Please turn in the solutions on Monday, October 27, in class.

Question 1 (Qual Exam from January 2014). Let $p\colon X \to Y$ be a quotient map. Assume that Y is connected and that, for all $y \in Y$, $p^{-1}(\{y\})$ is connected. Prove that X is connected.

Question 2 (Qual Exam from January 2013). Given a space X, we will say $x \sim y$ if there is no separation $X = A \cup B$ into disjoint open sets such that $x \in A$ and $y \in B$.

(a) Prove that this is an equivalence relation.

We will call the equivalence classes *quasicomponents*.

(b) Show that each quasicomponent is closed.

(c) Show that each connected component is contained in a quasicomponent.

Question 3 (Qual Exam from January 2010). Let $GL_n(\mathbb{R})$ be the set of all $n \times n$ matrices A with real entries and $\det(A) \neq 0$. Sending $A = (a_{ij})$ to the vector (b_k) with $b_k = a_{ij}$ for $k = (i-1)n + j$ represents $GL_n(\mathbb{R})$ as a subset of the Euclidean space $\mathbb{R}^{n^2}$ of dimension n^2. We give $GL_n(\mathbb{R})$ the subspace topology.

(i) Show that the determinant function $\det\colon GL_n(\mathbb{R}) \to \mathbb{R}$ is continuous.

(ii) Show that I and $-I$ lie in different path components of $GL_n(\mathbb{R})$ if n is odd. (Here I denotes the $n \times n$ identity matrix.) Show that I and $-I$ lie in the same path component if n is even. (Hint: try the case $n = 2$ first.)

Question 4 (Qual Exam from August 2010). Let $X = [0,1]/(0,1)$ and let

$$\pi\colon [0,1] \longrightarrow X$$

be the quotient map.

(a) Is X connected?

(b) Is X path-connected?

Consider the complement of $\pi(0)$ in X, with the subspace topology.

(c) Is it connected?

Consider the complement of $\pi(1/2)$ in X, with the subspace topology.

(d) Is it connected?

Question 5 (Qual Exam from June 2012). A property $\mathcal{P}$ of topological spaces is said to be hereditary if it carries over from a space to all its subspaces, i.e., whenever a space X has property $\mathcal{P}$, then each subspace Y of X has property $\mathcal{P}$. Which of the following properties are hereditary? If it is, give a proof. If not, give a counterexample.

(a) Hausdorff;

(b) Connectedness;

(c) Compactness;

(d) Metric space.