Question 3, part (ii). If n is odd, $\det(I) = 1$ and $\det(-I) = -1$, so any path connecting I to $-I$ would map to a path connecting 1 to -1 when composed with det. By the Intermediate Value Theorem, 0 would be in the image of that continuous composition. This contradicts the fact that $\det(M) \neq 0$ for all M in GL_n.

If n is even, we need to construct a path from I to $-I$ in GL_n. First, let's look at the case $n = 2$. The path will be a concatenation of two paths. The first is from the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ to } \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

given by

$$A(t) = \begin{pmatrix} 1-2t & -2t \\ 2t & 1-2t \end{pmatrix}$$

for all $t \in [0,1]$. Notice that $\det A(t) = (1-2t)^2 + 4t^2 > 0$ for all t. The second is from

$$\begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \text{ to } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

given by

$$B(t) = \begin{pmatrix} -1 & -2+2t \\ 2-2t & -1 \end{pmatrix}$$

for all $t \in [0,1]$. Notice that $\det B(t) = 1 + (2-2t)^2 > 0$ for all t. In the general case, I and $-I$ have $n/2$ instances of the 2-dimensional blocks on the diagonal. Use the deformations above within each block. Since the determinant is the product of the determinants of these blocks, it stays positive throughout the path.