) = a$ and $p(1) = b$. Since all A_i are closed, $X\backslash A_i$ are open and

$$X\backslash A = \bigcup_{i=1}^{\infty} X\backslash A_i.$$

Now $p([0,1])$ is a compact subset of X and $\{X\backslash A_i\}$ cover $p([0,1])$. There must be a finite subcovering. Take the largest $X\backslash A_k$ of the nested sets $X\backslash A_i$ in the subcovering. So $p([0,1]) \subset X\backslash A_k$. This contradicts the fact that A_k separates a from b.

Question 4. One option is to adjust the proof that $[0,1]$ is compact. Instead, I will mimic the proof that the product of two compact spaces is compact. Beware that I_ℓ^2 is certainly not a product. In any case, the top edge $[0,1] \times 1$ has subspace topology which is I_ℓ, and that is not compact.

What I have in mind is an analogue of the Tube Lemma. Given any $x \in [0,1]$, $x \times [0,1]$ has the usual interval topology and so is compact. So it is covered by finitely many sets $\{U_{k_x}\}$ from the given open covering $\{U\}$. A neighborhood of $x \times [0,1]$ in I_ℓ contains the set $(x - \varepsilon_x, x + \varepsilon_x) \times [0,1]$ for some ε_x specific to x. Choose a finite collection C of x in $[0,1]$ such that $(x - \varepsilon_x, x + \varepsilon_x)$, $x \in C$, cover $[0,1]$ (using compactness of $[0,1]$ for the second time). Now the collection

$$\bigcup_{x \in C} \{U_{k_x}\}$$

is a finite subcovering of $\{U\}$.